On Non-Homogeneous Cubic Diophantine Equation
\[4x^2 + 4y^2 - 7xy = 19z^3\]

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Abstract: Four different methods of the non-zero non-negative solutions of non-homogeneous cubic Diophantine equation \[4x^2 + 4y^2 - 7xy = 19z^3\] are observed. Some interesting relations among the special numbers and the solutions are determined.

Keywords: The method of factorization, integer solutions, linear transformation, relations and special numbers

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Notations used:
\[t_{m,n} = \text{Polygonal number of rank } n \text{ with sides } m.\]
\[P_{n}^{m} = \text{Pyramidal number of rank } n \text{ with size } m.\]
\[G_{n} = \text{Gnomonic number}\]

1. Introduction

The number theory is the king of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For a broad review of variety of problems, one may try to see [3-12]. Integer solutions of cubic Diophantine Equation has appeared in Jayakumar. P.,Meena, J [16,18]. Inspired by these, in this work, we are observed another interesting four different methods of the non-zero non-negative solutions the non-homogeneous cubic Diophantine equation \[4x^2 + 4y^2 - 7xy = 19z^3\] Further, some elegant properties among the special numbers and the solutions are observed

2. Method of Description

Consider the cubic Diophantine equation
\[4x^2 + 4y^2 - 7xy = 19z^3\] 
(1)

Take the linear transformations
\[x = u + v, \quad y = u - v, \quad u \neq v \neq 0\] 
(2)

Using (2) in (1), it gives to
\[u^2 + 15v^2 = 19z^3\] 
(3)

If take \[z = z (a, b) = a^2 + 15b^2\]
(4)

where \(a\) and \(b\) non-zero distinct integers, then we solve (1) through dissimilar method of solutions of (1) which are furnished below.

2.1 Method: I

We can write 19 as
\[19 = (2 + i\sqrt{15}) (10 - i\sqrt{15})\] 
(5)

Using (4) and (5) in (3) and applying the factorization process, this gives us
\[(u + i\sqrt{15}v) (u - i\sqrt{15}v) = (2 + i\sqrt{15}) (2 - i\sqrt{15}) (a + i\sqrt{15}b)^3 (a - i\sqrt{15}b)^3\] 
(6)

Equating the positive and negative factors, we get
\[(u + i\sqrt{15}v) = (2 + i\sqrt{15}) (a + i\sqrt{15}b)^3\] 
(7)

\[(u - i\sqrt{15}v) = (2 - i\sqrt{15}) (a - i\sqrt{15}b)^3\] 
(8)

In sight of (2), the solutions \(x, y\) are found to be
\[x = x (a, b) = 3a^3 - 135ab^2 - 39a^2b + 195b^3\] 
(9)

\[y = y (a, b) = a^3 - 45ab^2 + 6a^2b - 30b^3\] 
(10)

Hence (4), (8) and (9) gives us two parametric the non-zero different integral values of (1).

Observations

1. \(z (a, a)\) is a perfect square.
2. \(1/106 [y (a, a) - x (a, a)]\) is a cubic integer
3. \(x (1, a) - 390P_{a}^5 + 330P_{a}^3 = 0\) (Mod3)
4. \(x (a, 1) - 6P_{a}^5 + 42 P_{a} - G_{5a} = 0\) (Mod7)
5. \(y (a, 1) - 2P_{a}^5 + 52P_{a} - G_{5a} = 0\) (Mod7)
6. \(z (0, 2)\) is a nasty number
7. \(x (2,0)\) is a nasty number
8. \(x (0, 2)\) is a nasty number
9. \(y (a,0)\) is a cubic integer
10. \(y (0,1)\) is a perfect Square

2.2 Method: II

Let \[19 = \frac{(17 + i\sqrt{15})(17 - i\sqrt{15})}{16}\] 
(11)
Using (4) and (10) in (3) and applying the factorization process, this gives us

\[(u + i\sqrt{5})v = \frac{1}{16}(17 + i\sqrt{5})\]  
\[(17 - i\sqrt{5})(a + i\sqrt{5}b)^3(a - i\sqrt{5}b)^3\]  

Equating the positive and negative factors, we get

\[u = u(a, b) = \frac{1}{4}[-13a^3 + 585ab^2 - 135a^2b + 675b^3]\]  
\[v = v(a, b) = \frac{1}{4}[3a^3 - 135ab^2 - 39a^2b + 195b^3]\]  

In sight of (2), the values of x, y are given by

\[x = x(a, b) = \frac{1}{4}[-10a^3 + 450ab^2 - 174a^2b + 870b^3]\]  
\[y = y(a, b) = \frac{1}{4}[-16a^3 + 720ab^2 - 96a^2b + 480b^3]\]  

Since our intention is to find integer solutions, taking a as 2a and b as 2b in (4), (24) and (25), the related parametric integer values of (1) are found as

\[x = x(a, b) = -20a^3 + 900ab^2 - 338a^2b + 1740b^3\]  
\[y = y(a, b) = -32a^3 + 1440ab^2 - 192a^2b + 960b^3\]  
\[z = z(a, b) = 4a^2 + 60b^2\]  

Hence the above give us two parametric the non-zero different integral values of (1).

**Observations**

1. z (a, a) is a perfect square
2. x (1, 1) is an even integer
3. -1/2[x (1, 0)] is a cubic integer
4. x (a, 1) - 40P3 + 318P; - G50a = 1 (Mod2)
5. x (1, a) - 3480P3 + 840P; - G50a = 0 (mod 19)
6. -1/4[y (1, 0)] is a cubic integer
7. y (a, 1) + 1920P3 + 96a4 - G50a = 1 (mod 2)
8. y (1, a) - 1920P3 - 480P; - G50a = 0 (mod11)
9. x (a, a) - y(a, a) = 0 (mod 2)

Each of the following is a nasty number
10. \[\frac{1}{2} z(0, 1), \frac{3}{2} z(1, 0), \frac{5}{4} z(1, 0)\]

**2.4 Method: IV**

Instead of (16), write as

\[z = \frac{7(i\sqrt{15}) - (i\sqrt{15})}{64}\]  

Using (4), (10) and (21) in (15) and applying the factorization process, this gives us

\[u = u(a, b) = \frac{1}{8}[-a^3 + 405a^2b - 450ab^2 + 2025b^3]\]  
\[v = v(a, b) = \frac{1}{8}[9a^3 - 3a^2b - 405ab^2 + 15b^3]\]  

In true of (2), the values of x, y are found as

\[x = x(a, b) = \frac{1}{8}[39a^3 - 387ab^2 - 375a^2b - 375b^3]\]  
\[y = y(a, b) = \frac{1}{8}[48a^3 + 351ab^2 - 363a^2b + 363b^3]\]  

Since our intention is to find integer solutions, taking a as 7a and b as 7b in (4), (24) and (25), the related parametric integer values of (1) are found as
x = a (a, b) = 64a^3 - 3264 a^2 b - 2880ab^2 + 16320b^3
y = y (a, b) = - 80a^3 - 3216 a^2 b + 3600ab^2 + 3600b^3
z = z (a, b) = 16a^2 + 240b^2

Hence the above give us two parametric the non-zero different integral values of (1).

Properties:
1. z (a, a) is a perfect square
2. x (1, 0) is a cubing integer
3. \( \frac{1}{3} x(1, 1) \) is a perfect square
4. x (a, 1) = 128 P^3 + 3328 P_a - G_{224} = 1(mod2)
5. x (a, 1) = 97290P^3 + 19200P + C_{192a} = 0(mod5)
6. y (a, 1) = 160 P^3 + 3136 t_a = G_{180a} = 0(mod2)
7. y (1,a)-32160 P \equiv 0(mod3)

Each of the following is a nasty number
8. \( \frac{-3}{8} z (1, 0), \frac{1}{8} z (0, 1), \frac{3}{8} x (1, 0), \frac{-3}{10} y(1,0) \)

3. Conclusion

In this work, we observed various process of determining infinitely a lot of non-zero different integer values to the cubic Diophantine equation \( 4x^2 + 4 y^2 - 7xy = 13z^3 \). One may try to find non-negative integer solutions of the above equations together with their similar observations.

References


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