On The Homogeneous Biquadratic Equation With 5 Unknowns: \( x^4 - y^4 = 65 \left( z^2 - w^2 \right) R^2 \)

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Abstract: The Homogenous biquadratic equation with five unknowns given by \( x^4 - y^4 = 65 \left( z^2 - w^2 \right) R^2 \) is considered and analyzed for finding its non zero distinct integral solutions. Introducing the linear transformations \( x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1 \) and employing the method of factorization different patterns of non zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, Thabit-ibn-Kurrah number, Star number, Carol number, woodall number, kynea number, pentatope number, stellaoctangul number, octahedral number, Mersenne number are exhibited.

Keywords: Homogeneous equation, Integral solutions, Polygonal numbers, Pyramidal and special number.

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Notations used:
- \( T_{m,n} \) - Polygonal number of rank \( n \) with size \( m \)
- \( P_n \) - Pyramidal number of rank \( n \) with size \( m \)
- \( g_n \) - Gnomonic number of rank \( n \)
- \( Pr_n \) - Pronic number of rank \( n \)
- \( Ct_{16,n} \) - Centered hexadecagonal pyramidal number of rank \( n \)
- \( OH_n \) - Octahedral number of rank \( n \)
- \( SO_n \) - Stella octangular number of rank \( n \)
- \( ky_n \) - kynea number
- \( carl_n \) - carol number

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-12]. In this context one may refer [4-10] for various problems on the biquadratic Diophantine equations. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. This paper concern with the homogeneous biquadratic equation with five unknowns \( x^4 - y^4 = 65 \left( z^2 - w^2 \right) R^2 \) for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. Method of Analysis

The biquadratic equation with five unknowns to be solved for its non-zero distinct integral solution is

\[ x^4 - y^4 = 65 \left( z^2 - w^2 \right) R^2 \]  

Consider the transformations

\[ x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1 \]  

On substituting (2) in (1), we get

\[ u^2 + v^2 = 65R^2 \]  

2.1 Pattern: I

Assume \( 65 = (8 + i) (8 - i) \) and \( R = a^2 + b^2 = (a + i b)(a - i b) \)

Using (4) and (5) in (3) and employing the method of factorization we get.

\[ (u + i v) (u - iv) = (8 + i) (8 - i) (a + i b)^2 (a - i b)^2 \]

On equating the positive and negative factors, we have,

\[ (u + i v) = (8 + i) (a + i b)^2 \]
\[ (u - iv) = (8 - i) (a - i b)^2 \]

On equating real and imaginary parts, we get

\[ u = u(a, b) = 8a^2 - 8b^2 - 2ab \]
\[ v = v(a, b) = a^2 - b^2 + 16ab \]

On substituting \( u \) and \( v \) in (2) we get the values of \( x, y, z \) and \( w \). The non-zero distinct integrals values of \( x, y, z, w \) and \( R \) satisfying (1) are given by

\[ x = x(a, b) = 9a^2 - 9b^2 + 14ab \]
\[ y = y(a, b) = 7a^2 - 7b^2 - 18ab \]
\[ z = z(a, b) = 3(8a^2 + 8b^2 - 48a^2 b^2 + 126a^3 b - 126ab^3) + 1 \]
\[ w = w(a, b) = 3(8a^3 + 8b^3 - 48a^2 b^2 + 126a^3 b - 126ab^3) - 1 \]
\[ R = R(a, b) = a^2 + b^2 \]

Properties:
1. \( x(2, a) + y(2, a) + Ct_{16,a} + 8t_{t,2} - W_4 = 0 \)
2. \( R(a+1, a+1) - 2t_{t,2} - G_{23} \equiv 0 \pmod{3} \)

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3. \( z(1,b) - W(1,b) - OH_2 \equiv 0 (\text{mod } 4) \)

4. \( x [a(a+1),1] + 9y[a(a+1),1] = 260P_n = 0 \)

5. \( 7x [a(2a^2-1),1] - 9y[a(2a^2-1),1] - 260S_n = 0 \)

6. \( R(3,3) = P_3^5 = 0 \)

7. \( x(2,2) - y(2,2) = 0 (\text{mod} 2) \)

8. \( 7x[a, a(2a + 1)] - 9y[a, a(2a^2 + 1)] - 864OH_4 = 0 \)

9. \( z(1,1) + w(1,1) + T_{3,4} = 0 (\text{mod} 2) \)

10. \( 7x[a, a(a + 1)] - 9y[a, a(a + 1)] - 260(P_n)^2 = 0. \)

### 2.2 Pattern: II

Also 65 can be written in equation (3) as

\[ 65 = (1 + 8i) (1 - 8i) \]  

(6)

Using (5) and (6) in equation (3) it is written in factorizable form as

\[ (u + iv)(u - iv) = (1 + 8i)(1 - 8i)(a + ib)^2(a - ib)^2 \]

On equating the positive and negative factors, we get,

\[ (u + iv) = (1 + 8i)(1 - 8i)(a + ib)^2 \]

\[ (u - iv) = (1 - 8i)(a - ib)^2 \]

On equating real and imaginary parts, we have

\[ u = u(a,b) = a^2 - b^2 - 16ab \]

\[ v = v(a,b) = 8a^2 - 8b^2 + 2ab \]

Substituting the values of \( u \) and \( v \) in (2), the non-zero distinct values of \( x, y, z, w \) and \( R \) satisfying (1) are given by

\[ x = x(a,b) = 9a^2 - 9b^2 - 14ab \]

\[ y = y(a,b) = 7a^2 + 7b^2 - 18ab \]

\[ z = z(a,b) = 2(8a^2 + 8b^2 - 48a^2 b^2 - 126a^2 b + 126ab) + 1 \]

\[ w = w(a,b) = 2(8a^2 + 8b^2 - 48a^2 b^2 - 126a^2 b + 126ab) - 1 \]

\[ R = R(a,b) = a^2 + b^2 \]

### Properties:

1. \( 7x[2(a-1)^2,1] + 9y[2(a-1)^2,1] = 260(G_3)^2 = 0 \)

2. \( y(2a,1) + 7R(2a,1) + G_{10a} - W_i \equiv 0 (\text{mod} 2) \)

3. \( R(2a,2a) = 8t_{3,4} = 0 \)

4. \( x(a,a+1) + y(a,a+1) + Ct_{1,6a} + 24t_{1,4a} + G_{14a} = 0 \)

5. \( x(a,1) + R(1,1) + 10t_{1,4a} + G_{7a} + T_{K} \equiv 0 (\text{mod} 2) \)

6. \( \tau(2,2) + w(2,2) + T_{10,23} = \text{CarolutNumber}. \)

7. \( x(a(a+1),1) + 9y[a(a+1),1] + 288p = 0. \)

8. \( 7x[a(2a-1)^2] + 9y[a(2a-1)^2] + 260(G_3)^2 = 0 \)

9. \( 9y[a(2a^2 + 1)] - 7x[a(2a^2 + 1)] - 378(OH_4)^2 = -1920OH_4 + T_{10,6a} = 0 \)

10. \( 7x[l,1] + 9y[l,1] + CS_i = \text{woodallnumber}. \)

### 2.3 Pattern: III

Rewrite (3) as

\[ 1 * u = 65R^2 - u^2 \]

(7)

Assume \( u = 65a^2 - b^2 = (\sqrt{65} + a)(\sqrt{65} - a) \)

(8)

Write \( a \) as

\[ (\sqrt{65} + a)(\sqrt{65} - a) \]

(9)

Using (8) and (9) in (7) it is written in factorizable form as

\[ (\sqrt{65} + 8)(\sqrt{65} - 8) = \]

\[ = (\sqrt{65} R + v)(\sqrt{65} R - v) \]

On equating the rational and irrational parts, we get

\[ (\sqrt{65} + 8) = (\sqrt{65} R + v) \]

\[ (\sqrt{65} - 8) = (\sqrt{65} R - v) \]

On equating the real and imaginary parts, we get

\[ R = R(a,b) = 65a^2 + b^2 + 16ab \]

\[ v = v(a,b) = 520a^2 + 8b^2 + 130ab \]

Substituting the values of \( u \) and \( v \) in (2), the non-zero distinct integral values of \( x, y, z, w \) and \( R \) satisfying (1) are given by

\[ x = x(a,b) = 585a^2 + 7b^2 + 130ab \]

\[ y = y(a,b) = -452a^2 - 9b^2 - 130ab \]

\[ z = z(a,b) = 2(3380a^2 - 8b^2 + 3450ab) + 1 \]

\[ w = w(a,b) = 2(3380a^2 - 8b^2 - 3450ab) - 1 \]

\[ R = R(a,b) = 65a^2 + b^2 + 16ab \]

### Properties:

1. \( x[A, (2A^2 - 1)] + y[A, (2A^2 - 1)] = -30T_{4,4,4}^{2} + 2(SO_4)^2 = 0 \)

2. \( R(A, 2A - 1) - 101T_{4,4} + G_{10} = 0 \)

3. \( R(2A, 2A) = 328T_{4,4} = 0 \)

4. \( x[1, (A+A)] + 7R(A, A) + 10P_{T_{4,4}} + 18 P_{T_{4,4}} = 0 (\text{mod} 4) \)

5. \( y(A, 1) + 9R(A, 1) - 130T_{4,4} + G_{7a} - P_{T_{4,4}} = 0 \)

6. \( x[2A(2A + 1),1] + y[2A(2A + 1),1] - 390(OH_{4}) = 0 \)

7. \( x(l,1) + y(l,1) = 0 (\text{mod} 2) \)

8. \( x(a, a(a^2 + 1)] - 585T_{4,4} - 2l(OH_{4}) = 390OH_{4} = 0 \)

9. \( z(l,1) + w(l,1) = 47 (\text{mod} 3584) \)

10. \( x(a + a, 1) - 130P_{a} - 585P_{a} = -P_{a}^{2} = 0 \)

### 2.4 Pattern: IV Rewrite (3) as

\[ 1 * v^2 = 65R^2 - u^2 \]

(11)

Write 1 as

\[ \frac{1}{\sqrt{65} + 1(\sqrt{65} - 1)} \]

(12)

Assume \( v = 65a^2 - b^2 = (\sqrt{65} a - b)(\sqrt{65} a + b) \)

(13)

Using (12) and (13) in (11), it is written in factorizable form as

\[ \left(\frac{\sqrt{65} + 1}{\sqrt{65} - 1}\right) \]

\[ = \left(\sqrt{65} a - b\right)^2 \left(\sqrt{65} a + b\right)^2 \]

\[ = \left(\sqrt{65} R - v\right)^2 \left(\sqrt{65} R + v\right)^2 \]

On equating the rational and irrational factors we get,

\[ R = R(a,b) = \frac{1}{8} (65a^2 + b^2 + 2ab) \]

\[ u = u(a,b) = \frac{1}{8} (65a^2 + b^2 - 130ab) \]

Replacing \( a^a \) by 8A and \( b^b \) by 8B in the above equations (13) and (15), we get

\[ R = R(A, B) = 520A^2 + 8B^2 + 16AB \]

\[ u = u(A, B) = 520A^2 + 8B^2 + 1040AB \]

\[ v = v(A, B) = 4160A^2 - 64B^2 \]

On substituting the values of \( u \) and \( v \) in (2), the non-zero distinct integral values of \( x, y, z, w \) and \( R \) satisfying (1) are given by

\[ x = x(A, B) = 4680A^2 - 56B^2 + 1040AB \]

\[ y = y(A, B) = -3640A^2 + 72B^2 + 1040AB \]

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\[ z = z(A, B) = 2(216320A^4 - 512B^4 + 432640A^2B^2 - 66560AB^3 + 1) +
\]
\[ w = w(A, B) = 2(216320A^4 - 512B^4 + 432640A^2B^2 - 66560AB^3 + 1) -
\]
\[ R = R(A, B) = 520A^2 + 8B^2 + 16AB
\]

Properties:

1. \( x(A+1, 1) - y(A+1, 1) - 8320 T_{a} - G_{3200A} S_{12} \equiv (mod 2)
\]
2. \( y(1A) = 9R (1, A) - G_{4848} \equiv 0 (mod 3)
\]
3. \( R\{2A^2 - 1, 1\} - 520 (SO_{A})^2 + 16 (SO_{A}) - W_2 =
\]

Star number
4. \( z(A, 1) - W(A, 1) \equiv (mod 2)
\]
5. \( x(1B) + 7R (1, B) - P_{25} \equiv G_{516B} \equiv 0 (mod 2)
\]
6. \( R(1, 1) - 4 \text{ is Nasty number.}
\]
7. \( x(1, 1) - y(1, 1) - P_{26} - G_{N_{0}P} = 0
\]
8. \( 72[a(2a^2 + 1)] + 56[y(a(2a^2 + 1)] - 13312[w^2] - 499200O_1 = 0
\]
9. \( 8x(1, 2a + 1) + 224P_{e} - G_{w_{10}} - J_{s} = \text{Cullen number.}
\]
10. \( z(1, 1) + w(1, 1) = 32 (mod 402816)
\]

2.5 Pattern: V

Write (3) as \( u^2 - R^2 = 64R^2 - v^2 \)
\[ (u + R)(u - R) = (8R + v)(8R - v), \quad (16)
\]

which is expressed in the form of ratio as
\[ \frac{u + R}{8R + v} = \frac{u - R}{8R - v} \quad \frac{2}{2}, \quad B \neq 0
\]

This is equivalent to the following two equations
\[ -uA + R(8B^2 + A) - VB = 0
\]
\[ uB + R(B - 8A) = VA = 0
\]

On solving the above equations by the method of cross multiplication we get,
\[ u = u(A, B) = -A^2 - B^2 - 16AB
\]
\[ R = R(A, B) = -A^2 - B^2
\]
\[ v = v(A, B) = 8A^2 - 8B^2 - 2AB
\]

Substituting the values of \( u \) and \( v \) in (2), the non – zero distinct integral values of \( x, y, z, w \) and \( R \) satisfying (1) are given by,
\[ x = x(A, B) = 7A^2 - 9B^2 - 18AB
\]
\[ y = y(A, B) = -9A^2 + 7B^2 - 14AB
\]
\[ z = z(A, B) = 2[-8A^4 + 8B^4 + 32A^2 B^2 - 126 A^2 B^2 + 126AB^2] + 1
\]
\[ w = w(A, B) = 2[-8A^4 + 8B^4 + 32A^2 B^2 - 126 A^2 B^2 + 126AB^2] - 1
\]
\[ R = R(A, B) = -A^2 - B^2
\]

Properties:

1. \( 9x \{1, A(A+1)] + 7y \{1, A(A+1)] + 32(P_{A})^2 - 260T_{4A} - G_{3200A} \equiv \text{woodall number}
\]
2. \( R\{A+1, 1\} + T_{4A} + G_{4848} \equiv 0 (mod 3)
\]
3. \( y \{1, A(2A^2 - 1)] + x \{1, A(2A^2 - 1)] + 14 SO_{A} + W_{2} = Ky_{2} \equiv 0
\]
4. \( R\{2A, 2A\} + 8t_{4A} = 0
\]
5. \( x(1, 1) + 7R(1, 1) + P_{4} \equiv \text{Nasty number}
\]

6. \( x[a(2a^2 + 1)] - 21(OH_{2}) + 180O_{2} \text{ is a cubic integer}
\]
7. \( w(1, 1) + z(1, 1) + R(1, 1) - P_{4} = 0
\]
8. \( x(1, a + 1) - 7T_{n_{a}} \text{ is a perfect square}
\]
9. \( R[a(2a^2 + 1)] - 3(OH_{2})^2 \equiv \text{caroll number.}
\]
10. \( x(a + 1, a + 2) + 7y(a + 1, a + 2) + 292_{n_{a} - 1}
\]

\[ G_{4848} \equiv 0 (mod 2).\]

3. Conclusion

It is worth to note that in (2), the transformations for \( z \) and \( w \) may be considered as \( z = 2u + v \) and \( w = 2u - v \). For this case, the values of \( x, y \) and \( R \) are the same as above where as the values of \( z \) and \( w \) changes for every pattern. To conclude one may consider biquadratic equations with multivariables (\( \geq 5 \)) and search for their non-zero distinct integer solutions along with their corresponding properties.

References


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