2- Dominator Coloring for Various Graphs in Graph Theory

A. Sangeetha Devi¹, M. M. Shanmugapriya²

¹Research Scholar, Department of Mathematics, Karpagam University, Coimbatore-21, India

²Assistant Professor, Department of Mathematics, Karpagam University, Coimbatore-21, India

Abstract: Given a graph G, the dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graphG dominates at least 2-color class. In this paper, as an extension of Dominator coloring such that various graph using 2-dominator coloring has been discussed.

Keywords: 2-Dominator Coloring, Barbell Graph, Star Graph, Banana Tree, Wheel Graph

1. Introduction

In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number χ (G) of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If χ (G) = k, we say that G is k-chromatic.

A dominating set S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The domination number is the order of a minimum dominating set. Given a graph G and an integer k, finding a dominating set of order k is NP-complete on arbitrary graphs [2], [4].

Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. A graph has a dominator coloring if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G. A $\chi_d(G)$ - coloring of G is any dominator coloring with $\chi_d(G)$ colors. Our study of this was motivated by [9] and [7].

2. Various Kinds of Graphs in Graph Theory

Theorem 2.1

Let $S_{1,n}$ be a star graph, C [$S_{1,n}$] beacentralgraphof star graph, then $\chi_{d,2}$ {C [$S_{1,n}$] } $\leq n+2,n\geq 3$.

Proof:

Let U_0 be the root vertex of G. Let { $v_0, v_1, v_2, \dots, v_{n-1}$ } be the remaining vertices of G. By definition of central graph C(G), subdivide the each edge exactly once and joining all the non-adjacent vertices of G. Let $w_0, w_1, w_2, \dots, w_{n-1}$ be the subdividing vertices of G. By definition of 2- dominator coloring, assign color C_0 to the root vertices u_0 . Now color of all the pendent vertices of *G* by ,*n*" different colors. Introduce two new colors ,,*C_i* ,, ,,*C_j* "alternatively to the subdividing vertices of *G*, *C*₀ will dominates ,,*C_i*, ,,*C_j*". Also all the pendent vertices of *G* dominates at least two color class. i.e, $1+n+2 \le n+3$. Therefore $\chi_{d,2}$ {C [*S*_{1,n}]} $\le n+3$, $n \ge 3$.

Theorem 2.2

Let $S_{1,n}$ be a star graph, then $\chi_{d,2}(S_{1,n}) = 2, n \ge 2$.

Proof:

Let $S_{1,n}$ be a star graph by 2 dominator coloring and from theorem 3.2 and 3.7[9], the assignment of colors to the vertices of *G*, May dominates itself and it forms a color class. Thus $\chi_{d,2}(S_{1,n}) = 2$.

Theorem 2.3

Let G=B (K_n , K_n) where n=3, 4, 5, ... Then $\chi_{d,2}$ [B (K_n , K_n)]=n.

Proof:

Let $G = B(K_n, K_n)$ be a Barbell graph. Now divide the vertices of *G* be *U* and *V*. Let $U = \{u_0, u_1, u_2, \dots, u_{n-1}\}$ and $V = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ respectively. By definition of Barbell graph, *G* is obtained by connecting two complete graphs K_n by a_n bridge e_i . Using theorem 1.2 [2] and 3.1 [3],2-dominator coloring in any complete graph all the vertices must contain at least n colors. Let the color vertex u_0 by c_0, u_1 by c_1, u_2 by c_2, \dots, u_n by c_{n-1} . Similarly, color of the vertex v_0 by c_0, v_1 by c_1, v_2 by c_2, \dots, v_n by c_{n-1} . Thus a bridge $, p_i$ "between *U* and *V* will receive the color, $c_0 c_{n-1}$." Hence no two adjacent vertices have same color. Also every vertex of *G* dominates at least two color classes. i.e. $\chi_{d,2}$ [$B(K_n, K_n$)] = $n, n=3,4,5,\dots$

Theorem 2.4

Let $G=P_2 \square P_n$, $n \ge 2$, then $\chi_{d,2}(G) = 4$.

Proof:

Let $G = P_2$ \prod_n be a ladder graph , $n \ge 2$ (i.e. two rails or paths connected by rungs). Now color the vertices in path P_1 by two different colors c_0 , c_1 alternatively. Similarly color. The vertices in path P_2 by another new two different colors

 c_2 , c_3 alternatively. Therefore the every rung P_n between the paths P_1 and P_2 will receive the color c_0 , c_2 and c_1 , c_3 alternatively. Hence by definition of 2 dominator coloring $\chi_{d,2}(P_2P_n) = 4, n \ge 2$.

Theorem 2.5

Let $G=B_{m,n}$, $m,n \ge 2$ be a banana tree then $\chi_{d,2}(B_{m,n}) = 2$

Proof:

Let $B_{m,n}$ be a banana tree of order $m,n \ge 2$ ie (n,m)- banana tree is a graph created by connecting a leaf of each of m copies of n-star graph with a pendent vertex and it is different from all stars. Thus from theorem 3.2 and 3.7[9], $\chi_{d,2}(B_{m,n}) = 2$.

Theorem 2.6

If $G=F_{n,k}$, $n,k \ge 2$, then $\chi_{d,2}(G) = 2$

Proof:

Let $G=F_{n,k}$, $n,k \ge 2$, be a fire cracker graph i.e $F_{n,k}$ is a graph connecting n, k- states by an leaf. By definition of 2 dominator coloring and by use of theorem 3.2 and 3.7[9], every vertex may dominates its own class. Thus $\chi_{d,2}[F_{n,k}] = 2$.

Theorem 2.7

If $G=D_n^{(m)}$, *n* be the number of vertex *m* be the copies of complete graph, $n,m \ge 3$, then, $\chi_{d,2}[D_n^{(m)}] = n$.

Proof

Let $G=D_n^{(m)}$ be a wind mill graph and it is obtained of connecting m copies of K_n with the common vertex. By the definition of 2- dominator coloring from theorem 1.2 and $3.1[3]D_n^{(m)}$ needs at least n colors and no two adjacent vertices will receive same color. Thus $\chi_{d,2} [D_n^{(m)}] = n$, $n \ge 3,4,5, ...$

Theorem 2.8

If $G=G_n$ be any gear graph then $\chi_{d,2}(G) = 4, n = 3,4,5,7,...$

Proof:

Let $G=G_n$ be a gear graph or bipartite wheel graph. Let u_1 be the root vertex of G. Let v_1, v_2, \dots, v_n be the main vertex of G. Let w_1, w_2, \dots, w_n be the sub vertices of G. Procedure for 2- dominator coloring of G: Case 1: For n=3, $\chi_{d,2}$ (G_3) = 4.

Let a root vertex u is assigned by the color c_0 . Now assign color to vertex v_0 by c_0 , v_1 by c_1 , v_2 by c_2 Similarly w_0 by c_0 , w_1 by c_1 , w_2 by c_2 respectively. Therefore every vertex must dominate at least two color class. Hence $\chi_{d,2}(G) = 4$. Case 2: For $n = 4,5,6,..., \chi_{d,2}(G_3) = 4$.

Assign color C_0 to the vertex u_0 . Now color C_1, C_2 alternatively to the vertices $v_0, v_1, \ldots, v_{n-1}$. Hence a root vertex u_0 colored by C_0 will dominates the color classes C_1 and C_2 respectively. The vertices $w_0, w_1, \ldots, w_{n-1}$ is assigned by the color C_3 such that C_3 will dominates the color classes C_1 and C_2 in G. By definition of 2-Dominator coloring the vertices of G must dominates at least two color classes. Hence $\chi_{d,2}(G_3) = 4$, n = 4,5,6,...

Theorem 2.9

If $G=W_n$ be a wheel graph then $\chi_{d,2}(W_n) = \begin{cases} 3, n = 6, 8, 10, \dots \\ 4, n = 7, 9, 11, \dots \end{cases}$

Proof:

Let $G=W_n$ be a wheel graph. Let u_1 be the root vertex of G. Let v_1, v_2, \dots, v_n be the main vertex of G.

2- Dominator coloring of G as follows.

Case 1: $\chi_{d,2}(W_n) = 3, n = 6,8,10, ...$

First color c_0 to the root vertex u_1 . Assign color $,C_1, C_2$ "alternatively to the remaining verticies of *G*. i.e v_1v_2 receive color C_1C_2 , v_2v_3 receive color C_2C_1 , v_3v_4 will receive C_1C and so on. By coloring it forms a clique. Hence every vertex must dominate at least two color classes. Thus $\chi_{d,2}(W_n) = 3, n = 6,8,10, ...$

Case 2: $\chi_{d,2}(W_n) = 4$, n = 7,9,11, ...

Assign color C_0 to the root vertex u_1 . Now the colors $C_1 C_2$, C_3 will be assigned alternatively to the remaining vertices of G. i.e. v_1v_2 by C_1C_2 , v_2v_3 by C_2C_3 , v_3v_4 by C_3C_1 , v_4v_5 by C_1C_2 ,...,and so on.By coloring it forms a clique while assigning the color in the above alternative sequence, at last there may or may not remains some vertices of G, it depends up on $,n^{\circ}$ suppose the vertices of G remains, to color the remaining vertices of G using the above same three colors $c_1 c_2$, c_3 colored in any order, but there should not any two adjacent vertices receive the same color. Hence every vertex must dominate at least two color classes. Thus $\chi_{d,2}(W_n) = 4$, n = 7,9,11, ...

Theorem 2.10:

For any prism graph $Y_{2,n}, \chi_{d,2}(Y_{2,n}) \le n$, n=3,4,5,....

Proof:

Let $G = Y_{2,n}$ be a prism graph. Let $G = U \cup V$. Let $U = \{u_0, u_1, u_2, \dots, u_{n-1}\}$ be the inner vertices of G. $V = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ be the outer vertices of G respectively. Let $E(U) = \{u_i u_{i+1}; 0 \le i \le n-2\}$ and $E(V) = \{v_i v_{i+1}; 0 \le i \le n-2\}$. Thus for any $Y_{2,n}, E(G) = 3n$. Here x=1 be the edge joining the vertex of U and V.

Let color the vertices of U be $C=\{0,2x+1,2(2x+1),..., (M-1)(2I+1)\}$ by M different colors. Note $M=\lceil n/(2x+1)\rceil$, x=1 be the edge joining the vertices of U colored by another different colors 1,2,..., n-M in clockwise direction respectively. Similarly the vertices of V can be colored by the above said same M colors, but it starts from the vertices v_1 , it may not be end at the vertex v_0 .i.e., $v_1, v_4, v_7,...$ And so on. Also the remaining vertices of V colored by above said same coloring using same concept. Therefore by coloring, there exists a clique between the vertices U and Vof G. Thus,G dominates at least two color classes. Hence G admits $\chi_{d,2}(Y_{2,n}) \le n$, n=3,4,5,....

Theorem 2.11

For any graph G, $G = W_{3,n}$ then $\chi_{d,2}(G) \le n, n \ge 3$.

Proof:

Let W_n be a web graph.

Let $G = U \cup V \cup W$ be the vertex set.

Let $U = \{ u_0, u_1, \dots, u_{n-1} \}$ $V = \{ v_0, v_1, \dots, v_{n-1} \}$ and $W = \{ w_0, w_1, \dots, w_{n-1} \}$ respectively.

Let $E(U) = \{ u_i u_{i+1}; 0 \le i \le n-2 \}$ and $E(V) = \{ v_j v_{j+1}; 0 \le j \le n-2 \}$ be the edge set of U and V

Let $E(t) = \begin{cases} u_k v_k; & 0 \le K \le n-1 \\ v_k w_k; & 0 \le K \le n-1 \end{cases}$ be the edge between the vertex set UV and VW respectively. Thus for any web graph $W_{3,n}E(G)=3n+n$. Consider x=1 be the edge joining the vertex UV and VW.

Now the procedure for 2 -dominator coloring of G follows:

Let color the vertex $G \in U \cup V \cup W$ by $C=\{0,2x+1,2(2x+1),\ldots,(M-1)(2x+1)\}$ by M different colors

in clockwise direction. Note $M = \lceil n/(2x+1) \rceil$, x=1. Also the remaining vertices of $G \in U \cup V \cup W$ colored by n-*M* different color 1,2,..., *n*-*M* in clockwise direction. To color the vertices of *U*, starts from the vertex u_0 using $C = \{$ $0,2x+1,2(2x+1)\},....(M-1)(2x+1)\}$ and it may or may not ends at the vertex u_{n-1} and the remaining vertices of *G* colored by *n*-*M* different colors. Next color the vertices of *V* starts from vertex v_1 using above said the *M* colors $C=\{$ $0,2x+1,2(2x+1)\},.....(M-1)(2x+1)\}$ and it also may or may not ends at the vertex v_{n-1} then the remaining vertices of *v* colored by the same said*n*-*M* different colors in same previous manner. Similarly to color the remaining vertices of *W* needs those same *n*-*M* colors using same procedure as said above. Likewise coloring there exist a cliques in *G*.

Hence every vertex of G dominates at least two color classes. Thus for any W_n , $\chi_{d,2}(W_{3,n}) \le M + n - m \le n, n \ge 3$.

Theorem 2.12

If $G=H_n$ ba a helm graph, then

$$\chi_{d,2}(H_n) = \begin{cases} 3, n = 6, 8, 10, 12, \dots \\ 4, n = 7, 9, 11, \dots \end{cases}$$

Proof:

By the above theorem $\chi_{d,2}(W_n) = \begin{cases} 3, n = 6,8,10,12, ... \\ 4, n = 7,9,11, \end{cases}$ definition of helm graph, assign color c_0 to all the pendent vertices $w_0, w_1, ..., w_n$ in G, which was already colored to the root vertex u_0 in G. By definition of 2-dominator coloring $\chi_{d,2}(H_n) = \begin{cases} 3, n = 6,8,10,12, ... \\ 4, n = 7,9,11, \end{cases}$

Cayley Graph

Let G be a group, and let S be subset of G. Then the Cayley Digraph D(G, S) on G with connection set S is defined as follows:

1. The vertices are the elements of G

2. There is an arc joining g and h if and only if $h = s_g$ for some $s \in S$.

We can extend this idea to a Cayley Color graph, where *S* is a generating set for *G*, each $s_i \in S$ is assigned a color, and if $g = s_{ih}$, then the arc connecting them is colored s_i .

Theorem 2.12

Let *G* be a group with generating set *S* and let φ be a colorpreserving permutation on *V* (*D*(*G*, *S*)). Then φ is a color preserving automorphism of *D*(*G*, *S*) if and only if $\varphi(gh) = \varphi(g)h$.

Suppose that $\varphi(gh) = \varphi(g)h$.

Proof:

To show that φ is color-preserving, we need to show that if gh-1 = s, then $\varphi(gh-1) = s$. Suppose gh-1 = s. Then $\varphi(gh-1) = \varphi(g)gh-1 = \varphi(g)gh-1 = \varphi(ggh-1)s = s$.

3. Conclusion

In this paper, we found out some result of various graphs like, Central Graph of Star Graph C [$S_{1,n}$], Star Graph: $S_{1,n}$, Barbell Graph B (K_n , K_n), Ladder Graph, Banana Tree $B_{m,n}$, Fire cracks graph: $F_{n,k}$, $n,k \ge 2$, Wind Mill Graph $D_n^{(m)}$, Gear Graph G_n , Wheel Graph, Prism Graph $Y_{2,n}$, Web Graph W_n , Helm Graph H_n , Cayley Graph using 2-dominator coloring. This will be initiative study of extension of 2-Dominator coloring.

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