On Non-Homogeneous Cubic Diophantine Equation $4x^2 + 4y^2 - 7xy = 19z^3$

Dr. P. Jayakumar¹, V. Pandian², R. Venkatraman³

¹Professor of Mathematics, Periyar Maniammai University, Vallam, Thanajvur -613 403, T.N, India

²Assistant Professor of Mathematics, A.V.V.M. Sri Pushpam College, Poondi-613 503, Thanajyur, T.N, India

²Assistant Professor of Mathematics, SRM University Vadapalani Campus, Chennai - 600026. T.N, India

Abstract: Four different methods of the non-zero non-negative solutions of non- homogeneous cubic Diophantine equation $4x^2 + 4y^2 - 7xy = 19z^3$ are observed. Some interesting relations among the special numbers and the solutions are determined.

Keywords: The method of factorization, integer solutions, linear transformation, relations and special numbers

2010 Mathematics subject classification: 11D25

Notations used:

 $t_{m,n}$ = Polygonal number of rank n with sides m. $P_n^{\ m}$ = Pyramidal number of rank n with size m. G_n = Gnomonic number

1. Introduction

The number theory is the king of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For a broad review of variety of problems, one may try to see [3-12]. Integer solutions of cubic Diophantine Equation has appeared in Jayakumar. P.,Meena, J [16,18]. Inspired by these, in this work, we are observed another interesting four different methods of the non-zero non-negative solutions the non - homogeneous cubic Diophantine equation $4x^2 + 4y^2 - 7xy = 19z^3$ Further, some elegant properties among the special numbers and the solutions are observed

2. Method of Description

Consider the cubic Diophantine equation $4x^2+4y^2-7xy = 19z^3$ (1)

Take the linear transformations

$$x = u + v, y = u - v, u \neq v \neq 0$$
 (2)
Using (2) in (1) it gives to

$$u^{2} + 15v^{2} = 19z^{3}$$
(3)

If take
$$z = z$$
 (a, b) = $a^2+15b^2 = (a + i\sqrt{15}b)$
(a - $i\sqrt{15}b$)
where a and b non-zero distinct integers, then we

we solve (1) through dissimilar method of solutions of (1) which are furnished below.

2.1 Method: I

We can write 19 as

$$19 = (2 + i\sqrt{15}) (10 - i\sqrt{15})$$
(5)

Using (4) and (5) in (3) and applying the factorization process, this gives us

$$(u + i\sqrt{15} v) (u - i\sqrt{15} v) = (2 + \sqrt{15} i)(2 - i\sqrt{15})$$

 $(a + i\sqrt{15} b)^3(a - Ib)^3$

Equating the positive and negative factors, we get

$$(u + i\sqrt{15} v) = (2 + i\sqrt{15})(a + i\sqrt{15} b)^{3}$$
(6)
$$(u - i\sqrt{15} v) = (2 - i\sqrt{15})(a - i\sqrt{15} b)^{3}$$
(7)

It gives us

$$u = u (a, b) = 2a^3 - 90ab^2 - 45a^2b + 225b^3$$

v = v (a, b) = $a^3 - 45ab^2 + 6a^2b - 30b^3$

In sight of (2), the solutions *x*, *y* are found to be

$$x = x (a, b) = 3a^{3} - 135ab^{2} - 39a^{2}b + 195b^{3}$$
 (8)
 $y = y (a, b) = a^{3} - 45ab^{2} - 51a^{2}b + 225b^{3}$ (9)
 $z = z (a, b) = a^{2} + 15b^{2}$
Hence (4), (8) and (9) gives us two parametric the non-zero

different integral values of (1).

Observations

1. z (a, a) is a perfect square. 2. 1/106 [y (a, a) - x (a, a)] is a cubic integer 3. x (1, a) $-390P_a^{5} + 330P_A \equiv 0 (Mod3)$ 4. x (a, 1) $-6P_a^{5} + 42 P_a + G_{5a} \equiv 0 (Mod7)$ 5. y (a, 1) $-2P_b^{5} + 52P_a - G_a \equiv 0 (Mod 7)$ 6. z (0, 2) is a nasty number 7. x (2,0) is a nasty number 8. x (0, 2) is a nasty number 9. y (a,0) is a cubic integer 10.y (0,1) is a perfect Square

2.2 Method: II

Let
$$19 = \frac{(17 + i\sqrt{15})(17 - i\sqrt{15})}{16}$$
 (10)

(4)

Using (4) and (10) in (3) and applying the factorization process, this gives us

$$(u + i\sqrt{15} v)(u - i\sqrt{15} v) = \frac{1}{16} [(17 + i\sqrt{15}) (u + i\sqrt{15})^3 (u - i\sqrt{15})^3]$$

$$(17 - i\sqrt{15})(u + i\sqrt{15})^3 (u - i\sqrt{15})^3]$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{15} v) = \frac{1}{4} \left[(17 + i\sqrt{15})(a + i\sqrt{15} b)^3 \right]$$
(11)

$$(u - i\sqrt{15}v) = \frac{1}{4} [(17 - i\sqrt{15})(a - i\sqrt{15}b)^3]$$
 (12)

It gives us

$$u = u (a, b) = \frac{1}{4} [17 a^3 - 765ab^2 - 45a^2b + 225b^3]$$
 (13)

$$v = v(a, b) = \frac{1}{4} [a^3 + 51ab^2 + 45a^2b - 255b^3]$$
 (14)

In true of (2), the values of x, y are given by

$$x = x (a, b) = \frac{1}{4} [18a^3 - 810ab^2 + 6a^2b - 30b^3]$$
 (15)

$$y = y(a, b) = \frac{1}{4} [16a^3 - 7205ab^2 - 96a^2b + 480b^3]$$
 (16)

Since our intension is to find integer solutions, taking a as 2a and b as 2b in (4), (13) and (14), the related parametric integer values of (1) are found as

$$\mathbf{x} = \mathbf{x} (\mathbf{a}, \mathbf{b}) = 36a^3 - 162ab^2 + 12a^2b - 60b^3]$$
(17)

$$y = y(a, b) = 32a^3 - 1440ab^2 - 192a^2b + 192b^3$$
 (18)

$$z = z (a, b) = 4a^2 + 60b^2$$
 (19)

Hence (17), (18) and (19) gives us two parametric the non-zero different integral values of (1).

Observations

1. z (a, a) is a perfect square. 2. x (1,1) is an even integer. 3. y (1, 1) is an even integer. 4. x (a, 1) - 72P_a⁵ + G_{69a} +24P_b = 1 (Mod 2) 5. x (1, a) - 120 P_a⁵ +102P_a - G_{57a} = 1(mod 2) 6. y (a,1) -64 P_a⁵+224P_a+G_{68a} = 1 (mod 2) 7. z (2a,a)-76t_{4, a}= 0 8. z (a, 2a) -256 t_{4,a} = 0 9. y (a, a) -x (a, a)-2508 P_a⁵+1254 t_{4, a}= 0 Each of the following is a nasty number 10. $\frac{1}{2}$ z (0,1), $\frac{1}{6}$ z(3,0), $\frac{3}{2}$ z(2,0)

2.3 Method: III

We can write (3) as
$$u^2 + 15v^2 = 19z^3 * 1$$

$$u^{2} + 15v^{2} = 19z^{3} * 1$$
(20)
Take 1 as $1 = (1 + i\sqrt{15})(1 - i\sqrt{15})$ (21)

Using (4), (5) and (21) in (20) and applying the factorization process, this gives us

$$(u + i\sqrt{15} v)(u - i\sqrt{15} v) = \frac{1}{16} [(1 + i\sqrt{15})(1 - i\sqrt{15}) (1 - i\sqrt{15})(1 - i\sqrt{15})(1 - i\sqrt{15})(1 - i\sqrt{15})(a + i\sqrt{15} b)^3(a - i\sqrt{15} b)^3]$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{15}v) = \frac{1}{4}(1 + i\sqrt{15})(10 + i\sqrt{15})(a + i\sqrt{15}b)$$
(22)

$$(\mathbf{u} - i\sqrt{15}\,\mathbf{v}) = \frac{1}{4}(1 - i\sqrt{15})(10 - i\sqrt{15})(\mathbf{a} - i\sqrt{15}\mathbf{b})^3$$
(23)

It gives us

$$u = u (a, b) = \frac{1}{4} [-13a^3 + 585ab^2 - 135a^2b + 675b^3]$$

v = v (a, b) = $\frac{1}{4} [3a^3 - 135ab^2 - 39a^2b + 195b^3]$

In sight of (2), the values of x, y are given by

$$x = x (a, b) = \frac{1}{4} [-10a^{3} + 450ab^{2} - 174a^{2}b + 870b^{3}]$$
(24)
$$y = y (a, b) = \frac{1}{4} [-16ab^{3} + 720ab^{2} - 96a^{2}b + 480b^{3}]$$
(25)

Since our intension is to find integer solutions, taking a as 2a and b as 2b in (4), (24) and (25), the related parametric integer values of (1) are found as

 $x = x (a, b) = -20a^{3} + 900ab^{2} - 338a^{2}b + 1740b^{3}$ $y = y (a, b) = -32ab^{3} + 1440ab^{2} - 192a^{2}b + 960b^{3}$ $z = z (a, b) = 4a^{2} + 60b^{2}$

Hence the above give us two parametric the non-zero different integral values of (1).

Observations

1. z (a, a) is a perfect square 2. x (1, 1) is an even integer 3. -1/2[x (1, 0)] is a cubic integer 4. x (a,1) $-40P_a^{5}+318P_a-G_{609a} \equiv 1 (Mod2)$ 5. x (1, a) $-3480P_a^{5}+840P_a-G_{25a} \equiv 0 \pmod{19}$ 6. -1/4[y (1,0)] is a cubic integer 7. y (a,1) $+192P_a^{3}+96t_{4,a}-G_{103a} \equiv 1 \pmod{2}$ 8. y(1, a) $-1920P_a^{3}-480P_a-G_{361a} \equiv 0 \pmod{11}$ 9. x (a, a) $-y(a, a) \equiv 0 \pmod{2}$ Each of the following is a nasty number 10. $\frac{1}{2} z (0, 1), \frac{3}{2} z (1, 0), -\frac{3}{2} z (1, 0), \frac{3}{4} z (1, 0)$

2.4 Method: IV

Instead of (16), write1as

$$1 = \frac{(7 + i\sqrt{15})(7 - i\sqrt{15})}{64} \tag{26}$$

Using (4), (10) and (21) in (15) and applying the factorization process, this gives us

$$(u + i\sqrt{15}v)(u - i\sqrt{15}v) = \frac{1}{64} [(2+i\sqrt{15}(2-i\sqrt{15})(2-i\sqrt{15})(7+i\sqrt{15})(1+i\sqrt{15})(2+i\sqrt{15})(3-i\sqrt{15})(3-i\sqrt{15})(3-i\sqrt{15})(3-i\sqrt{15})(3-i\sqrt{15})(3-i\sqrt{15})(2-i\sqrt{$$

y = y (a, b) =
$$\frac{1}{8}$$
 [-48a³ + 351ab² - 363a²b + 363b³] (25)

Since our intension is to find integer solutions, taking a as 7a and b as 7b in (4), (24) and (25), the related parametric integer values of (1) are found as

Volume 5 Issue 4, April 2016

<u>www.ijsr.net</u>

Licensed Under Creative Commons Attribution CC BY

1747

 $\begin{aligned} x &= x \ (a, b) = 64a^3 - 3264 \ a^2 \ b - 2880ab^2 + 16320b^3 \\ y &= y \ (a, b) = -80a^3 - 3216 \ a^2b + 3600a \ b^2 + 3600b^3 \\ z &= z \ (a, b) = 16a^2 + 240b^2 \end{aligned}$

Hence the above give us two parametric the non-zero different integral values of (1).

Properties:

z (a, a) is a perfect square
 x (1. 0) is a cubing integer
 1/10 x(1, 1) is a perfect square

4. x (a, 1) - 128 P_a^{5} + 3328 P_a - $G_{224a} \equiv 1 \pmod{2}$ 5. x (1, a) - 97920 P_a^{3} + 19200 P_a + $G_{192a} \equiv 0 \pmod{5}$ 6. y (a, 1) +160 P_a^{5} +3136 $t_{4,a}$ - $G_{1800a} \equiv 0 \pmod{2}$ 7. y (1,a)-32160 P_a^{5} +1248 $t_{4,a}$ + $G_{1608a} \equiv 0 \pmod{3}$

Each of the following is a nasty number

8.
$$\frac{3}{8}$$
 z (1, 0), $\frac{1}{8}$ z (0, 1), $\frac{3}{8}$ x (1, 0), $-\frac{3}{10}$ y(1,0)

3. Conclusion

In this work, we observed various process of determining infinitely a lot of non-zero different integer values to the cubic Diophantine equation $4x^2 + 4y^2 - 7xy = 13z^3$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

References

- [1] Dickson, L.E., History of theory of numbers, Vol.11, Chelsea publishing company, New –York (1952).
- [2] Mordell, L.J., Diophantine equation, Academic press, London (1969) Journal of Science and Research, Vol (3) Issue 12, 20-22 (December -14)
- [3] Jayakumar. P, Sangeetha, K "Lattice points on the cone $x^2 + 9y^2 = 50z^2$ " International Journal of Science and Research, Vol (3), Issue 12, 20-22 December2014)
- [4] Jayakumar P, Kanaga Dhurga, C," On Quadratic Diophantine equation $x^2 + 16y^2 = 20z^{2n}$ Galois J. Maths, 1(1) (2014), 17-23.
- [5] Jayakumar. P, Kanaga Dhurga. C, "Lattice points on the cone $x^2 + 9y^2 = 50 z^2$ " Diophantus J. Math,3(2) (2014), 61-71
- [6] Jayakumar. P, Prabha. S " On Ternary Quadratic Diophantine equation $x^2 + 15y^2 = 14 z^2$ " Archimedes J. Math., 4(3) (2014), 159-164.
- [7] Jayakumar. P, Meena, J "Integral solutions of the Ternary Quadratic Diophantine equation: $x^2 + 7y^2 = 16z^2$ International Journal of Science and Technology, Vol.4, Issue 4, 1-4, Dec 2014.
- [8] Jayakumar . P, Shankarakalidoss, G "Lattice points on Homogenous cone $x^2 +9y^2 = 50z^2$ " International journal of Science and Research, Vol (4), Issue 1, 2053-2055, January -2015.
- [9] Jayakumar. P, Shankarakalidoss. G "Integral points on the Homogenous cone $x^2+ y^2 = 10z^2$ International Journal for Scienctific Research and Development, Vol (2), Issue 11, 234-235, January -2015

- [10] Jayakumar.P, Prapha.S "Integral points on the cone x^2 +25 y^2 =17 z^2 " International Journal of Science and Research Vol(4), Issue 1, 2050 2052, January 2015.
- [11] Jayakumar.P, Prabha. S, "Lattice points on the cone x^2 + $9y^2 = 26z^2$ "International Journal of Science and Research Vol (4), Issue 1, 2050-2052, January -2015
- [12] Jayakumar. P, Sangeetha. K, "Integral solution of the Homogeneous Biquadratic Diophantine equation with six unknowns: $(x^3 y^3) z = (W^2 P^2) R^4$ "International Journal of Science and Research, Vol(3), Issue 12, December-2014)
- [13] Jayakumar. P, Meena. J " Ternary Quadratic Diophantine equation: $8x^2 + 8y^2 15xy=40z^2$ International Journal of Science and Research, Vol.4, Issue 12, 654 – 655, December - 2015.
- [14] Jayakumar. P, Meena.J "On the Homogeneous Biquadratic Diophantine equation with 5 Unknown " x^4 - y^4 =26(z^2 · w^2)R² International Journal of Science and Rearch, Vol.4, Issue 12, 656 658, December-2015.
- [15] Jayakumar. P, Meena. J "On the Homogeneous Biquadratic Diophantine equation with 5 unknown x^4 $y^4=40(z^2-w^2)R^2$ International Journal of Scientific Research and Development, Vol.3, Issue10 204 – 206, 2015.
- [16] Jayakumar.P, Meena.J "Integer Solution of Non Homogoneous Ternary Cubic Diophantine equation: x^2 + y^2 –xy=103 z^3 International Journal of Science and Research,Vol.5, Issue 3, 1777-1779, March -2016
- [17] Jayakumar. P, Meena. J "On Ternary Quadratic Diophantine equation: $4x^2 + 4y^2 7xy=96z^2$ International Journal of Scientific Research and Development, Vol.4, Issue 01, 876-877, 2016.
- [18] Jayakumar. P, Meena. J ,On Cubic Diophantine Equation $x^2 + y^2 - xy= 39z^3$ International Journal of Research and Engineering and Technology,Vol.05, Issue 03, 499-501,March-2016.
- [19] Jayakumar. P, Venkatraman. R "On Homogeneous Biquadratic Diophantine equation $x^4-y^4=17(z^2-w^2)R^2$ International Journal of Research and Engineering andTechnology,Vol.05,Issue03,502-505, March- 2016
- [20] Jayakumar.P, Venkatraman.R "Lattice Points On the Homogoneous cone $x^2 + y^2 = 26z^2$: International Journal of Science and Research, Vol.5, Issue 3, 1774 1776, March 2016
- [21] Jayakumar. P, Venkatraman. R "On the Homogeneous Biquadratic Diophantine equation $x^4-y^4 = 65(z^2-w^2)R^2$ with 5 unknown International Journal of Science and Research, Vol.5, Issue 3, 1863 - 1866, March - 2016

Author Profile

Dr. P. Jayakumar received the B. Sc, M.Sc degrees in Mathematics from Madras University in 1980 and 1983 and the M. Phil, Ph.D degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 1988 and 2010.Who is now working as Professor of Mathematics, Periyar Maniammai University, Vallam, Thanajvur-613 403, Tamil Nadu, India.

V. Pandian received the B.Sc, M.Sc, and MPhil degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 2002, 2004 and 2006. Who is now working as Assistant Professor of Mathematics, A.V.V.M. Sri Pushpam College (Autonomous), Poondi -613 503, Thanajvur.

R. Venkatraman received the B.Sc, M.Sc, and MPhil degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 2002, 2004 and 2006. Who is now working as Assistant Professor of Mathematics, SRM University Vadapalani Campus, Chennai-600026, India.