Painlevé Analysis, Bäcklund and Cole-Hopf Transformations of the (2+1) and (3+1)-dimensional Burgers Equations

T. Shanmuga Priya[1], B. Mayil Vaganan[2]

Department of Applied Mathematics and Statistics, Madurai Kamaraj University, Madurai-625021, India

Abstract: *We apply the Painlevé analysis to the (2+1) and (3+1)-dimensional Burgers Equations respectively given by $u_t + \left(\frac{2}{F(y,t)}\right)uu_x = \gamma(u_{xx} + u_{yy})$ and $u_t + \left(\frac{2}{F(y,z,t)}\right)uu_x = \gamma(u_{xx} + u_{yy} + u_{zz})$ to determine the Bäcklund transformations and further deduce their respective Cole-Hopf transformations.*

Keywords: Integrability, Painlevé property, Bäcklund transformation, Cole-Hopf transformation, Burgers equation

1. Introduction

The Burgers equation $u_t = 2uu_x + u_{xx}$ has attracted much attention since, it was first proposed by Bateman. Then Burgers gave some special solutions in 1940. Later on, Cole and Hopf independently pointed out that any of the solutions of the heat equation $\xi_t = \xi_{xx}$ can be mapped to a solution of the Burgers equation. Also the Painlevé property (PP) [3] ,[5] ,[7] has shown its usefulness in studying the partial differential equation (PDE), especially in the sense of integrability [4]. Some connections of PP with inverse scattering transform (IST) [1] were found. Through Painlevé analysis for an integrable model, we could probably find its various interesting properties like the Bäcklund transformation [6, [8], Cole - Hopf transformations, Schwarzian derivative and Lax pairs[2].

In this paper, we introduce the (2+1) and (3+1)-dimensional Burgers equations respectively by

$$u_t + \left(\frac{2}{F(y,t)}\right)uu_x = \gamma(u_{xx} + u_{yy})$$

and $u_t + \left(\frac{2}{F(y,z,t)}\right)uu_x = \gamma(u_{xx} + u_{yy} + u_{zz})$

which Possess the Painlevé property, Bäcklund transformation and Cole-Hopf transformation. Thus, by using the WTC [9] method, we confirm whether the (2+1) and (3+1)-dimensional Burgers equations pass the Painlevé test or not. Secondly, we obtain the Bäcklund transformation. Finally, we deduce the Cole-Hopf transformation.

2. Painlevé Analysis of the (2+1)-dimensional Burgers Equation

We investigate the singularity structure of (2+1)-dimensional Burgers equation,

$$u_t + \left(\frac{2}{F(y,t)}\right)uu_x = \gamma(u_{xx} + u_{yy}) \qquad (1)$$

We effect a local Laurent expansion in the neighbourhood of a non-characteristic singular manifold, $\varphi(x,y,t) = 0$, where $\varphi_x \neq 0$ and $\varphi_y \neq 0$. Assume that the leading order analysis of the (2+1)-dimensional Burgers equation has the form
$u(x,y,t) = \varphi^a(x,y,t)\, u_0(x,y,\text{t})$, (2)

where u_0 is an analytic function of x, y, t and a is an integer to be determined later. Substituting (2) into (1) and balancing the nonlinear terms against the dominant linear terms, we get $a = -1$.

Consider the Laurent Series expansion of the solutions in the neighbourhood of the singular manifold

$$u = \sum_{j=0}^{\infty} u_j\, \varphi^{j-1}. \qquad (3)$$

Substituting (3) into (1) yields the recursion relation for u_j given by

$$u_{j-2,t} + (j-2)u_{j-1}\varphi_t + \left(\frac{2}{F(y,t)}\right)\sum_{m=0}^{j} u_{j-m}(u_{m-1,x} +$$
$$m-1\varphi x\,um = \gamma uj-2,xx+$$
$$2j-2uj-1,x\varphi x+j-2uj-1\varphi xx+j-1j-2uj\,\varphi x$$
$$2+uj-2,yy+2j-2uj-1,y\varphi y+j-2uj-1\varphi yy+j-1j-2uj$$
$$\varphi y\,2. \qquad (4)$$

Collecting the terms involving u_j, we found that
$\gamma(j+1)(j-2)u_j\,\varphi_x^2$
$= F(y,t)\big(u_{j-1}, \ldots, u_0, \varphi_t, \varphi_x, \varphi_y, \varphi_{xx}, \varphi_{yy}, \ldots\big)$
$\qquad\qquad$ for j=0,1,2,… $\qquad (5)$

The resonances at $j = -1$ represent the arbitrariness of the singularity manifold $\varphi(x,y,t) = 0$. Following that, we prove the existence of arbitrary function for the other cases j=1, 2 successively. At j=2, we introduce an arbitrary function u_2 and a "compatability condition" on the function (φ, u_0, u_1) that requires the right hand side of (4) to vanish identically.
For the (2+1)-dimensional Burgers equation, we find from (4) that

$$j = 0, u_0 = \frac{-2\gamma(\varphi_x^2 + \varphi_y^2)}{\varphi_x\left(\frac{2}{F(y,t)}\right)}. \qquad (6)$$

When j=1, we get
$\varphi_t + \left(\frac{2}{F(y,t)}\right)\varphi_x u_1 -$
$2\left(\frac{\gamma\varphi_{xx}(\varphi_x^2+\varphi_y^2)}{\varphi_x^2} - \frac{\gamma(2\varphi_{xx}\varphi_x+2\varphi_{xy}\varphi_y)}{\varphi_x}\right) -$
$\left(\frac{\varphi_x}{\varphi_x^2+\varphi_y^2}\right)\left[\frac{\gamma\varphi_{xx}(\varphi_x^2+\varphi_y^2)}{\varphi_x} - 2\varphi_x\left(\frac{\gamma\varphi_{xx}(\varphi_x^2+\varphi_y^2)}{\varphi_x^2} -\right.\right.$
$\left.\frac{\gamma(2\varphi_{xx}\varphi_x+2\varphi_{xy}\varphi_y)}{\varphi_x}\right) + \frac{\gamma\varphi_{yy}(\varphi_x^2+\varphi_y^2)}{\varphi_x} -$

244

$2\varphi_y \left(-\frac{\gamma F_y(\varphi_x{}^2 + \varphi_y{}^2)}{F \varphi_x} + \frac{\gamma \varphi_{xy}(\varphi_x{}^2 + \varphi_y{}^2)}{\varphi_x{}^2} - \frac{\gamma(2\varphi_{xy}\varphi_x + 2\varphi_{yy}\varphi_y)}{\varphi_x} \right) \Big] = 0. \,(7)$

When j=2, we obtain

$\partial_x \left(\varphi_t + \left(\frac{2}{F(y,t)}\right)\varphi_x u_1 - \right.$

2γφxxφx 2+φy 2φx 2−γ2φxxφx +2φxyφy φx−φxφx 2+φy 2γφxxφx 2+φy 2φx−2φxyφxxφx 2+φy 2φx 2−γ2φxxφx+2φxyφy φx+γφyyφx 2+φy 2φx−2φy−γFyφx 2+φy 2Fφx+γφxyφx 2+φy 2φx 2−γ2φxyφx+2φyyφy φx=0. 8

Where $F(y,t)$ satisfies the heat equation, that is, $F_t = \gamma F_{yy}$. By the eqn.(8), the compatability condition (7) at j=2 is satisfied identically. Thus the (2+1)-dimensional Burgers equation possesses the Painlevé property. Furthermore, if we set the arbitrary function u_2 equal to zero and require that

$$u_{1,t} + \left(\frac{2}{F(y,t)}\right) u_1 u_{1,x} = \gamma(u_{1,xx} + u_{1,yy}), \qquad (9)$$

then $u_j = 0, j \geq 2$.

In this case, we find the following Bäcklund transformation for the (2+1)-dimensional Burgers equation

$$u = -\frac{2\gamma \varphi_x}{\varphi} + u_1, \qquad (10)$$

where (u, u_1) satisfy the (2+1)-dimensional Burgers equation and

$$\varphi_t + \left(\frac{2}{F(y,t)}\right) u_1 \varphi_x = \gamma(\varphi_{xx} + \varphi_{yy}). \qquad (11)$$

When $u_1 = 0$, the Cole-Hopf transformation is obtained. Thus $u = -\frac{2\gamma \varphi_x}{\varphi}$ is the Cole-Hopf transformation of the (2+1)-dimensional Burgers equation.

Painlevé Analysis of the (3+1)- Dimensional Burgers Equation

To investigate the singularity structure of the (3+1)-dimensional Burgers Equation,

$$u_t + \left(\frac{2}{F(y,z,t)}\right) u u_x = \gamma(u_{xx} + u_{yy} + u_{zz}). \qquad (12)$$

We effect the local Laurent expansion in the neighbourhood of a non-characteristic singular manifold,
$\varphi(x,y,z,t) = 0$, where $\varphi_x \neq 0, \varphi_y \neq 0$ and $\varphi_z \neq 0$.

Assume that the leading order analysis of the (3+1)-dimensional Burgers equation has the form

$$u(x,y,z,t) = \varphi^a(x,y,z,t)\, u_0(x,y,\text{z},\text{t}), \qquad (13)$$

where u_0 is an analytic function of x, y, z, t and a is an integer to be determined later. Substituting (13) in (12) and balancing the nonlinear terms against the dominant linear terms, we get $a = -1$.

Consider the Laurent Series expansion of the solutions in the neighbourhood of the singular manifold, namely

$$u = \sum_{j=0}^{\infty} u_j \, \varphi^{j-1}. \qquad (14)$$

Substituting (14) into (12) yields the recursion relation for u_j given by

$u_{j-2,t} + (j-2)u_{j-1}\varphi_t + \left(\frac{2}{F(y,z,t)}\right)\sum_{m=0}^{j} u_{j-m}\big(u_{m-1,x} + (m-1)\varphi_x u_m\big) = \gamma\big(u_{j-2,xx} + 2(j-2)u_{j-1,x}\varphi_x + (j-2)u_{j-1}\varphi_{xx} + (j-1)(j-2)u_j\,\varphi_x{}^2 + u_{j-2,yy} + 2(j-2)u_{j-1,y}\varphi_y + (j-2)u_{j-1}\varphi_{yy} + (j-1)(j-2)u_j\,\varphi_y{}^2 + u_{j-2,zz} + 2(j-2)u_{j-1,z}\varphi_z + (j-2)u_{j-1}\varphi_{zz} + (j-1)(j-2)u_j\,\varphi_z{}^2\big). \qquad (15)$

Collecting the terms involving u_j, we found that
$\gamma(j+1)(j-2)u_j\,\varphi_x{}^2 =$
$F(y,z,t)\big(u_{j-1}, \dots, u_0, \varphi_t, \varphi_x, \varphi_y, \varphi_z, \varphi_{xx}, \varphi_{yy}, \dots\big)$
$\text{for } j = 0,1,\dots \qquad (16)$

The resonances at $j = -1$ represent the arbitrariness of the singularity manifold $\varphi(x,y,z,t) = 0$. Following that, we prove the existence of arbitrary function for the other cases j=1, 2 successively. At j=2, we introduce an arbitrary function u_2 and a "compatability condition" on the function (φ, u_0, u_1) that requires the right hand side of (16) to vanish identically.

For the (3+1)-dimensional Burgers equation, we find from (15) that

$$j = 0, u_0 = \frac{-2\gamma(\varphi_x{}^2 + \varphi_y{}^2 + \varphi_z{}^2)}{\varphi_x\left(\frac{2}{F(y,z,t)}\right)} \qquad . (17)$$

When j=1, we get

$\varphi_t + \left(\frac{2}{F(y,z,t)}\right)\varphi_x u_1 - 2\left(\frac{\gamma\varphi_{xx}(\varphi_x{}^2 + \varphi_y{}^2 + \varphi_z{}^2)}{\varphi_x{}^2} - \right.$

γ2φxxφx+2φxyφy +2φxzφz φx−φxφx 2+φy 2+φz 2γφxxφx 2+φy 2+φz 2φx−2φxyφxxφx 2+φy 2+φz 2φx 2−γ2φxxφx+2φxyφy +2φxzφz φx+γφyyφx 2+φy 2+φz 2φx−2φy−γFyφx 2+φy 2+φz 2Fφx+γφxyφx 2+φy 2+φz 2φx 2−γ2φxyφx+2φyyφy +2φyzφz φx+γφzzφx 2+φy 2+φz 2φx−2φz−γFzφx 2+φy 2+φz 2Fφx+γφxzφx 2+φy 2+φz 2φx 2−γ2φxzφx+2φyzφy +2φzzφz φx=0. 18

When j=2, we obtain

$\partial_x\left(\varphi_t + \left(\frac{2}{F(y,z,t)}\right)\varphi_x u_1 - 2\left(\frac{\gamma\varphi_{xx}(\varphi_x{}^2 + \varphi_y{}^2 + \varphi_z{}^2)}{\varphi_x{}^2} - \right.\right.$

γ2φxxφx+2φxyφy +2φxzφz φx−φxφx 2+φy 2+φz 2γφxxφx 2+φy 2+φz 2φx−2φxyφxxφx 2+φy 2+φz 2φx 2−γ2φxxφx+2φxyφy +2φxzφz φx+γφyyφx 2+φy 2+φz 2φx−2φy−γFyφx 2+φy 2+φz 2Fφx+γφxyφx 2+φy 2+φz 2φx 2−γ2φxyφx+2φyyφy +2φyzφz φx+γφzzφx 2+φy 2+φz 2φx−2φz−γFzφx 2+φy 2+φz 2Fφx+γφxzφx 2+φy 2+φz 2φx 2−γ2φxzφx+2φyzφy +2φzzφz φx=0. (19)

where $F(y,z,t)$ satisfies the heat equation, that is,
$$F_t = \gamma(F_{yy} + F_{zz}).$$
By the eqn.(18), the compatability condition (19) at j=2 is satisfied identically. Thus the (3+1)-dimensional Burgers equation possesses the Painlevé property. Furthermore, if we set the arbitrary function u_2 equal to zero and require that

$u_{1,t} + \left(\frac{2}{F(y,z,t)}\right) u_1 u_{1,x} = \gamma(u_{1,xx} + u_{1,yy} + u_{1,zz}), (20)$

245

then $u_j = 0, j \geq 2$.

In this case, we find the following Bäcklund transformation for the (3+1)-dimensional Burgers equation as

$$u = -\frac{2\gamma \varphi_x}{\varphi} + u_1, \qquad\qquad (21)$$

where (u, u_1) satisfy the (3+1)-dimensional Burgers equation and

$$\varphi_t + \left(\frac{2}{F(y,z,t)}\right) u_1 \varphi_x = \gamma\left(\varphi_{xx} + \varphi_{yy} + \varphi_{zz}\right), \qquad (22)$$

when $u_1 = 0$, the Cole-Hopf transformation is obtained.

Thus $u = -\frac{2\gamma \varphi_x}{\varphi}$ is the Cole-Hopf transformation of the (3+1)-dimensional Burgers equation.

References

[1] M .J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.

[2] T. Brugarino, "Painlevé property, auto - Bäcklund transformation, Lax pairs and reduction to the standard form for the Korteweg-de Vries equation with non uniformities", J. Math. Phys., 30, 1013 - 1015, 1989.

[3] N. Joshi, Painlevé property of general variable-coefficient versions of the Kortewegde Vries and non-linear Schrodinger equations", Phys. Lett. A , V.125, 456-460, 1987.

[4] A. Ramani, B. Grammaticos and T. Bountis, The Painlevé Property and Singularity Analysis of Integrable and Non integrable Systems, Phys. Rep.,180, (3), 159 - 245, 1989.

[5] M.Tabor, Painlevé Property for Partial Differential Equations, in Soliton Theory: A Survey of Results, Editor: Fordy.A.P., Manchester University Press, 427-446, (1990).

[6] J. Weiss., The Painlevé property for partial differential equations . II : Backlund transformation, Lax pairs and the Schwarzian derivative, J. Math. Phys.,24, 1405 - 1413, 1983.

[7] J.Weiss, On classes of integrable systems and the Painlevé property, J. Math. Phys., 25, 13 -24, 1984.

[8] J.Weiss, The Painlevé property and Backlund transformations for the sequence of Boussinesq equations, J. Math. Phys.,26, 258 - 269, 1985.

[9] J.Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys., 24, 522 – 526, 1983.