Asymptotic Behavior of Solution of a Periodic Mutualistic System

Mohamed Ahmed Abdallah

Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk, KSA
Department of Basic Science, Faculty of Engineering, Sinnar University, Sinnar, Sudan

Abstract: We focus on the system of reaction-diffusion equations. We prove the existence of steady state solution of mutualistic system with constant coefficients. And our purpose is to estimates for periodic solutions of periodic system. We derive the asymptotic behavior of periodic system.

Keywords: Existence of steady state, Estimates, Asymptotic behavior of periodic mutualistic system.

2000 MSC: 35Q80, 34C99, 35K55, 92D25

1. Introduction

In this paper, we consider the system of reaction diffusion equations [2, 3]. The equations is given by the following system:

\[
\begin{align*}
(u, v) &= \Delta u + u[a(x, t) - b(u, v)] + c(u, v), \\
(v, v) &= \Delta v + v[d(x, t) + e(u, v) - f(t, v)].
\end{align*}
\]

(1.1)

where \(a, b, c, d, e\) and \(f\) are sufficiently smooth functions defined on a cylinder \(\Omega \times [0, T]\), where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^n\), \(\Delta\) denotes the Laplacian with respect to the variables \(x = (x_1, x_2, \ldots, x_n) \in \Omega\), \(\partial/\partial v\) denotes derivative in the direction of the outer normal to \(\partial \Omega\) at \(x \in \partial \Omega\) and \(u(t, x), v(t, x)\) is a solution of (1.1).

We assume that \(a, \ldots, f\) are strictly positive and periodic in the time variable \(t\) with period \(T > 0\). The boundary condition is supposed by

\[
\frac{\partial u}{\partial n}\bigg|_{\partial \Omega \times [0, T]} = \frac{\partial v}{\partial n}\bigg|_{\partial \Omega \times [0, T]} = 0
\]

(1.2)

and the initial condition is given by

\[
u(x, t)|_{t=0} = u_0(x), \quad v(x, t)|_{t=0} = v_0(x).
\]

(1.3)

2. The Existence of Steady State Solutions

Consider the following steady state problem:

\[
\begin{align*}
(u, v) &= \Delta u + u[a - bu + cv], \\
(v, v) &= \Delta v + v[d + eu - f v].
\end{align*}
\]

(2.1)

here \(a, b, c, d, e\) and \(f\) are positive constants and \(u(t, x), v(t, x)\) is a solution of (2.1).

Steady state solution satisfies the following equations:

Email address: mohamed.ah.abd@hotmail.com (Mohamed Ahmed Abdallah)

Preprint submitted to April 7, 2016

\[
f_1(u, v) = u(a - bu + cv) = 0, \\
f_2(u, v) = v(d + eu - f v) = 0
\]

Volume 5 Issue 4, April 2016

www.ijsr.net

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These four regions are closely related to the following corresponding differential inequalities.

\[ \Gamma_1: p^r \geq f_1(p,q), q \geq f_2(p,q), \Gamma_2: p^r \geq f_1(p,q), q \leq f_2(p,q), \Gamma_3: p^r \leq f_1(p,q), q \geq f_2(p,q), \Gamma_4: p^r \leq f_1(p,q), q \leq f_2(p,q) \]

then there exists a pair of smooth functions \((p(t), q(t))\) with values on \(\Gamma_i\) for all \(t \geq 0\) such that:

1. \((p(0), q(0)) = (u_1, v_1)\); \(\lim_{t \to \infty} (p(t), q(t)) = (u_2, v_2)\)
2. \((p, q)\) satisfies the corresponding differential inequalities in \(\Gamma_i\)

Proof: we show the lemma for \(\Gamma_{2i}\) in \(R_2\) since it is representative and is more relevant to later applications. Consider the case where \(\Gamma_2\) is the straight line \(PQ\). Since \(P, Q\) are in \(R_2\), there exists \(\delta > 0\) such that:

\[ \{ (u, v) \in \mathbb{R}^2 : \min \{ f_1(u, v), (u, v) \in \mathbb{R}^2 \} \geq \delta \} \]

Define:

\[ p(t) = u_2 + (u_1 - u_2)e^{-\epsilon t}, (t \geq 0) \]

where \(\epsilon > 0\) is a constant to be chosen. Then \((p, q)\) lies on \(\Gamma_2\) for all \(t\) and satisfies property (1) choose \(\epsilon \leq \min \{ \delta | \delta u_1 - u_2, \delta v_1 - v_2 \}. \)

Lemma 2 Assume that \(g\) and \(k\) are continuous and \(T\) - periodic on \(\Omega \times \mathbb{R}\), \(\partial \Omega\) is of class \(C^2, w \in C^{2, 1}(\Omega \times \mathbb{R}) \cap C^1(\bar{\Omega} \times \mathbb{R}), k(t, x) > 0, w(t, x) > 0 \) on \(\Omega \times \mathbb{R}\), \(w\) is a solution of:

\[ \frac{\partial w}{\partial t} = k \Delta w + gw \]
where $w_i = \min_{\Omega \times [0, T]} w(t, x)$; $w_m = \max_{\Omega \times [0, T]} w(t, x)

**Proof:** Suppose that there exists $(t_1, x_1) \in \Omega \times \mathbb{R}$ such that $w(t_1, x_1) = w_i$ then $w(t_1, x_1) = 0$ and $\Delta w(t_1, x_1) \geq 0$. Therefore from (3.3) we see that $g(t_1, x_1)w_i < 0$ and hence $g(t_1, x_1) < 0$. If $w$ does not assume the value $w_i$ anywhere on open set $\Omega \times \mathbb{R}$ then there exists $(t_1, x_1) \in \partial \Omega \times \mathbb{R}$ such that $w(t_1, x_1) = w_i$ if it were the case that $g(t_1, x_1) > 0$ then there would exist an open ball $D$ centered at $(t_1, x_1)$ such that $g(t_1, x_1) > 0$ on $D \cap (\Omega \times \mathbb{R})$ since $w(t_1, x_1) > w(t_1, x_1)$ for $(t, x) \in D \cap (\Omega \times \mathbb{R})$. Let $\partial \Delta w = -w_i = -\mathbb{R}g < 0$ at points in $D \cap (\Omega \times \mathbb{R})$ at $(t_1, x_1) < 0$ which contradicts (3.4). From this contradiction we see that $g(t_1, x_1) \leq 0$. If this exists a point $(t_2, x_2)$ in $\partial \Omega \times \mathbb{R}$ such that $w(t_2, x_2) = w_m$ then it follows from (3.3) and the same reasoning as used above that $g(t_2, x_2) \geq 0$. If there exist $(t_2, x_2)$ in $\partial \Omega \times \mathbb{R}$ such that $w(t_2, x_2) < w(t_2, x_2)$ for all $(t, x) \in \Omega \times \mathbb{R}$ and if it were the case that $g(t_2, x_2) < 0$ we would have $\Delta w = -w_i > 0$ for points in $\Omega \times \mathbb{R}$ near $t_2, x_2$ in this case the maximum principle would imply that $\Delta g(t_2, x_2) > 0$ contradicting (3.4). Show that $g(t_2, x_2) \geq 0$.

**Theorem 3.1**

**Assume that** $b_1 > \frac{c_m c_m}{f_i}$ (3.7)

**And** $\partial \Omega$ is of class $C^2$. If $u, v \in C^{1, 0}(\Omega \times \mathbb{R}) \cap C^{1, 0}(\Omega \times \mathbb{R})$, $(u, v)$ is a solution of (3.1) - (3.2) on $\Omega \times \mathbb{R}$, $(u, t, x)$ and $v(t, x)$ are periodic solution and are $T$ - periodic in $t$, $u(t, x) > 0$ and $v(t, x) > 0$ for $(x, t) \in (\Omega \times \mathbb{R})$, $u$ and $v$ satisfy the boundary conditions:

$$
\frac{\partial u}{\partial \nu} |_{\partial \Omega \times [0, T]} = \frac{\partial v}{\partial \nu} |_{\partial \Omega \times [0, T]} = 0
$$

Then for $(x, t) \in (\Omega \times \mathbb{R})$:

$$
a_i f_m + d_i c_t \leq u(t, x), x \leq f_i a_m + c_m c_m b_i f_i - c_m c_m,
$$

$$
a_i e_i + b_m d_i \leq v(t, x), x \leq f_i a_m + c_m c_m b_i f_i - c_m c_m,
$$

**Proof:** Suppose that $u$ and $v$ are as in the statement of the theorem from (2) and the equation (3.1) we suppose the existence of $(t_m, x_1)$ in $(\Omega \times \mathbb{R})$ such that

$$
u(t_m, x_1) = u_m = \min_{\Omega \times \mathbb{R}} u(t_1, x_1)
$$

And

$$
a(t_1, x_1) = -(t_1, x_1)u(t_1, x_1) + c(t_1, x_1)\nu(t_1, x_1) \leq 0
$$

Similarly from lemma (2) and equation (3.2) we suppose the existence of $(t_2, x_2)$ in $(\Omega \times \mathbb{R})$ such that

$$
\nu(t_2, x_2) = v_m = \min_{\Omega \times \mathbb{R}} v(t_2, x_2)
$$

And

$$
d(t_2, x_2) = e(t_2, x_2)u(t_2, x_2) - f(t_2, x_2)\nu(t_2, x_2) \leq 0
$$

Multiply (3.11) by $e_1$ and (3.12) by $b_m$ then we obtain

$$
a_i e_1 \leq b_m e_1 u_i + c_m c_m v_i \leq 0
$$

Then

$$
a_i e_1 + b_m d_i \leq (c_i e_i - b_m f_m) v_i \leq 0
$$
Finally we consider initial boundary value problem for

\[ \frac{e_1 a_1 + b_m d_1}{b_m f_m - c_m d_1} \leq v(t, x) \leq \frac{e_m a_m + b_1 d_m}{b_1 f_1 - c_m d_m} \]

this proves theorem (3.1)

4. Asymptotic Behavior of Periodic System

Theorem 4.1

If \( b_1 > \frac{c_m a_m}{f_1} \) (4.1)

Then there exist pairs \((\hat{u}, \hat{v})\) and \((\hat{u}^*, \hat{v}^*)\) with components in \( C^{2+\infty, 1+\infty}(\Omega \times \mathbb{R})\) such that the components of both pairs are strictly positive and T-periodic in \( t \) each pair is a solution of (1.1) and satisfies the boundary conditions (1.2).

Moreover

\[ \hat{u}(t, x) \leq \hat{u}^*(t, x); \hat{v}(t, x) \leq \hat{v}^*(t, x) \text{ on } \Omega \times \mathbb{R} \]

and if \((u, v)\) is a solution of the initial boundary value problem given by (1.1) \( \setminus (1.2) \) with \( u(0, x) = \Phi(x); v(0, x) = \Psi(x) \) such that \( \Phi, \Psi \in C^{2+\infty}(\Omega) \)

\( \Phi(x) \geq 0; \Psi(x) \geq 0; \Phi(x) \neq 0; \Psi(x) \neq 0 \)

and:

\[ \frac{\partial \Phi}{\partial n} |_{\partial \Omega} = \frac{\partial \Psi}{\partial n} |_{\partial \Omega} = 0 \]

then for any \( \epsilon > 0 \):

\[ \hat{u}(t, x) - \epsilon < u(t, x) < \hat{u}^* + \epsilon \]

\[ \hat{v}(t, x) - \epsilon < v(t, x) < \hat{v}^* + \epsilon \]

for \( x \) in \( \Omega \) and all sufficiently large \( t \). If \( a, b, c, d, e \) and \( f \) are functions of \( t \) alone.

Then \( \hat{u} \equiv \hat{u}^*; \hat{v} \equiv \hat{v}^* \) and \( \hat{u}, \hat{v} \) are functions of \( t \) alone.

**Proof:** Choose constant \( k_1, k_2 \) such that

\[ a_m - b_1 k_1 + c_m k_2 < 0, (4.2) \]

\[ d_m + e_m k_1 - f_1 k_2 < 0, (4.3) \]

using (4.1), \( k_1 \) and \( k_2 \) exist, then choose \( 0 < \delta_1 < k_1, 0 < \delta_2 < k_2 \) such that

\[ a_1 - b_m \delta_1 + c_m \delta_2 > 0, (4.4) \]

\[ d_1 + e_1 \delta_1 - f_1 \delta_2 > 0, (4.4) \]

in fact if \( \delta_1 < \frac{a_1}{b_m}; \delta_2 < \frac{d_1}{f_1}; \delta_1, \delta_2 \) are suitable. It is obvious that \((k_1, k_2)\) and \((\delta_1, \delta_2)\) are periodic upper and lower solutions. There exist two pairs periodic solutions of original

periodic boundary value problem

\[ (1.2) \setminus (2.1) (\hat{u}, \hat{v}), (\hat{u}^*, \hat{v}^*) \]

and:

\[ \delta_1 \leq \hat{u} \leq \hat{u}^* \leq k_1, (4.6) \]

\[ \delta_2 \leq \hat{v} \leq \hat{v}^* \leq k_2, (4.7) \]

we consider following initial boundary value problem

\[ (1.2) \setminus (2.1) \text{ with:} \]

\[ u(x, t_0) = \delta_1; v(x, t_0) = \delta_2, (4.8) \]

\[ u(x, t_0) = k_1; v(x, t_0) = k_2, (4.9) \]

**Denote:** \((u_1, v_1)\) and \((u_2, v_2)\) are the corresponding solutions of

\[ (1.2) \setminus (2.1), (4.8) \] and \((1.2) \setminus (2.1), (4.9) \]

then \( \lim_{t \to \infty} u_1(v_1, v_2) = (\hat{u}, \hat{v}) \) and \( \lim_{t \to \infty} u_2(v_2, v_2) = (\hat{u}^*, \hat{v}^*) \).

Finally we consider initial boundary value problem for

\[ u_0(x) \geq 0; v_0(x) \geq 0 \]

and \( \lim_{t \to \infty} u_0(x) \neq 0; v_0(x) \neq 0 \) by strong maximum principle

\[ u_0(t_0 + 1, x) > 0, v_0(t_0 + 1, x) > 0 (4.10) \]

we choose \((\delta_1, \delta_2)\) and \((k_1, k_2)\) such that

\[ \delta_1 \leq u_0(t_0 + 1, x) \leq k_1; \delta_2 \leq v_0(t_0 + 1, x) \leq k_2 \]

we obtain following comparison relation:

\[ u_1(t - 1, x) \leq u(t, x) \leq u_2(t - 1, x), v_1(t - 1, x) \leq v(t, x) \leq v_2(t - 1, x), t \geq t_0 + 1 (4.12) \]

Therefore we have

\[ \hat{u}(t, x) \leq \lim_{t \to \infty} u(t, x) \leq \lim_{t \to \infty} u(t, x) \leq \hat{u}^*(t, x), (4.13) \]

and

\[ \hat{v}(t, x) \leq \lim_{t \to \infty} v(t, x) \leq \lim_{t \to \infty} v(t, x) \leq \hat{v}^*(t, x). (4.14) \]

For uniqueness:

\[ \int_0^T [a(t) - b(t) \hat{u}(t) + c(t) \hat{v}(t)] dt = \int_0^T [a(t) - b(t) \hat{u}^*(t) + c(t) \hat{v}^*(t)] dt. \]

By means of \[ \int_0^T a(t) \hat{u}^*(t) dt = \int_0^T a(t) \hat{u}^*(t) dt \]

\[ \int_0^T b(t)(\hat{u}^*(t) - \hat{u}(t)) dt = \int_0^T c(t)(\hat{v}^*(t) - \hat{v}(t)) dt = 0 (4.15) \]

If \((\hat{u}^*) \geq \hat{u}(t)\) and \((\hat{u}^*) \equiv \hat{u}(t)\) we have

\[ \frac{b_1}{c_m} \leq \int_0^T (\hat{v}^*(t) - \hat{v}(t)) dt \]

similarly have:

\[ \frac{a_1}{b_m} \leq \int_0^T (\hat{v}^*(t) - \hat{v}(t)) dt \]

it follows from (4.16) \( - (4.17) \)

it is contradiction with (4.1), therefore the uniqueness holds.

5. Acknowledgement

The author is grateful to the professor Zhou Li for his helpful suggestions which improved the presentation of this paper.

References

