Asymptotic Behavior of Solution of a Periodic Mutualistic System

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Abstract: We focus on the system of reaction-diffusion equations. We prove the existence of steady state solution of mutualistic system with constant coefficients. And our purpose is to estimates for periodic solutions of periodic system. We derive the asymptotic behavior of periodic system.

Keywords: Existence of steady state, Estimates, Asymptotic behavior of periodic mutualistic system.

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1. Introduction

In this paper, we consider the system of reaction diffusion equations [2, 3]. The equations is given by the following system:

\[
\begin{align*}
(u_t, v_t) &= \Delta u + u(a(t, x) - b(t, x)u + c(t, x)v), \\
(v_t) &= \Delta v + v(d + e(t, x)u - f(t, x)v).
\end{align*}
\]  (1.1)

here \(a, b, c, d, e, f\) and \(f\) are sufficiently smooth functions defined on a cylinder \(\Omega \times [0, T]\), where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), \(\Delta\) denotes the Laplacian with respect to the variables \(x = (x_1, x_2, ..., x_n) \in \Omega\), \(\partial/\partial x_i\) denotes derivative in the direction of the outer normal to \(\partial \Omega\) at \(x \in \partial \Omega\) and \(u(t, x), v(t, x)\) is a solution of (1.1). We assume that \(a, b, c, d, e, f\) are strictly positive and periodic in the time variable \(t\) with period \(T > 0\).

The boundary condition is given by

\[
\frac{\partial u}{\partial n}\big|_{\partial \Omega \times [0, T]} = \frac{\partial v}{\partial n}\big|_{\partial \Omega \times [0, T]} = 0
\]  (1.2)

and the initial condition is given by

\[
u(x, t)|_{t=0} = u_0(x), \quad v(t, t)|_{t=0} = v_0(x).
\]  (1.3)

2. The Existence of Steady State Solutions

Consider the following steady state problem:

\[
\begin{align*}
(u, v) &= \Delta u + u(a - b u + c v), \\
v_t &= \Delta v + v(d + e u - f v).
\end{align*}
\]  (2.1)

and then we compute the Jacobian

\[
A = \begin{bmatrix}
a - bu + cv & bu \\
v & d + eu - fv - f
\end{bmatrix}
\]

Now we find intersection points:

\[
L_1 = a \| - bu + cv = 0,
\]

\[
L_2 = d + eu - fv = 0.
\]

from Eq (2.2) in \(u\) axis the point is \((\frac{c}{b}, 0)\), from Eq (2.3) in \(v\) axis the point is \((0, \frac{d}{f})\), now we solving the simultaneous equations (2.2) and (2.3) and then we find \((u^*, v^*) = (\frac{ca + bd - ce + f}{bf - ce} - \frac{e}{f})\) as \(\frac{c}{b} > \frac{d}{f}\) there are four equilibriums points \((0, 0), \left(\frac{c}{b}, 0\right), \left(0, \frac{d}{f}\right)\) and \((u^*, v^*)\) now we discuss the stability for these points:

(i) \(\det(A - \lambda I)|_{(u^*, 0)} = \det\begin{bmatrix}
a - \lambda & 0 \\
0 & d - \lambda
\end{bmatrix} = 0\) then \(\lambda_1, \lambda_2 > 0\) then this point is unstable.

(ii) \(\det(A - \lambda I)|_{(0, v^*)} = \det\begin{bmatrix}
a - 2b & c - e \\
0 & d + e\frac{a}{b} - \lambda
\end{bmatrix} = 0\) then \(\lambda_1 > 0, \lambda_2 > 0\) the point is unstable.

(iii) \(\det(A - \lambda I)|_{(\frac{c}{b}, \frac{d}{f})} = \det\begin{bmatrix}
a + \frac{ca}{b} - \lambda & 0 \\
\frac{e}{f} & -d - \lambda
\end{bmatrix} = 0\)

then \(\lambda_1 > 0, \lambda_2 > 0\) the point is unstable.

(iv) \(\det(A - \lambda I)|_{(u^*, v^*)} = \det\begin{bmatrix}
bu - \lambda & cv \\
v & -f v - \lambda
\end{bmatrix} = 0\)

\[= (-bu - \lambda)(-fv - \lambda - ceu v) = 0\]

\[= \lambda^2 + (bu + fv)\lambda + (bf - ce)uv^* = 0\]

\[
\lambda_{1,2} = \frac{-bv^* - f v^* \pm \sqrt{(bu + fv)^2 - 4(bf - ce)uv^*}}{2}.
\]

Re\(\lambda_{1,2} < 0\) if \(\frac{b}{c} > \frac{e}{f}\) under this condition the point \((u^*, v^*)\) is stable.

Lemma 1 Let \(P = (u_1, v_1), Q = (u_2, v_2)\) be any two distinct points in \(R_i\) and let \(\Gamma_i\) be any smooth curve lying in \(R_i\) with end points \(P, Q\) where \(R_i\) is any one of the four regions as following:
Consider the case where

\[
\text{Proof:}
\]

in (2) \((p, q) \in R_4\) such that:

\[
u_2 > u^* - \epsilon; \quad v_2 > v^* - \epsilon\quad \text{and} \quad \overline{u}_2 < u^* + \epsilon; \quad \overline{v}_2 < v^* + \epsilon.
\]

That is, we have the following properties:

\[
\begin{align*}
\text{Property (1)}: & \quad p(t) > u^* - \epsilon, \quad q(t) > v^* - \epsilon, \\
\text{Property (2)}: & \quad \overline{u}_2 < u^* + \epsilon, \quad \overline{v}_2 < v^* + \epsilon.
\end{align*}
\]

We conclude by letting \(t \to \infty\) in \(p(t) \leq u(t, x) \leq \overline{p}(t), q(t) \leq v(t, x) \leq \overline{q}(t)\) and the arbitrariness of \(\epsilon\):

\[
\lim_{\epsilon \to 0} (u(t, x), v(t, x)) = (u^*, v^*).
\]

3. Estimates for Periodic Solutions of Periodic System

Optimal upper and lower bounds for multiplicity of coexistence states and conditions for existence of coexistence states we consider the system

\[
\begin{align*}
&u_t = u_{xx} + u[a(x,t) - b(t,x)u + c(x,v)](x, t), \quad (3.1)
\end{align*}
\]

\[
\begin{align*}
v_t = v_{xx} + v[d(x,t) + e(x,t)u - f(x,v)](x, t), \quad (3.2)
\end{align*}
\]

where it is only assumed that the functions \(a, b, c, d, e, f\) are constant positive, and T-periodic on \(\Omega \times \mathbb{R}\) under certain conditions on \(a, b, c, d, e, f\) we shall show in a following that these also imply the existence of coexistence states.

Lemma 2 Assume that \(g \in C^2, \omega \in C^1, \Omega \in \mathbb{R^n} \cap C^1, a, b, c, d, e, f \in \mathbb{R}\) and \(k(x, t) > 0\) on \(\Omega \times \mathbb{R}\) such that

\[
\frac{\partial w}{\partial t} = k\Delta w + gw
\]

on \(\Omega \times \mathbb{R}, w(x, t) \equiv w(x, t + T)
\]

\[
\frac{\partial w}{\partial t} = 0
\]

Then there exist points \((t_1, x_1)\) and \((t_2, x_2)\) in \(\Omega \times \mathbb{R}\) such that

\[
w(x_1, t_1) < 0, \quad g(t_1, x_1) \leq 0
\]

\[
w(x_2, t_2) > 0, \quad g(t_2, x_2) \geq 0
\]
where \( w_t = \min_{\partial \Omega \times [0,T]} w(t,x) \), \( w_m = \max_{\partial \Omega \times [0,T]} w(t,x) \)

**Proof:** Suppose that there exists \((t_1,x_1) \in \Omega \times \mathbb{R}^n \) such that \( w(t_1,x_1) = w_1 \) then \( w_t(t_1,x_1) = 0 \) and \( \partial w(t_1,x_1) \geq 0 \). Therefore from (3.3) we see that \( g(t_1,x_1) = w_0 \) and hence \( g(t_1,x_1) \leq 0 \). If \( w_0 \) does not assume the value \( x_1 \) anywhere on \( \partial \Omega \times \mathbb{R}^n \) then there exists \((t_1,x_1) \in \partial \Omega \times \mathbb{R}^n \) such that \( w(t_1,x_1) = x_1 \) if were the case that \( g(t_1,x_1) > 0 \) then there would exist an open ball centered at \((t_1,x_1)\) such that \( g(t_1,x_1) > 0 \) on \( \partial \Omega \times \mathbb{R}^n \) since \( w(t_1,x_1) = w(t_1,x_1) \) for \( (t,x) \in \partial \Omega \times (\partial \Omega \times \mathbb{R}^n) \). And \( g(t_1,x_1) \leq 0 \) and the same reasoning as used above that \( g(t_2,x_2) \geq 0 \) if there exists \((t_2,x_2)\) in \( \partial \Omega \times \mathbb{R}^n \) such that \( g(x_1) \leq 0 \) for \( (t,x) \in \partial \Omega \times \mathbb{R}^n \) and if it were the case that \( g(t_2,x_2) < 0 \) we would have \( \partial g(t_2,x_2) > 0 \) for points in \( \mathbb{R}^n \) near \( t_2,x_2 \) in the case the maximum principle would imply that \( \partial g(t_2,x_2) > 0 \) contradicting (3.4) that show \( g(t_2,x_2) \geq 0 \) 

**Theorem 3.1**

Assume that \( b_1 > \frac{c_m - c_m}{d_1} \) (3.7)

And \( \partial \Omega \) is of class \( C^2 \). If \( u,v \in C^{2,1}(\Omega \times \mathbb{R}) \cap C^{1,0}(\partial \Omega \times \mathbb{R}) \), \((u,v)\) is a solution of (3.1) – (3.2) on \( (\Omega \times \mathbb{R}) \), \( u(t,x) \) and \( v(t,x) \) are periodic solution and are T – periodic in \( t \) and \( u(t_1,x) > 0 \) and \( v(t_1,x) > 0 \) for \( (x,t) \in \Omega \times (\partial \Omega \times \mathbb{R}) \), \( u \) and \( v \) satisfy the boundary conditions:

\[
\frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega \times [0,T]} = \frac{\partial v}{\partial \nu} \bigg|_{\partial \Omega \times [0,T]} = 0 \tag{3.8}
\]

Then for \( (x,t) \in (\Omega \times \mathbb{R}) \):

\[
a_t f_m + d_1 e_t - c_t e_l \\
\frac{b_m f_m - c_t e_l}{b_m f_m - c_t e_l} \leq u(t,x) \leq f_t a_m + c_t e_m
\]

\[
\frac{a_t e_t + d_t d_t - c_t e_l}{b_m f_m - c_t e_l} \leq v(t,x) \leq \frac{e_t a_m + b_t d_t}{b_t f_t - e_t c_m}
\]

**Proof:** Suppose that \( u \) and \( v \) are as in the statement of the theorem from \( \text{lemma (2)} \) and the equation \( (3.1) \) we suppose the existence of \((t_0,x_0)\) in \( (\Omega \times \mathbb{R}) \) such that

\[
u(t_0,x_0) = u(t_0,x_0) = \min_{t \in [0,T]} u(t_0,x_0)
\]

And \( a_t(t_0,x_0) = b_t(t_0,x_0) = c_t(t_0,x_0) = 0 \) \((3.11)\)

Similarly from \( \text{lemma (2)} \) and equation \( (3.2) \) we suppose the existence of \((t_0,x_0)\) in \( (\Omega \times \mathbb{R}) \) such that

\[
u(t_0,x_0) = v(t_0,x_0) = \min_{t \in [0,T]} v(t_0,x_0)
\]

And \( d(t_0,x_0) = e(t_0,x_0) = f(t_0,x_0) = 0 \) \((3.12)\) by \( d_0 \) then we obtain

\[
a_t e_t + b_m e_t + c_t e_l - d_t f_m v_t \leq 0
\]

\[
\frac{a_t e_t + b_m e_t + c_t e_l - d_t f_m v_t}{b_m f_m - c_t e_l} \leq 0
\]

Then \( a_t e_t + b_m e_t + c_t e_l - d_t f_m v_t \leq 0 \)

\[
(b_m f_m - c_t e_l) v_t \geq a_t e_t + d_t f_m v_t \geq 0
\]

\[
\frac{a_t e_t + b_m e_t + c_t e_l - d_t f_m v_t}{b_m f_m - c_t e_l} \geq 0 \tag{3.13}
\]

and multiply \((3.11)\) by \( b_t \) and \((3.12)\) by \( c_t \) then we obtain

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_m f_m - c_t e_l} \leq 0
\]

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_m f_m - c_t e_l} \geq 0
\]

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_m f_m - c_t e_l} \geq 0 \tag{3.14}
\]

from \( \text{lemma (2)} \) and the equation \( (3.1) \) we suppose the existence of \((t_0,x_0)\) in \( (\Omega \times \mathbb{R}) \) such that

\[
u(t_0,x_0) = u(t_0,x_0) = \min_{t \in [0,T]} u(t_0,x_0)
\]

and from \( \text{lemma (2)} \) and the equation \( (3.2) \) we suppose the existence of \((t_0,x_0)\) in \( (\Omega \times \mathbb{R}) \) such that

\[
u(t_0,x_0) = v(t_0,x_0) = \min_{t \in [0,T]} v(t_0,x_0)
\]

we multiply \( (3.15) \) by \( e_t \) then we obtain

\[
(e_t a_m + e_t b_m + e_t c_m) \geq 0
\]

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_t f_t - c_t e_m} \geq 0
\]

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_t f_t - c_t e_m} \geq 0 \tag{3.17}
\]

we multiply \((3.15)\) by \( f_t \) and \((3.16)\) by \( c_t \) then we obtain

\[
f_t a_m + b_t f_t e_t + c_t e_m \geq 0
\]

\[
f_t a_m + b_t f_t e_t + c_t e_m \geq 0
\]

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_t f_t - c_t e_m} \geq 0 \tag{3.18}
\]

Since \( u_t \leq u(t,x) \leq u_m \) and \( v_t \leq v(t,x) \leq v_m \) from inequalities \((3.13)\), \((3.14)\) and \((3.18)\) we obtain:

\[
\frac{a_t e_t + b_t e_t + c_t e_l - d_t f_m v_t}{b_t f_t - c_t e_m} \leq 0
\]
\[ \frac{e_m a_1 + b_m d_1}{b_m f_{\alpha} - c_m e_{\alpha}} \leq v(t, x) \leq \frac{e_m a_m + b_l d_m}{b_l f_{\beta} - c_m e_{\beta}} \]

this proves theorem (3.1).

4. Asymptotic Behavior of Periodic System

Theorem 4.1

If \( b_1 > \frac{c_m}{f_{\beta}} \) (4.1)

Then there exist pairs \((\bar{u}, \bar{v})\) and \((\bar{u}^*, \bar{v}^*)\) with components in \(C^{2+\infty,1+\infty}(\Omega \times \mathbb{R})\) such that the components of both pairs are strictly positive and \(T\)-periodic in \(t\) each pair is a solution of (1.1) and satisfies the boundary conditions (1.2).

Moreover

\[ \bar{u}(t,x) \leq \bar{u}^*(t,x); \bar{v}(t,x) \leq \bar{v}^*(t,x) \text{ on } \Omega \times \mathbb{R} \]

and if \((u, v)\) is a solution of the initial boundary value problem given by (1.1) \(- (1.2)\) with \(u(0,x) = \Phi(x); v(0,x) = \Psi(x)\) such that \(\Phi, \Psi \in C^{1+\infty}(\Omega)\)

\(\Phi(x) \geq 0; \Psi(x) \geq 0; \Phi(x) \neq 0; \Psi(x) \neq 0\)

and:

\[ \frac{\partial \Phi}{\partial \eta} \bigg|_{\partial \Omega} = \frac{\partial \Psi}{\partial \eta} \bigg|_{\partial \Omega} = 0 \]

then for any \(\varepsilon > 0\):

\[ \bar{u}(t,x) - \varepsilon < u(t,x) < \bar{u}^* + \varepsilon \]

\[ \bar{v}(t,x) - \varepsilon < v(t,x) < \bar{v}^* + \varepsilon \]

for \(x\) in \(\Omega\) and all sufficiently large \(t\). If \(a, b, c, d, e\) and \(f\) are function of \(t\) alone.

Then \(\bar{u} \equiv \bar{u}^*; \bar{v} \equiv \bar{v}^*\) and \(\bar{u}, \bar{v}\) are functions of \(t\) alone.

Proof: Choose constant \(k_1, k_2\) such that

\[ a_m - b_m k_1 + c_m k_2 < 0, (4.2) \]

\[ d_m + e_m k_1 - f_k k_2 < 0, (4.3) \]

using (4.1), \(k_1\) and \(k_2\) exist, then choose \(0 < \delta_1 < k_1, 0 < \delta_2 < k_2\) such that

\[ a_1 - b_1 \delta_1 + c_1 \delta_2 > 0, (4.4) \]

\[ d_1 + e_1 \delta_1 - f_1 \delta_2 > 0, (4.4) \]

in fact if \(\delta_1 < \frac{a_1}{b_1}; \delta_2 < \frac{d_1}{f_1}; \delta_1, \delta_2\) are suitable. It is obvious that \((k_1, k_2)\) and \((\delta_1, \delta_2)\) are periodic upper and lower solutions. There exist two pairs periodic solutions of original periodic boundary value problem

\[ (1.2) \big| - (2.1) (\bar{u}, \bar{v}), (\bar{u}^*, \bar{v}^*)\] and:

\[ \delta_1 \leq \bar{u} \leq \bar{u}^* \leq k_1, (4.6) \]

\[ \delta_2 \leq \bar{v} \leq \bar{v}^* \leq k_2, (4.7) \]

we consider following initial boundary value problem

\[ (1.2) \big| - (2.1)\] with:

\[ u(x,t_0) = \delta_1; u(x,t_0) = \delta_2, (4.8) \]

\[ u(x,t_0) = \delta_1; u(x,t_0) = \delta_2, (4.9) \]

Denote: \((u_1, v_1)\) and \((u_2, v_2)\) are the corresponding solutions of \((1.2) \big| - (2.1)\), (4.8) and \((1.2) \big| - (2.1)\), (4.9) then

\[ \lim_{t \to +\infty} (u_1, v_1) = (\bar{u}, \bar{v}) \] and

\[ \lim_{t \to +\infty} (u_2, v_2) = (\bar{u}^*, \bar{v}^*)\]

Finally we consider initial boundary value problem for

\[ u_0(x) \geq 0; u_0(x) \geq 0 \] and \(u_0(x) \neq 0; u_0(x) \neq 0\) by strong maximum principle

\[ u_0(t_0 + 1, x) > 0, v_0(t_0 + 1, x) > 0 \]

we choose \((\delta_1, \delta_2)\) and \((k_1, k_2)\) such that

\[ \delta_1 \leq u_0(t_0 + 1, x) \leq k_1; \delta_2 \leq v_0(t_0 + 1, x) \leq k_2 \]

we obtain following comparison relation:

\[ u_1(t - 1, x) \leq u(t, x) \leq u_2(t - 1, x), v_1(t - 1, x) \leq v(t, x) \leq v_2(t - 1, x), t \geq t_0 + 1 \]

Therefore we have

\[ \bar{u}(t,x) \leq \liminf_{t \to +\infty} u(t,x), \limsup_{t \to +\infty} u(t,x) \leq \bar{u}^*(t,x), (4.13) \]

and

\[ \bar{v}(t,x) \leq \liminf_{t \to +\infty} v(t,x), \limsup_{t \to +\infty} v(t,x) \leq \bar{v}^*(t,x). (4.14) \]

For uniqueness:

\[ \int_0^T [a(t) - b(t) \bar{u}(t) + c(t) \bar{v}(t)] dt = \int_0^T [a(t) - b(t) \bar{u}^*(t) + c(t) \bar{v}^*(t)] dt \]

By means of \(\int_0^T \frac{b(t)}{c(t)} \frac{\bar{u}(t) - \bar{v}(t)}{\bar{u}^*(t) - \bar{v}^*(t)} dt = \int_0^T c(t)(\bar{v}^*(t) - \bar{v}(t)) dt = 0 \) (4.15)

If: \((\bar{u}^*(t) \geq \bar{u}(t) \text{ and } \bar{u}^*(t) \neq \bar{u}(t)\) we have

\[ \frac{b_1}{c_m} \leq \frac{\int_0^T (\bar{v}^*(t) - \bar{v}(t)) dt}{\int_0^T (\bar{u}^*(t) - \bar{u}(t)) dt} \]

similarly we have:

\[ \frac{e_m}{f_1} \leq \frac{\int_0^T (\bar{v}^*(t) - \bar{v}(t)) dt}{\int_0^T (\bar{u}^*(t) - \bar{u}(t)) dt} \]

it follows from (4.16) – (4.17)

\[ \frac{b_1}{c_m} \leq \frac{e_m}{f_1} \]

it is contradiction with (4.1), therefore the uniqueness holds.

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References


