Painlevé Analysis, Bäcklund and Cole-Hopf Transformations of the (2+1) and (3+1)-dimensional Burgers Equations

T. Shanmuga Priya¹, B. Mayil Vaganan²

Department of Applied Mathematics and Statistics, Madurai Kamaraj University, Madurai-625021, India

Abstract: We apply the Painlevé analysis to the (2+1) and (3+1)-dimensional Burgers Equations respectively given by

\[ \frac{2}{F_y(y,t)} u_{t} = \gamma (u_{xx} + u_{yy}) \]

and

\[ \frac{2}{F(y,t)} u_{t} = \gamma (u_{xx} + u_{yy} + u_{zz}) \]

where \( u_0 \) is an analytic function of \( x, y, t \) and \( \alpha \) is an integer to be determined later. Substituting (2) into (1) and balancing the nonlinear terms against the dominant linear terms, we get \( \alpha = -1 \).

Consider the Laurent Series expansion of the solutions in the neighbourhood of the singular manifold

\[ u = \sum_{j=0}^{\infty} u_j \phi^{j-1}. \]  

Substituting (3) into (1) yields the recursion relation for \( u_j \) given by

\[ u_{j-2,t} + (j-2)u_{j-1} \phi_t + \left( \frac{2}{F(y,t)} \right) \sum_{m=0}^{j-1} u_m - u_{1,x} + m - 1 \rho \sigma \ u_m = \gamma j - 2 \sigma x + 2j - 2u_{j-1} \phi_x + 2j - 2u_{j-1} \phi_y + j - 1 - 2u_j \phi x + j - 2u_j \phi y. \]  

Collecting the terms involving \( u_j \), we found that

\[ \gamma (j + 1) (j - 2) u_0 \phi_x^2 = F(y,t) \left( \frac{u_{j-1} - u_0}{\phi_x}, \frac{u_{j-2} - u_0}{\phi_x}, \frac{u_{j-3} - u_0}{\phi_x}, \frac{u_{j-4} - u_0}{\phi_x}, \ldots \right) \]

for \( j = 0, 1, 2, \ldots \).

The resonances at \( j = -1 \) represent the arbitrariness of the singularity manifold \( \phi(x, y, t) = 0 \). Following that, we prove the existence of arbitrary function for the other cases \( j = 1, 2 \) successively. At \( j = 2 \), we introduce an arbitrary function \( u_2 \) and a “compatibility condition” on the function \( (\phi, u_0, u_1) \) that requires the right hand side of (4) to vanish identically. For the (2+1)-dimensional Burgers equation, we find from (4) that

\[ j = 0, u_0 = \frac{-2 \gamma (\phi_x^2 + \phi_y^2)}{\phi_x (\phi_x \phi_y)}. \]

When \( j = 1 \), we get

\[ \phi_t + \left( \frac{2}{F(y,t)} \right) \phi_x u_1 = - \left( \gamma \phi_x \phi_y \right) \phi_x + 2 \left( \frac{\phi_x \phi_y}{\phi_x^2 + \phi_y^2} \right) \frac{\phi_x \phi_y}{\phi_x^2} - 2 \left( \gamma \phi_x \phi_y \phi_x \phi_y \right) \phi_x + \left( \gamma \phi_x \phi_y \phi_x \phi_y \right) \phi_x. \]

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In this case, we find the following Bäcklund transformation
to be determined later. Substituting (13) in (12) and
We effect the local Laurent expansion in the neighbourhood
of the singular manifold, namely

\[ \psi_x(x, y, z, t) = 2\varphi_x + 2\varphi_y + 2\varphi_z, \]

(15)

where \( \psi_x \) is an arbitrary function for the other cases
satisfy the (2+1)-dimensional Burgers equation
sufficiently. At j=2, we introduce an arbitrary function \( u_2 \) and a "compatibility condition" on the function
\( (u_0, u_1) \) that requires the right hand side of (16) to vanish identically.

For the (3+1)-dimensional Burgers equation, we find from
(15) that

\[ j = 0, u_0 = -\frac{2\varphi_x + \varphi_y + \varphi_z}{2\varphi_x + \varphi_y + \varphi_z}, \]

(17)

When \( j=1 \), we get

\[ \psi_x + \frac{2}{\varphi_x + \psi_y + \psi_z} u_1 \psi_x - \frac{2\varphi_x + \varphi_y + \varphi_z}{2\varphi_x + \varphi_y + \varphi_z} = 0. \]

(18)

When \( j=2 \), we obtain

\[ \psi_x + \frac{2}{\varphi_x + \psi_y + \psi_z} u_1 \psi_x - \frac{2\varphi_x + \varphi_y + \varphi_z}{2\varphi_x + \varphi_y + \varphi_z} = 0. \]

(19)

where \( F(y, z, t) \) satisfies the heat equation, that is,

\[ F_t = \gamma (F_y + F_z). \]

By the eqn.(18), the compatibility condition (19) at \( j=2 \) is satisfied identically. Thus the (3+1)-dimensional Burgers equation possesses the Painlevé property. Furthermore, if we
set the arbitrary function \( u_2 \) equal to zero and require that

\[ u_{1,x} + \frac{2}{\varphi_x + \psi_y + \psi_z} u_1 u_{1,x} = \frac{2\varphi_x + \varphi_y + \varphi_z}{2\varphi_x + \varphi_y + \varphi_z}, \]

(20)

Painlevé Analysis of the (3+1)- Dimensional Burgers Equation

To investigate the singularity structure of the (3+1)-
dimensional Burgers Equation,

\[ u_t + \left( \frac{2}{\varphi_x + \psi_y + \psi_z} \right) u_{1,x} = \gamma (u_{1,x} + u_{1,y} + u_{1,z}). \]

(12)

We effect the local Laurent expansion in the neighbourhood of a non-characteristic singular manifold,

\[ \psi(x, y, z, t) = 0, \]

where \( \varphi_x \neq 0, \varphi_y \neq 0 \) and \( \varphi_z \neq 0 \).

Assume that the leading order analysis of the (3+1)-
dimensional Burgers equation has the form

\[ u(x, y, z, t) = \varphi^a(x, y, z, t), \]

(13)

where \( \varphi^a \) is an analytic function of \( x, y, z, t \) and \( a \) is an integer to be determined later. Substituting (13) in (12) and balancing the nonlinear terms against the dominant linear terms, we get \( a = -1 \).

Consider the Laurent Series expansion of the solutions in the neighbourhood of the singular manifold, namely

\[ u = \sum_{j=0}^{\infty} u_j \varphi^j. \]

(14)

Substituting (14) into (12) yields the recursion relation for \( u_j \) given by

\[ u_{j-2} + \frac{(j - 2)u_{j-2} - \varphi_t + \frac{2}{\varphi_x + \psi_y + \psi_z}}{\varphi_x + \psi_y + \psi_z} = 0. \]

(7)

When \( j=2 \), we obtain

\[ \psi_x + \frac{2}{\varphi_x + \psi_y + \psi_z} u_1 \psi_x - \frac{2\varphi_x + \varphi_y + \varphi_z}{2\varphi_x + \varphi_y + \varphi_z} = 0. \]

(16)

The resonances at \( j = -1 \) represent the arbitrariness of the singularity manifold \( \gamma (x, y, z, t) = 0 \). Following that, we
prove the existence of arbitrary function for the other cases
j=1, 2 successively. At \( j=2 \), we introduce an arbitrary function \( u_2 \) and a "compatibility condition" on the function
\( (u_0, u_1) \) that requires the right hand side of (16) to vanish identically.
then $u_j = 0, j \geq 2$.

In this case, we find the following Bäcklund transformation for the (3+1)-dimensional Burgers equation as

$$u = -\frac{2\varphi_x}{\varphi} + u_1,$$

where $(u, u_1)$ satisfy the (3+1)-dimensional Burgers equation and

$$\varphi_t + \left(\frac{2}{f(y,x,t)}\right) u_1 \varphi_x = \gamma (\varphi_{xx} + \varphi_{yy} + \varphi_{zz}),$$

when $u_1 = 0$, the Cole-Hopf transformation is obtained.

Thus $u = -\frac{2\varphi_x}{\varphi}$ is the Cole- Hopf transformation of the (3+1)-dimensional Burgers equation.

References


