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On The Uniqueness Problems of Entire Functions Concerning Differential Polynomials

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Abstract: ***We discuss the uniqueness of entire functions concerning differential polynomials. The results extend and generalise recent results of Jiang-Tao Li and Ping Li.***

Keywords: Entire functions, Uniqueness, Differential Polynomials, Deficient value, Weighted sharing.

1. Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane C. For $a \in C \cup \{\infty\}$ we say that f and g share the value aCM (counting multiplicities) if the a-points of f and g coincide in locations and multiplicities. If we do not consider the multiplicities, we say that f and g share the value aIM(ignoring multiplicities). For standard definitions and notations of the value distribution theory we refer to [1].

We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ is called small function of f if $T(r,a) = S(r,f)$.

In [2] and [3] the idea of weighted sharing is introduced which measures how close a shared value is to being shared IM or to being shared CM. We now explain the idea of weighted sharing of values.

Definition 1.1. [2, 3] Let k be a nonnegative integer or infinity. For $a \in C$ we denote by $E_k(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m \geq k$. If $E_k(a,f) = E_k(a,g)$ we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\le k)$ and z_0 is a zero of $f-a$ with multiplicity $m(> k)$ if and only if it is a zero of $g-a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integers p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.2. Let f be a non-constant meromorphic function and $a \in C \cup \{\infty\}$. For a positive integer k we denote by $N_{k)}(r,a;f)$ $(N_{(k}(r,a;f))$ the counting function of those a-points of f whose multiplicities are less than or equal to k (greater than or equal to k), where an a-point is counted according to its multiplicity.
Also by $\bar{N}_{k)}(r,a;f)$ and $\bar{N}_{(k}(r,a;f))$ we denote the corresponding reduced counting functions.
We put

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^{+} T(r,f)}{\log r}$$

and

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}.$$

$\lambda(f)$ and $\delta(a,f)$ are called the order of $f(z)$ and the deficiency of a with respect to $f(z)$ respectively.

Definition 1.3. Let f and g be two non-constant meromorphic functions for any positive integer k we denote by $N^{L}_{(k}(r,1;f)$ the counting function of zeros of $f-1$ and $g-1$ whose multiplicities are not less than k and about which $f-1$ has larger multiplicity than $g-1$. By $N^{E}_{(k}(r,1;f)(N^{E}_{k)}(r,1;f))$ we denote the counting function of zeros of $f-1$ and $g-1$ whose multiplicities are not less (greater) than k and about which $f-1$ and $g-1$ have equal multiplicities. By $\bar{N}^{L}_{(k}(r,1;f)$, $\bar{N}^{E}_{(k}(r,1;f)$ and $\bar{N}^{E}_{k)}(r,1;f)$ we denote the corresponding reduced counting functions.

Definition 1.4. Let f be a non-constant meromorphic function we denote by $N_0(r,0;f')$ the counting function of zeros of f' which are not zeros of $f(f-1)$.

In 1976 C. C. Yang [7] asked: If two transcendental entire functions f and g assume the same zeros with the same multiplicities and that their first derivatives assume the same 1-points with the same multiplicities then what can be said about the relationship between f and g ?

To solve the above question in 1981 K. Shibazaki [6] proved the following:

Theorem A. [6] Let f and g be two entire functions of finite order. If f' and g' share the value 1 CM with $\delta(0,f) > 0$ and 0 being lacunary for g then either $f \equiv g$ or $f'g' \equiv 1$.

In 1990 H. X. Yi [10] proved the following theorem:

Theorem B. [10] Let f and g be two non-constant entire functions and let $k$be a non-negative integer.If f and g share the value 0 CM, $f^{(k)}$ and $g^{(k)}$ share the value 1CM and $\delta(0,f) > \frac{1}{2}$, then either $f \equiv g$ or $f^{(k)}g^{(k)} \equiv 1$.

Let f be a non-constant meromorphic function. We denote by $P(f) = \sum_{i=0}^{k} a_i f^{(i)}$ the differential polynomial of f, where $a_i, i = 0,1,2,\ldots\ldots,k$ are finite complex numbers and k is a positive integer.

Recently Jiang-Tao Li and Ping Li [4] proved the following theorem:

Theorem C. [4] Let f and g be two non-constant entire functions. Suppose that $f$and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 CM and $\delta(0,f) > \frac{1}{2}$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f)P(g) \equiv 1$.

Theorem D. [4]. Let f and g be two non-constant entire functions. Suppose that $f$and g share the value 0 CM, $P(f)$ and $P(g)$ share the value 1IMand $\delta(0,f) > \frac{4}{5}$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f)P(g) \equiv 1$.

In this paper we prove the following theorem which extend and generalise Theorem C and Theorem D.

Theorem 1.1. Let f and g be two non-constant entire functions. Suppose that $f$and g share the value 0 CM, $P(f)$ and $P(g)$share $(1,l)$, where $l = 0,1,2,\ldots\ldots,\infty$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f)P(g) \equiv 1$if one of the following holds:

(i)$l = 0$ and $\delta(0,f) > \frac{4}{5}$

(ii) $0 < l < \infty$ and $\delta(0,f) > \frac{l+1}{2l+1}$

(iii) $l = \infty$ and $\delta(0,f) > \frac{1}{2}$

2. Lemmas

Lemma 2.1. [5] Let f be a non-constant meromorphic function and let $k$be a nonnegative integer. Then

$$T(r,P(f)) \leq T(r,f) + k\,\overline{N}(r,\infty;f) + S(r,f) \quad (2.1)$$

Lemma 2.2. [4] Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and $a(z)$ is a small function of $f(z)$. If $f(z)$ is not a polynomial, then

$$N(r,0;P(f)-P(a)) \leq T(r,P(f)) - T(r,f) + N(r,a;f) + S(r,f)$$
(2.2)

and

$$N(r,0;P(f)-P(a)) \leq N(r,a;f) + k\overline{N}(r,\infty;f) + S(r,f) \quad (2.3)$$

Lemma 2.3. [4] Let f and g be two non-constant meromorphic functions such that $f$and g share the value 1 IM. Let

$$h = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}$$

If $h \not\equiv 0$, then

$$T(r;f) \leq N(r,0;f) + 2\,\overline{N}(r,\infty;f) + 2\,\overline{N}_{(1}^{L}(r,1;f) + N(r,0;g) + 2\,\overline{N}(r,\infty;g) + \overline{N}_{(1}^{L}(r,1;g) + S(r,f) + S(r,g).$$

Lemma 2.4. Let f and g be two non-constant meromorphic functions such that f and g share $(1,l)$ where l is a positive integer. Let

$$h = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1} \quad (2.4)$$

If $h \not\equiv 0$, then

$$T(r,f) \leq N(r,0;f) + 2\,\overline{N}(r,\infty;f) + \overline{N}_{(l+1}^{L}(r,1;f) + N(r,0;g) + 2\,\overline{N}(r,\infty;g) + \overline{N}_{(l+1}^{L}(r,1;g) + S(r,f) + S(r,g) \quad (2.5)$$

Proof. We get from Nevanlinna's second fundamental theorem

$$T(r,f) + T(r,g) \leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,f) + S(r,g). \quad (2.6)$$

Since f and g share $(1,l)$, we get

$$\overline{N}(r,1;f) = N_{l)}^{E}(r,1;f) + \overline{N}_{(l+1}^{L}(r,1;f) + \overline{N}_{(l+1}^{L}(r,1;g) + \overline{N}_{(l+1}^{E}(r,1;f) + S(r,f) + S(r,g) = \overline{N}(r,1;g) + S(r,f) + S(r,g).$$

Hence

$$\begin{aligned}\overline{N}(r,1;f) + \overline{N}(r,1;g) &= \overline{N}_{1)}(r,1;f) + \overline{N}_{(2}(r,1;f) + \overline{N}(r,1;g) + S(r,f) + S(r,g)\\ &\leq \overline{N}_{1)}(r,1;f) + \overline{N}_{(2}(r,1;g) + \overline{N}(r,1;g) + S(r,f) + S(r,g)\\ &\leq \overline{N}_{1)}(r,1;f) + N(r,1;g) + S(r,f) + S(r,g)\\ &\leq \overline{N}_{1)}(r,1;f) + T(r,g) + S(r,f) + S(r,g) \quad (2.7)\end{aligned}$$

From (2.6) and (2.7) we get

$$\begin{aligned}T(r,f) + T(r,g) &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,f) + S(r,g)\\ &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + \overline{N}_{1)}(r,1;f) + T(r,g) + S(r,f) + S(r,g)\end{aligned}$$

Therefore

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g')$$

$+\overline{N}_{1)}(r,1;\ f)+S(r,f)+S(r,g)$ (2.8)

From (2.4) it can be easily calculated that the possible poles of h occur at (i) multiple zeros of f and g, (ii) those 1 points of f and g whose multiplicities are different (iii) those poles of f and g whose multiplicities are different, (iv) zeros $f'(g')$ which arenot the zeros of $f(f-1)(g(g-1))$.

Since h has only simple poles, we have

$$N(r,\infty;\ h)\le \overline{N}(r,\infty;\ f)+\overline{N}(r,\infty;\ g)+\overline{N}^{L}_{(l+1}(r,1;\ f)+\overline{N}^{L}_{(l+1}(r,1;\ g)+\overline{N}_{(2}(r,0;\ f)+\overline{N}_{(2}(r,0;\ g)+N_0(r,0;\ f')+N_0(r,0;\ g')+S(r,f)+S(r,g).\ (2.9)$$

Let z_0 be a common zero of $f-1$ and $g-1$ then by (2.4), we have z_0 is a zero of h.
Also from (2.4), we have
$m(r,h)=S(r,f)+S(r,g)$.

Hence by (2.9) we have

$$\overline{N}_{1)}(r,1;\ f)\le N(r,0;\ h)\le T(r,h)+O(1)\le N(r,\infty;\ h)+S(r,f)+S(r,g)\le \overline{N}(r,\infty;\ f)+\overline{N}(r,\infty;\ g)+\overline{N}^{L}_{(l+1}(r,1;\ f)+\overline{N}^{L}_{(l+1}(r,1;\ g)+\overline{N}_{(2}(r,0;\ f)+\overline{N}_{(2}(r,0;\ g)+N_0(r,0;\ f')+N_0(r,0;\ g')+S(r,f)+S(r,g).\ (2.10)$$

From (2.8) and (2.10) we have

$$T(r,f)\le \overline{N}(r,0;\ f)+\overline{N}(r,\infty;\ f)-N_0(r,0;\ f')+\overline{N}(r,0;\ g)+\overline{N}(r,\infty;\ g)-N_0(r,0;\ g')+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\ g)+\overline{N}^{L}_{(l+1}(r,1;f)+\overline{N}^{L}_{(l+1}(r,1;\ g)+\overline{N}_{(2}(r,0;f)+\overline{N}_{(2}(r,0;g)+N_0(r,0;f')+N_0(r,0;\ g')+S(r,f)+S(r,g)$$
$$\le \overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+\overline{N}(r,0;\ g)+2\overline{N}(r,\infty;\ g)+\overline{N}^{L}_{(l+1}(r,1;f)+\overline{N}^{L}_{(l+1}(r,1;\ g)+\overline{N}_{(2}(r,0;\ f)+\overline{N}_{(2}(r,0;\ g)+S(r,f)+S(r,g)$$
$$\le N(r,0;f)+2\overline{N}(r,\infty;f)+N(r,0;\ g)+2\overline{N}(r,\infty;\ g)+\overline{N}^{L}_{(l+1}(r,1;f)+\overline{N}^{L}_{(l+1}(r,1;\ g)+S(r,f)+S(r,g)$$

This completes the proof.

Lemma 2.5. [8] Let $f_j(j=1,2,\ldots,m+1)$ and $g_j(j=1,2,\ldots,m)$ are entire functions satisfying the following conditions:
(i)$\sum_{j=1}^{m} f_j(z)e^{g_j(z)}\equiv f_{m+1}(z)$;
(ii) The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1\le j\le m+1$, $1\le k\le m$ andthe order of $f_j(z)$ is less than the order of $e^{g_l(z)-g_k(z)}$ for $m\ge 2$ and $1\le j\le m+1, 1\le l,k\le m, l\ne k$.
Then $f_j\equiv 0(j=1,2,\ldots,m+1)$.

Lemma 2.6. Let f and g be two non-constant entire functions. Suppose that f and $g$share the value 0 CM, $P(f)$ and $P(g)$ share the value 1IM. If $\lambda(f)\neq 1$and $P(f)\equiv P(g)$, then $f\equiv g$.

Proof. It can be proved easily with the help of the proof of the Theorem D.

3. Proof of the Theorem 1.1

Proof. If $l=0,\infty$ then by Theorem D and Theorem C we get the result.
Now suppose that $0<l<\infty$. Then we have to consider the following two cases:

Case 1 Any one or both of $P(f)$and $P(g)$ is constant. Suppose that $P(f)=c$; where c is a finite complex constant.Then

$$f\equiv d+\sum_{i=1}^{m} p_i(z)e^{\alpha_i z},$$

whered is a finite complex constant, $p_i(i=1,2,\ldots,m)$ are polynomials and $\alpha_i(i=1,2,\ldots,m)$are distinct finite complex constants. Since $\lambda(f)\neq 1$therefore $f\equiv d+\sum_{i=1}^{m}p_i(z)$, that is f is apolynomial.Suppose the degree of $f$is n. Then
$N(r,0;f)=n\log r$and$T(r,f)=n\log r+O(1)$.

Therefore

$$\delta(0,f)=1-\limsup_{r\to\infty}\frac{N(r,0;\ f)}{T(r,f)}=0,$$

which is a contradiction.

Case 2$P(f),P(g)\not\equiv c$ where c is a finite complex constant.
Let $F=P(f)$, $G=P(g)$ and $H=\frac{F''}{F'}-\frac{2F'}{F-1}-\frac{G''}{G'}+\frac{2G'}{G-1}$then F and G share $(1,l)$.

By Milloux's basic result we have

$$T(r,f)\le N(r,0;f)+\overline{N}(r,\infty;\ f)+\overline{N}(r,1;\ P(f))+S(r,f)\le N(r,0;\ g)+\overline{N}(r,1;\ P(g))+S(r,f)\le T(r,g)+T(r,P(g))+S(r,f).$$

Hence by Lemma 2.1 we have
$T(r,f)\le 2T(r,g)+S(r,f)+S(r,g)$ (3.1)

Similarly we have
$T(r,g)\le 2T(r,f)+S(r,f)+S(r,g)$ (3.2)

From (3.1) and (3.2) we have
$S(r,f)=S(r,g)$ (3.3)

Now we have to consider the following two subcases:

Subcase1$H\equiv 0$.
In this case we have
$F=\frac{PG+Q}{RG+S}$, (3.4)
whereP,Q,R and S are finite complex constants such that $PS-QR\neq 0$:
Now we have to consider the following three cases:

Case I $P = 0$ and $R \neq 0$.

In this case $F = \frac{Q}{RG+S}$

If $S \neq 0$; then

$N\left(r, -\frac{S}{R}; G\right) = N(r, \infty; F)$.

By Nevanlinna's Second Fundamental Theorem we have

$$T(r,G) \leq N(r,0;G) + N\left(r, -\frac{S}{R}; G\right) + N(r,\infty;G) + S(r,G) = N(r,0;G) + N(r,\infty;F) + N(r,\infty;G) + S(r,G) = N(r,0;G) + S(r,G) \quad (3.5)$$

By Lemma2.2 from (3.5) and (3.3) we have

$$T(r,P(g)) \leq T(r,P(g)) - T(r,g) + N(r,0;g) + S(r,f)$$

Hence

$$T(r,g) \leq N(r,0;g) + S(r,f) = N(r,0;f) + S(r,f) \quad (3.6)$$

By (3.1) and (3.6) we have

$T(r,f) \leq 2N(r,0;f) + S(r,f)$,

which contradicts the condition $\delta(0,f) > \frac{l+1}{2l+1}$.

Hence $S = 0$ and so $FG = \frac{Q}{R}$. If 1 is a Picard exceptional value of F then $\frac{Q}{R} = 1$ otherwise it contradicts the Deficiency Theorem [9]. So $FG \equiv 1$.

If 1 is not a Picard exceptional value of F, then there exist a complex number z_0 such that $F(z_0) = G(z_0) = 1$. So $\frac{Q}{R} = 1$.

Hence $FG \equiv 1$.

Case II $P \neq 0$ and $R = 0$

In this case we have $F = \frac{P}{S}G + \frac{Q}{S}$.

If $Q \neq 0$, then

$N\left(r, \frac{Q}{S}; F\right) = N(r,0:G)$.

Hence by Nevanlinna's Second Fundamental Theorem we have

$$T(r,F) \leq N(r,0;F) + N\left(r, \frac{Q}{S}; F\right) + N(r,\infty;F) + S(r,F) = N(r,0;F) + N(r,0;G) + S(r,F) \quad (3.7)$$

By Lemma 2.2 from (3.7) and (3.3) we have

$$T(r,P(f)) \leq T(r,P(f)) - T(r,f) + N(r,0;f) + N(r,0;g) + S(r,f)$$

Hence

$$T(r,f) \leq N(r,0;f) + N(r,0;g) + S(r,f) = 2N(r,0;f) + S(r,f),$$

which contradicts the condition $\delta(0,f) > \frac{l+1}{2l+1}$.

Hence $Q = 0$ and so $F = \frac{P}{S}G$.

If 1 is a Picard exceptional value of F then $\frac{P}{S} = 1$ otherwise it contradicts the Deficiency Theorem [9]. So $F \equiv G$.

If 1 is not a Picard exceptional value of F, then there exist a complex number z_0 such that $F(z_0) = G(z_0) = 1$.

Hence $F \equiv G$.

By Lemma 2.6 we have $f \equiv g$.

Case III $PR \neq 0$

From (3.4) it is clear that $\frac{P}{R}$ is a Picard exceptional value of F. Hence by Nevanlinna's Second Fundamental Theorem we have

$$T(r,F) \leq N(r,0;F) + N\left(r, \frac{P}{R}; F\right) + N(r,\infty;F) + S(r,F) = N(r,0;F) + S(r,F) \quad (3.8)$$

By Lemma 2.2 and (3.8) we have

$T(r,f) \leq N(r,0;f) + S(r,f)$, which contradicts the condition $\delta(0,f) > \frac{l+1}{2l+1}$.

Subcase 2 $H \not\equiv 0$.

By Lemma 2.4 we have

$$T(r,F) \leq N(r,0;F) + \overline{N}_{(l+1}^{L}(r,1;F) + N(r,0;G) + \overline{N}_{(l+1}^{L}(r,1;G) + S(r,F) + S(r,G) \quad (3.9)$$

Hence by (2.3) we have

$$\overline{N}_{(l+1}^{L}(r,1;F) + \overline{N}_{(l+1}^{L}(r,1;G) \leq \frac{1}{l}N(r,0;F') \leq \frac{1}{l}N(r,0;F) + \frac{1}{l}\overline{N}(r,\infty;F) + S(r,F) \quad (3.10)$$

From (3.9) and (3.10) we have

$$T(r,F) \leq \frac{l+1}{l}N(r,0;F) + N(r,0;G) + S(r,F) + S(r,G) \quad (3.11)$$

Hence by Lemma 2.2 and (3.11) we have

$$T(r,P(f)) \leq T(r,P(f)) - T(r,f) + N(r,0;f) + \frac{1}{l}N(r,0;f) + N(r,0;g) + S(r,f) + S(r,g)$$

Therefore

$$T(r,f) \leq N(r,0;f) + \frac{1}{l}N(r,0;f) + N(r,0;g) + S(r,f) + S(r,g) \leq \frac{2l+1}{l}N(r,0:f) + S(r,f),$$

which contradicts the condition $\delta(0,f) > \frac{l+1}{2l+1}$.

This completes the proof.

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