Painlevé Analysis, Bäcklund and Cole-Hopf Transformations of the (2+1) and (3+1)-dimensional Burgers Equations

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Abstract: We apply the Painlevé analysis to the (2+1) and (3+1)-dimensional Burgers Equations respectively given by

\[ \frac{2}{F(y,t)} u_t = \gamma(u_{xx} + u_{yy}) \]

and

\[ \frac{2}{F(y,t)} u_t = \gamma(u_{xx} + u_{yy} + u_{zz}) \]

to determine the Bäcklund transformations and further deduce their respective Cole-Hopf transformations.

Keywords: Integrability, Painlevé property, Bäcklund transformation, Cole-Hopf transformation, Burgers equation

1. Introduction

The Burgers equation, \( u_t + (\frac{2}{F(y,t)} u)_x = \gamma(u_{xx} + u_{yy}) \)

has attracted much attention since, it was first proposed by Bateman. Then Burgers gave some special solutions in 1940. Later on, Cole and Hopf independently pointed out that any of the solutions of the heat equation, \( \xi = \xi_x \) can be mapped to a solution of the Burgers equation. Also the Painlevé property (PP) \([3],[5],[7]\) has shown its usefulness in studying the partial differential equation (PDE), especially in the sense of integrability \([4]\). Some connections of PP with inverse scattering transform (IST) \([1]\) were found. Through Painlevé analysis for an integrable model, we could probably find its differential equation (PDE), especially in the sense of integrability \([4]\). Cole-Hopf independently pointed out that any of the solutions of the heat equation

\[ \xi = \xi_x \]

possess the Painlevé property, Bäcklund transformation and Cole-Hopf transformation. Thus, by using the WTC \([9]\) method, we confirm whether the (2+1) and (3+1)-dimensional Burgers equations pass the Painlevé test or not. Secondly, we obtain the Bäcklund transformation. Finally, we deduce the Cole-Hopf transformation.

2. Painlevé Analysis of the (2+1)-dimensional Burgers Equation

We investigate the singularity structure of (2+1)-dimensional Burgers equation,

\[ u_t + \left( \frac{2}{F(y,t)} \right) u_x = \gamma(u_{xx} + u_{yy}) \]

We effect a local Laurent expansion in the neighbourhood of a non-characteristic singular manifold, \( \varphi(x,y,t) = 0 \), where \( \varphi_x \neq 0 \) and \( \varphi_y \neq 0 \). Assume that the leading order analysis of the (2+1) dimensional Burgers equation has the form

\[ u(x,y,t) = \varphi^a(x,y,t) u_0(x,y,t), \]

where \( u_0 \) is an analytic function of \( x, y, t \) and \( a \) is an integer to be determined later. Substituting (2) into (1) and balancing the nonlinear terms against the dominant linear terms, we get \( a = -1 \).

Consider the Laurent Series expansion of the solutions in the neighbourhood of the singular manifold

\[ u = \sum_{j=0}^{\infty} u_j \varphi^{j-1}. \]

Substituting (3) into (1) yields the recursion relation for \( u_j \) given by

\[ \varphi_{j+1} + (j - 2) \varphi_j + \left( \frac{2}{F(y,t)} \right) \sum_{m=0}^{j} u_{m-1} \varphi_{m+j} + \cdots = 0, \]

where \( \gamma(j+1)(j-2)u_0 \varphi_0^2 = F(y,t)(u_{j-1}, \ldots, u_0, \varphi_1, \varphi_x, \varphi_y, \varphi_{xx}, \varphi_{yy}, \ldots) \) for \( j=0,1,2,\ldots \)

The resonances at \( j = -1 \) represent the arbitrariness of the singularity manifold \( \varphi(x,y,t) = 0 \). Following that, we prove the existence of arbitrary function for the other cases \( j=1, 2 \) successively. At \( j=2 \), we introduce an arbitrary function \( u_2 \) and a “compatibility condition” on the function \( (\varphi, u_0, u_1) \) that requires the right hand side of (4) to vanish identically. For the (2+1)-dimensional Burgers equation, we find from (4) that

\[ j = 0, u_0 = \frac{-2\varphi_x (\varphi_x^2 + \varphi_y^2)}{\varphi_x (\varphi_x^2 + \varphi_y^2)}. \]
2\varphi_y \left(-\frac{\gamma F_y(\varphi_y^2 + \varphi_z^2)}{\varphi_x^2} + \frac{\gamma \varphi_y(\varphi_x^2 + \varphi_y^2)}{\varphi_x^2}\right) = 0. \quad (7)

When \( j = 2 \), we obtain
\[
\partial_x \left( \varphi_x + \left( \frac{2}{F(\varphi_x)} \right) u_1 u_{1,1} \right) = 2\varphi_x \varphi_x \varphi_z + 2\varphi_y \varphi_z \varphi_y \varphi_z + 2\varphi_x \varphi_z 2\varphi_x \varphi_y \varphi_z.
\]

By the eqn.(8), the compatibility condition (7) at \( j = 2 \) is satisfied identically. Thus the (2+1)-dimensional Burgers equation possesses the Painlevé property. Furthermore, if we set the arbitrary function \( u_2 \) equal to zero and require that
\[
\varphi_x + \left( \frac{2}{F(\varphi_x)} \right) u_1 \varphi_x = 0, \quad (9)
\]
then \( u_1 = 0, j \geq 2 \).

In this case, we find the following Bäcklund transformation for the (2+1)-dimensional Burgers equation
\[
u = -\frac{2\varphi_x}{\varphi} + u_1, \quad (10)
\]
where \( (u, u_1) \) satisfy the (2+1)-dimensional Burgers equation and
\[
\varphi_x + \left( \frac{2}{F(\varphi_x)} \right) u_1 \varphi_x = \gamma (\varphi_{xx} + \varphi_{yy}). \quad (11)
\]
When \( u_1 = 0 \), the Cole-Hopf transformation is obtained. Thus \( \nu = -\frac{2\varphi_x}{\varphi} \) is the Cole-Hopf transformation of the (2+1)-dimensional Burgers equation.

Painlevé Analysis of the (3+1)-Dimensional Burgers Equation

To investigate the singularity structure of the (3+1)-dimensional Burgers Equation,
\[
u_1 + \left( \frac{2}{F(\varphi_x)} \right) u_{1,1} \varphi_x = \gamma (\varphi_{xx} + \varphi_{yy} + \varphi_{zz}). \quad (12)
\]
We effect the local Laurent expansion in the neighbourhood of a non-characteristic singular manifold, \( \varphi(x, y, z, t) = 0 \), where \( \varphi_x \neq 0, \varphi_y \neq 0 \) and \( \varphi_z \neq 0 \).

Assume that the leading order analysis of the (3+1)-dimensional Burgers equation has the form
\[
u(x, y, z, t) = \varphi^a(x, y, z, t) \varphi_0(x, y, z, t), \quad (13)
\]
where \( \varphi_0 \) is an analytic function of \( x, y, z, t \) and \( a \) is an integer to be determined later. Substituting (13) in (12) and balancing the nonlinear terms against the dominant linear terms, we get \( a = -1 \).

Consider the Laurent Series expansion of the solutions in the neighbourhood of the singular manifold, namely
\[
u_1 = \sum_{j \geq 0} u_j \varphi^j. \quad (14)
\]
Substituting (14) into (12) yields the recursion relation for \( u_j \) given by
\[
u_{1,1} + \left( \frac{2}{F(\varphi_x)} \right) u_1 \varphi_1 \varphi_x = \gamma (\varphi_{1,xx} + \varphi_{1,yy} + \varphi_{1,zz}), \quad (15)
\]
where \( F(y, z, t) \) satisfies the heat equation, that is,
\[
F_t = \gamma (F_{yy} + F_{zz}). \quad (16)
\]

By the eqn.(18), the compatibility condition (19) at \( j = 2 \) is satisfied identically. Thus the (3+1)-dimensional Burgers equation possesses the Painlevé property. Furthermore, if we set the arbitrary function \( u_2 \) equal to zero and require that
\[
u_{1,1} + \left( \frac{2}{F(\varphi_x)} \right) u_1 \varphi_1 \varphi_x = \gamma (\varphi_{1,xx} + \varphi_{1,yy} + \varphi_{1,zz}), \quad (20)
\]
then $u_j = 0, j \geq 2$.

In this case, we find the following Bäcklund transformation for the (3+1)-dimensional Burgers equation as

$$ u = -2\frac{\psi_{xx}}{\varphi} + u_1, \quad (21) $$

where $(u, u_1)$ satisfy the (3+1)-dimensional Burgers equation and

$$ \varphi_t + \left( \frac{2}{f(y, z, t)} u_1 \varphi_x \right) = \gamma (\varphi_{xx} + \varphi_{yy} + \varphi_{zz}), \quad (22) $$

when $u_1 = 0$, the Cole-Hopf transformation is obtained.

Thus $u = -2\frac{\psi_{xx}}{\varphi}$ is the Cole-Hopf transformation of the (3+1)-dimensional Burgers equation.

References