The Extended Sech Function Method and The Cole-Hopf Transformation for Solving the Nonlinear Korteweg-de Vries Equation

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Abstract: In this paper, we are using the extended sech function method along with a type of cole-Hopf transformation to obtain the solutions for the nonlinear Korteweg -de Vries (KdV) equation. These types of solutions are represented as the hyperbolic function solutions including the solitary wave solution, shock wave solution and trigonometric function solution when the modulus M approaches to 1 and 0. Mathematica software is used in calculation and graphics.

Keywords: Cole-Hopf transformation, Extended Sech function, Korteweg -de Vries (KdV) equation, Solitary travelling wave solutions

1. Introduction

It is well known that the Korteweg –de Vries equation is the generic outcome of a weakly nonlinear long-wave asymptotic analysis of many physical systems. It is categorized by its

family of solitary wave solutions, with the familiar "Sech²" profile. However, in those circumstances when they are two wave modes with nearly coincident linear long wave speeds. Nonlinear Partial differential equations (PDE's) have a crucial role in various Scientific fields. The nonlinear wave phenomena such as dispersion, dissipation, diffusion... etc, are essential in nonlinear wave equations. Those waves are normally kink shaped with tanh-solutions and bell shaped with the sech-solutions. Last decades, a variety of powerful methods are proposed and established for obtaining an explicit solitary traveling wave solution of the nonlinear PDE's [4], [9], [23]-[25], [29] who are interested in nonlinear physical phenomena. Tanh-sech methods[1], [6], [7]. extended tanh method [11]. For integrable nonlinear differential equations, the inverse scattering transformation method [3], the Hirota method [9], the truncated painlevé expansion method [23], the Backlund transformation method [15], the exp-Function method and the Sech method [2]-[7], [10], [17], are used in looking for exact solution. There are many different methods to look for the exact solutions of those equations.

In the presented work, we implemented the Extended sech function method and the Cole-Hopf transformation to obtain the solitary travelling wave solutions of the nonlinear Korteweg –de Vries equation (KdV) of the form:

$$u_t + \alpha u u_x + \mu u_{xxx} = 0 \tag{1}$$

Where α and μ are real constants and to be determined later.

2. Description of the used Method

2.1 The Methodology of The Extended Sech function method

The extended sech function method proposes that a given nonlinear Partial differential equation (PDE) in one dimension of the form:

$$\Phi\left(u, u_{t}, u_{x}, u_{xx}, \ldots\right) = 0 \tag{2}$$

is transformed into a nonlinear ordinary differential equation (ODE) (3), using the wave variable $\xi = kx - \omega t$ so that $u(x,t) = U(\xi)$. Therefore:

$$\Psi(U,U',U'',U''',...) = 0$$
(3)

A new independent variable is introduced of the form :

$$Y = \operatorname{sech}(\xi) \tag{4}$$

The first and all higher derivatives is derived similarly to be:

$$\frac{d \bullet}{d\xi} = Y\sqrt{1-Y^2} \frac{d \bullet}{dY}$$

$$\frac{d^2 \bullet}{d\xi^2} = Y\left[(1-2Y^2)\frac{d \bullet}{dY} + Y(1-Y^2)\frac{d^2 \bullet}{dY^2}\right]$$

$$\frac{d^3 \bullet}{d\xi^3} = Y\sqrt{1-Y^2}\left[(1-6Y^2)\frac{d \bullet}{dY} + (5) + 3Y(1-2Y^2)\frac{d^2 \bullet}{dY^2} + Y^2(1-Y^2)\frac{d^3 \bullet}{dY^3}\right]$$
:

The extended sech function method admits the solution of (3) takes the form:

$$u(x,t) = U(\xi) = \sum_{i=0}^{n} a_i Y^i(\xi) + \sum_{i=1}^{n} b_i Y^{-i}(\xi)$$
(6)

Then we define the degree of $U(\xi)$ as $D[U(\xi)] = n$, which gives rise to the degree of other expression as

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$$D\left\lfloor \frac{d'U}{d\xi'} \right\rfloor = n + r, D\left\lfloor \left(\frac{d'U}{d\xi'} \right)^r \right\rfloor = q(n+r) \text{ and}$$
$$D\left\lfloor U'\left(\frac{d'U}{d\xi'} \right)^r \right\rfloor = np + q(n+r) \text{ . Thus we can obtain the value of}$$

n in (6) by balancing the derivative term of the highest order with the nonlinear term. Substituting (6) into (3) and setting all coefficients of Y^i , Y^{-i} to zero, an algebraic system for the unknown coefficients a_i and b_i is generated and to be solved. Consider, the KdV equation (1) and using $\xi = kx - \omega t$, $u(x,t) = U(\xi)$ with the change of derivatives (5). The obtained ODE takes the form:

$$-\omega U' + \alpha k U^2 U' + \mu k^3 U''' = 0 \tag{7}$$

All terms in the ODE (7) contain derivatives, for simplicity, it is integrated so that the constant of integration is set to be zero, gives:

$$-\omega U + \frac{\alpha k}{3} U^3 + \mu k^3 U'' = 0 \tag{8}$$

Balancing the highest order derivative (U'') with nonlinear term (U^3) in (8) gives n = 2. Thus, (6) admits the expansion:

$$U(\xi) = a_0 + a_1 Y(\xi) + a_2 Y^2(\xi) + b_1 Y^{-1}(\xi) + b_2 Y^{-2}(\xi) \quad (9)$$

Substituting (9) into (8) the following algebraic system is obtained for $Y^{-i}(\xi)$, $-4 \le i \le 4$:

$$\frac{1}{2}k\alpha b_{2}^{2} = 0,$$

$$k\alpha b_{1}b_{2} = 0,$$

$$\frac{1}{2}k\alpha b_{1}^{2} + 4k^{3}\mu b_{2} - \omega b_{2} + k\alpha a_{0}b_{2} = 0,$$

$$k^{3}\mu b_{1} - \omega b_{1} + k\alpha a_{0}b_{1} + k\alpha a_{1}b_{2} = 0,$$

$$-\omega a_{0} + \frac{1}{2}k\alpha a_{0}^{2} + k\alpha a_{1}b_{1} - 2k^{3}\mu b_{2} + k\alpha a_{2}b_{2} = 0,$$

$$k^{3}\mu a_{1} - \omega a_{1} + k\alpha a_{0}a_{1} + k\alpha a_{2}b_{1} = 0,$$

$$\frac{1}{2}k\alpha a_{1}^{2} + 4k^{3}\mu a_{2} - \omega a_{2} + k\alpha a_{0}a_{2} = 0,$$

$$-2k^{3}\mu a_{1} + k\alpha a_{1}a_{2} = 0,$$
(10)

The parameters k, ω, α and μ are assumed to be nonzero, solitary wave solutions are built with the solution of (10), by aid of Mathematica software:

$$\omega = -4k^{3}\mu, a_{2} = \frac{12k^{2}\mu}{\alpha}, a_{0} = -\frac{8k^{2}\mu}{\alpha}, \qquad (11)$$
$$b_{2} = a_{1} = b_{1} = 0$$

 $-6k^3\mu a_2 + \frac{1}{2}k\alpha a_2^2 = 0$

$$\omega = 4k^{3}\mu, a_{2} = \frac{12k^{2}\mu}{\alpha}, b_{2} = a_{0} = a_{1} = b_{1} = 0$$
 (12)

Returning the values of ω , a_0 , a_1 , a_2 , b_1 , b_2 from (11) and (12) into (9), the solitary wave solutions are built in the following forms:

$$U(\xi) = -\frac{8k^{2}\mu}{\alpha} + \frac{12k^{2}\mu}{\alpha}Y(\xi)^{2}, \quad \xi = kx + 4\mu k^{3}t \quad (13)$$

$$U(\xi) = \frac{12k^2\mu}{\alpha}Y(\xi)^2, \quad \xi = kx - 4\mu k^3 t$$
(14)

Taking in consideration (4) The graphics of the obtained surfaces in (13) and (14) are presented in figure.2 and figure.3 respectively, for given values.



Figure 1 The Extended Sech Function of (13) with



Figure 2 The Extended Sech Function of (14) with $\alpha = 6, k = 0.25, \mu = 1, -10 \le x \le 10$ and $2 \times 10^{-7} \le t \le 10 \times 10^{-7}$.

2.2 The Cole Hopf Transformation and the analytical solution of the KdV equation

In this section, we discuss the Korteweg –de Vries (KdV) equation (1) which is transformed into the nonlinear ordinary differential equation (ODE) after integrating with zero constant of integration, to become:

$$-\omega U' + \alpha k U U' + \mu k^3 U''' = 0 \tag{15}$$

Integrating with respect to ξ and taking into account that the integration constant to be zero, the ordinary differential equation (15) takes the following form;

$$pU - qU^2 - rU'' = 0 (16)$$

Where :

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$$p = \omega, \quad q = \frac{\alpha k}{2}, \quad r = \mu k^3$$
 (17)

Consider the Cole-Hopf transformation :

$$U(\xi) = \frac{\partial^2}{\partial \xi^2} \operatorname{Ln} G = G^{-1} G_{\xi\xi} - G^{-2} G_{\xi}^2$$
(18)

The first and the second derivatives would have the form:

$$U' = G^{-1}G_{3\xi} - 3G^{-2}G_{\xi}G_{\xi\xi} + 2G^{-3}G_{\xi}^{3}$$
(19)

$$U'' = G^{-1}G_{4\xi} - 4G^{-2}G_{\xi}G_{3\xi} - 3G^{-2}G_{\xi\xi}^{2} + 12G^{-3}G_{\xi}^{2}G_{\xi\xi} - 6G^{-4}G_{\xi}^{4}$$
(20)

Substituting by the transformation $U(\xi)$, (18), and its derivatives (19) and (20). The Equation in (16) is reduced to the following:

$$p\left(G^{-1}G_{\xi\xi} - G^{-2}G_{\xi}^{2}\right) + q\left(G^{-1}G_{\xi\xi} - G^{-2}G_{\xi}^{2}\right)^{2} + r\left(G^{-1}G_{4\xi} - 4G^{-2}G_{\xi}G_{3\xi} - 3G^{-2}G_{\xi\xi}^{2}\right) + 12G^{-3}G_{\xi}^{2}G_{\xi\xi} - 6G^{-4}G_{\xi}^{4} = 0$$
(21)

The power classification for G^i , $-1 \le i \le -4$ is to be:

$$G^{-1}: \quad pG_{\xi\xi} - rG_{4\xi} = 0 \tag{22}$$

$$G^{-2}: -pG_{\xi}^{2} + (-q+3r)G_{\xi\xi}^{2} - 4rG_{\xi}G_{\xi\xi\xi} = 0 \quad (23)$$

$$G^{-3}: \quad 2(-q+6r)G_{\xi}^{2}G_{\xi\xi} = 0$$
 (24)

$$G^{-4}: (q-6r)G_{\xi}^{4} = 0$$
(25)

From (24) and (25) we obtain the condition q = 6r, which leads to the undetermined nonlinear coefficient:

$$\alpha = 12\mu k^2 \tag{26}$$

From (22) by integrating twice with considering the integrating constants equal to zero, to take the form:

$$G_{\xi\xi} - \lambda^2 G = 0, \quad \lambda^2 = \frac{p}{r} = \frac{\omega}{\mu k^3}$$
(27)

Its solution is:

$$G = c_1 e^{\lambda \xi} + c_2 e^{-\lambda \xi}, \quad \lambda = \sqrt{\frac{\omega}{\mu k^3}}$$
(28)

Thus,

$$U(\xi) = \frac{GG_{\xi\xi} - G_{\xi}^{2}}{G^{2}} = \frac{4c_{1}c_{2}\lambda^{2}}{\left(c_{1}e^{\lambda\xi} + c_{2}e^{-\lambda\xi}\right)^{2}}$$
(29)

Let $\frac{c_2}{c_1} = m^2$, $0 < m^2 \le 1$, then (29) becomes, the Jacobi-

Glaisher functions for elliptic function which can be found [5], [16] :

$$U(\xi) = \frac{4\lambda^2 m^2}{\left(e^{\lambda\xi} + m^2 e^{-\lambda\xi}\right)^2},$$

$$m = \sqrt{\frac{c_2}{c_1}}, \quad \lambda = \sqrt{\frac{\omega}{\mu k^3}}, \quad \alpha = 12\mu k^2$$
(30)

Noting that as a modulus $m \to 1$, the solution of the Korteweg –de Vries equation tends to the hyperbolic function sech² ($\lambda \xi$).

A graphical representation of the solution (30) is shown in figure .3, figre .4 and figure .5 for some values of the modulus m.



Figure 3: The Cole- hopf surface (30) with $m = 1, k = 0.25, \mu = 1, \alpha = 0.75, \omega = 0.015625 - 10 \le x \le 10$



Figure 5 The Cole- hopf surface (30) with $m = 0.0001, k = 0.25, \mu = 1, \alpha = 0.75, \omega = 0.015625$ $-10 \le x \le 10$ and $2 \times 10^{-7} \le t \le 10 \times 10^{-7}$.

Volume 5 Issue 4, April 2016 www.ijsr.net $\sim 10^{-7}$

3. Conclusion

In this study, we employed the extended sech function method and the Cole-Hopf transformation for finding the solitary travelling wave solutions of the Korteweg –de Vries equation (KdV). The methods have been shown to computationally efficient in solving the equation of our interest. The last observation is that, with an appropriate value of the modulus m in (30) the solution tends to the hyperbolic function sech. By means of Mathematica all Graphics and computations are achieved.

4. A Appendix A : Properties of the Jacobi elliptic functions

- I. They satisfy The identities: $cn^2 \xi + sn^2 \xi = 1$, $dn^2 \xi + m^2 sn^2 \xi = 1$
- II. Derivatives of the Jacobi elliptic functions $\operatorname{sn}' \xi = \operatorname{cn} \xi \operatorname{dn} \xi$, $\operatorname{cn}' \xi = -\operatorname{sn} \xi \operatorname{dn} \xi$,

 $dn'\xi = -m^2 \operatorname{sn} \xi \operatorname{cn} \xi .$

Where m is a modulus. The Jacobi –Glaisher functions for elliptic function can be found in [5], [16].

III. A modulus *m* tends to one gives, $\lim_{m \to 1} \operatorname{sn} \xi = \tanh \xi$,

 $\lim_{m \to 1} \operatorname{cn} \xi = \operatorname{sech} \xi, \lim_{m \to 1} \operatorname{dn} \xi = \operatorname{sech} \xi, \text{ and for } m$ tends to zero we get $\lim_{m \to 0} \operatorname{sn} \xi = \sin \xi,$ $\lim_{m \to 0} \operatorname{cn} \xi = \cos \xi, \quad \lim_{m \to 0} \operatorname{dn} \xi = 1$

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Volume 5 Issue 4, April 2016 www.ijsr.net

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