On The Uniqueness Problems of Entire Functions Concerning Differential Polynomials

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Abstract: We discuss the uniqueness of entire functions concerning differential polynomials. The results extend and generalize recent results of Jiang-Tao Li and Ping Li.

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1. Introduction

Let $f$ and $g$ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that $f$ and $g$ share the value $a$CM (counting multiplicities) if the $a$-points of $f$ and $g$ coincide in locations and multiplicities. If we do not consider the multiplicities, we say that $f$ and $g$ share the value $a$IM (ignoring multiplicities). For standard definitions and notations of the value distribution theory we refer to [1].

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ is called small function of $f$ if $T(r, a) = S(r, f)$.

In [2] and [3] the idea of weighted sharing is introduced which measures how close a shared value is to being shared IM or to being shared CM. We now explain the idea of weighted sharing of values.

**Definition 1.1.** [2, 3] Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C}$ we denote by $E_k(a, f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$ we say that $f$, $g$ share the value $a$ with weight $k$.

The definition implies that if $f, g$ share a value $a$ with weight $k$ then $z_0$, a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and $z_0$ is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where $m$ is not necessarily equal to $n$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$ then $f, g$ share $(a, p)$ for all integers $p$, $0 \leq p < k$. Also we note that $f, g$ share a value $a$ CM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$ respectively.

**Definition 1.2.** Let $f$ be a non-constant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer $k$ we denote by $N_k(a; f)$ the counting function of those $a$-points of $f$ whose multiplicities are less than or equal to $k$ greater than or equal to $k$, where an $a$-point is counted according to its multiplicity.

Let $N_k(a; f)$ and $N_k(r, a; f)$ denote the corresponding reduced counting functions. We put

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}$$

and

$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}.$$

$\lambda(f)$ and $\delta(a, f)$ are called the order of $f(z)$ and the deficiency of $a$ with respect to $f(z)$ respectively.

**Definition 1.3.** Let $f$ and $g$ be two non-constant meromorphic functions for any positive integer $k$ we denote by $N_k(r, 1; f)$ the counting function of zeros of $f - 1$ and $g - 1$ whose multiplicities are not less than $k$ and about which $f - 1$ has larger multiplicity than $g - 1$. By $N_k^2(r, 1; f)(N_k^2(r, 1; f))$ we denote the counting function of zeros off $- 1$ and $g - 1$ whose multiplicities are not less (greater) than $k$ and about which $f - 1$ and $g - 1$ have equal multiplicities. By $N_k^2(r, 1; f), N_k^2(r, 1; f)$ and $N_k^2(r, 1; f)$ we denote the corresponding reduced counting functions.

**Definition 1.4.** Let $f$ be a non-constant meromorphic function we denote by $N_k(r, 0; f)$ the counting function of zeros of $f^k$ which are not zeros of $f^k - 1$.

In 1976 C. C. Yang [7] asked: If two transcendental entire functions $f$ and $g$ assume the same zeros with the same multiplicities and that their first derivatives assume the same $1$-points with the same multiplicities then what can be said about the relationship between $f$ and $g$?

To solve the above question in 1981 K. Shibazaki [6] proved the following:

**Theorem A.** [6] Let $f$ and $g$ be two entire functions of finite order. If $f^p$ and $g^p$ share the value $1$ CM with $\delta(0, f) > 0$ and $0$ being lacunary for $g$ then either $f \equiv g$ or $f g^p \equiv 1$.

In 1990 H. X. Yi [10] proved the following theorem:
Theorem B. [10] Let \( f \) and \( g \) be two non-constant entire functions and let \( k \) be a non-negative integer. If \( f \) and \( g \) share the value \( 0 \) CM, \( f^{(k)} \) and \( g^{(k)} \) share the value \( 1 \) CM and \( \delta(0, f) > \frac{\lambda}{2} \), then either \( f \equiv g \) or \( f^{(k)}g^{(k)} \equiv 1 \).

Let \( f \) be a non-constant meromorphic function. We denote by \( P(f) = \sum_{i=0}^{k} a_i f^{(i)} \) the differential polynomial of \( f \), where \( a_i, i = 0, 1, 2, \ldots, k \) are finite complex numbers and \( k \) is a positive integer.

Recently Jiang-Tao Li and Ping Li [4] proved the following theorem:

Theorem C. [4] Let \( f \) and \( g \) be two non-constant entire functions. Suppose that \( f \) and \( g \) share the value \( 0 \) CM, \( P(f) \) and \( P(g) \) share the value \( 1 \) CM and \( \delta(0, f) > \frac{\lambda}{2} \). If \( \lambda \neq 1 \), then \( f \equiv g \) unless \( P(f)P(g) \equiv 1 \).

Theorem D. [4]. Let \( f \) and \( g \) be two non-constant entire functions. Suppose that \( f \) and \( g \) share the value \( 0 \) CM, \( P(f) \) and \( P(g) \) share the value \( \lambda \) IM and \( \delta(0, f) > \frac{\lambda}{2} \). If \( \lambda \neq 1 \), then \( f \equiv g \) unless \( P(f)P(g) \equiv 1 \).

In this paper we prove the following theorem which extend and generalise Theorem C and Theorem D.

Theorem 1.1. Let \( f \) and \( g \) be two non-constant entire functions. Suppose that \( f \) and \( g \) share the value \( 0 \) CM, \( P(f) \) and \( P(g) \) share the value \( \lambda \) IM and \( \delta(0, f) > \frac{\lambda}{2} \). If \( \lambda \neq 1 \), then \( f \equiv g \) unless \( P(f)P(g) \equiv 1 \).

2. Lemmas

Lemma 2.1. [5] Let \( f \) be a non-constant meromorphic function and let \( k \) be a negative integer. Then

\[
T(r, P(f)) \leq T(r, f) + kN(r, \infty; f) + S(r, f) \quad (2.1)
\]

Lemma 2.2. [4] Suppose that \( f(z) \) is a non-constant meromorphic function in the complex plane and \( a(z) \) is a small function of \( f(z) \). If \( f(z) \) is not a polynomial, then

\[
N(r, 0; P(f) - P(a)) \leq T(r, P(f)) - T(r, f) + N(r, a; f) + S(r, f) \quad (2.2)
\]

and

\[
N(r, 0; P(f) - P(a)) \leq N(r, a; f) + kN(r, \infty; f) + S(r, f) \quad (2.3)
\]

Lemma 2.3. [4] Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share the value \( \lambda \) IM. Let \( h = \frac{f''}{f} - \frac{2f'}{f-1} - \frac{g''}{g} - \frac{2g'}{g-1} \). If \( h \equiv 0 \), then

\[
T(r; f) \leq N(r, 0; f) + 2N(r, \infty; f) + 2N_1(r, 1; f) + N(r, 0; g) + 2N(r, \infty; g) + N_1(r, 1; g) + S(r, f) + S(r, g)
\]

Lemma 2.4. Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share \( (1, l) \) where \( l \) is a positive integer. Let

\[
h = \frac{f''}{f} - \frac{2f'}{f-1} - \frac{g''}{g} + \frac{2g'}{g-1} \quad (2.4)
\]

If \( h \equiv 0 \), then

\[
T(r, f) \leq N(r, 0; f) + 2N(r, \infty; f)
\]

\[
+ \frac{N_1}{2}(r, 1; f) + N(r, 0; g)
\]

\[
+ 2N(r, \infty; g) + \frac{N_1}{4}(r, 1; g)
\]

\[
+ S(r, f) + S(r, g) \quad (2.5)
\]

Proof. We get from Nevanlinna’s second fundamental theorem

\[
T(r, f) + T(r, g) \leq N(r, 0; f) + N(r, 1; f)
\]

\[
+ N(r, \infty; f) + N(r, 1; g)
\]

\[
+ S(r, f) + S(r, g)
\]

Since \( f \) and \( g \) share \((1, l)\), we get

\[
N(r, 1; f) = N_1^2(r, 1; f) + N_2^2(r, 1; f) + N(r, 1; g)
\]

\[
+ S(r, f) + S(r, g)
\]

Hence

\[
N(r, 1; f) + N(r, 1; g) = N_1^2(r, 1; f) + N_2^2(r, 1; f) + N(r, 1; g)
\]

\[
+ S(r, f) + S(r, g)
\]

\[
\leq N_1(r, 1; f) + N_2(r, 1; g) + N(r, 1; g)
\]

\[
+ S(r, f) + S(r, g)
\]

\[
\leq N_1(r, 1; f) + T(r, g)
\]

\[
+ S(r, f) + S(r, g) \quad (2.7)
\]

From (2.6) and (2.7) we get

\[
T(r, f) + T(r, g) \leq N(r, 0; f) + N(r, 1; f)
\]

\[
+ N(r, \infty; f) + N(r, 1; g)
\]

\[
+ S(r, f) + S(r, g)
\]

\[
\leq N(r, 0; f) + N(r, \infty; f)
\]

\[
- N_0(r, 0; f') + N(r, 0; g')
\]

\[
+ N(r, \infty; g) - N_0(r, 0; g')
\]

\[
+ N_1(r, 1; f) + T(r, g)
\]

\[
+ S(r, f) + S(r, g)
\]

Therefore

\[
T(r, f) \leq N(r, 0; f) + N(r, \infty; f) - N_0(r, 0; f')
\]

\[
+ N(r, 0; g) + N(r, \infty; g) - N_0(r, 0; g')
\]
From (2.4) it can be easily calculated that the possible poles of $h$ occur at (i) multiple zeros of $f$ and $g$, (ii) those points of $f$ and $g$ whose multiplicities are different (iii) those poles of $f$ and $g$ whose multiplicities are different, (iv) zeros $f'(g')$ which are not the zeros of $f(f - 1)(g(g - 1))$.

Since $h$ has only simple poles, we have

$$N(r, \infty; h) \leq N(r, \infty; f) + N(r, \infty; g) + S(r, f) + S(r, g).$$

Let $z_0$ be a common zero of $f - 1$ and $g - 1$ then by (2.4), we have $z_0$ a zero of $h$.

Also from (2.4), we have $m(r, h) = S(r, f) + S(r, g)$.

Hence by (2.9) we have

$$N_0(r, 0; f) + N_0(r, 0; g) \leq T(r, h) + O(1).$$

From (2.8) and (2.10) we have

$$T(r, f) \leq N(r, 0; f) + N(r, 0; g) - N_0(r, 0; f') + S(r, f) + S(r, g).$$

Lemma 2.6. Let $f$ and $g$ be two non-constant entire functions. Suppose that $f$ and $g$ share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 CM. If $\lambda(f) \neq \lambda(g)$, then $f \equiv g$.

Proof. It can be proved easily with the help of the proof of the Theorem D.

3. Proof of the Theorem 1.1

Proof. If $l = 0, \infty$ then by Theorem D and Theorem C we get the result.

Now suppose that $0 < l < \infty$. Then we have to consider the following two cases:

**Case 1** Any one or both of $P(f)$ and $P(g)$ is constant. Suppose that $P(f) = c$, where $c$ is a finite complex constant. Then

$$f \equiv d + \sum_{i=1}^{m} p_i(z) e^{\alpha_i z},$$

where $d$ is a finite complex constant, $p_i(z) (i = 1, 2, ..., m)$ are polynomials and $\alpha_i (i = 1, 2, ..., m)$ are distinct finite complex constants. Since $\lambda(f) \neq \lambda(g)$ therefore $f \equiv d + \sum_{i=1}^{m} p_i(z)$, that is $f$ is a polynomial. Suppose the degree of $f$ is $m$. Then

$$N(r, 0; f) = n \log r + \mathcal{O} \left( \frac{r}{\log^2 r} \right).$$

Hence by Lemma 2.1, we have

$$S(r, f) = S(r, g) = 0,$$

which is a contradiction.

**Case 2** $P(f)$ and $P(g)$ are two non-constant entire functions. Suppose that $P(f) = c$, where $c$ is a finite complex constant. Let $F = P(f)$ and $G = P(g)$ and

$$H = \frac{F}{P \quad - \quad \frac{P}{R} \quad - \quad \frac{G}{R} \quad + \quad \frac{12}{C} \quad - \quad \frac{6}{C} \quad - \quad \frac{6}{C} \quad - \quad \frac{6}{C},$$

then $F$ and $G$ share (1, 1). By Milloux’s basic result we have

$$T(r, f) \leq N(r, 0; f) + N(r, 0; g) + S(r, f) + S(r, g).$$

Hence by Lemma 2.1, we have

$$T(r, f) \leq 2T(r, g) + S(r, f) + S(r, g).$$

Similarly we have

$$T(r, g) \leq 2T(r, f) + S(r, f) + S(r, g).$$

From (3.1) and (3.2) we have

$$S(r, f) = S(r, g).$$

Now we have to consider the following two subcases:

**Subcase 1**

$$\sum_{i=1}^{m} f_i(z) e^{\theta_i(z)} \equiv f_{m+1}(z).$$

(i) The order of $f_j(z)$ is less than the order of $e^{\theta_j(z)}$ for $1 \leq j \leq m + 1, 1 \leq k \leq m$ and the order of $f_j(z)$ is less than the order of $e^{\theta_j(z)}$ for $m + 2 \leq n \leq 2$.

Then $f_j \equiv 0 (j = 1, 2, ..., m + 1)$.

Lemma 2.5. [8] Let $f_j (j = 1, 2, ..., m + 1)$ and $g_j (j = 1, 2, ..., m)$ are entire functions satisfying the following conditions:

(i) $\sum_{j=1}^{m} f_j(z) e^{\theta_j(z)} \equiv f_{m+1}(z)$.

(ii) The order of $f_j(z)$ is less than the order of $e^{\theta_j(z)}$ for $1 \leq j \leq m + 1, 1 \leq k \leq m$ and the order of $f_j(z)$ is less than the order of $e^{\theta_j(z)}$ for $m + 2 \leq n \leq 2$.

Then $f_j \equiv 0 (j = 1, 2, ..., m + 1)$. 

**Subcase 1**

$$f \equiv g \quad (3.4)$$

where $P, Q, R$ and $S$ are finite complex constants such that $PS - QR \neq 0$.

Now we have to consider the following three cases:

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Case IP = 0 and R ≠ 0.
In this case $F = \frac{P}{G}$.
If $S = 0$, then
$N\left(r, -\frac{S}{R}; G\right) = N(r, \infty; F)$.
By Nevanlinna’s Second Fundamental Theorem we have
$T(r, G) \leq N(r, 0; G) + N\left(r, -\frac{S}{R}; G\right)$
+ $N(r, \infty; G) + S(r, f)$
= $N(r, 0; G) + N(r, \infty; F) + S(r, f)$
+ $N(r, 0; G) + S(r, G)$
= $N(r, 0; G) + S(r, G)$.

By Lemma 2.2 from (3.5) and (3.3) we have
$T(r, P(g)) \leq T(r, P(g)) - T(r, g)$
+ $N(r, 0; g) + S(r, f)$
Hence
$T(r, g) \leq N(r, 0; g) + S(r, f)$
= $N(r, 0; f) + S(r, f)$.

By (3.1) and (3.6) we have
$T(r, f) \leq 2N(r, 0; f) + S(r, f)$.
which contradicts the condition $\delta(0, f) > \frac{1+1}{2l+1}$.
Hence $S = 0$ and so $F = \frac{P}{G}$.
If 1 is a Picard exceptional value of $F$, then there exist a complex number $z_0$ such that $F(z_0) = G(z_0) = 1$. So $\frac{P}{G} = 1$.
Hence $F \equiv G$.

Case III PR ≠ 0.
From (3.4) it is clear that $p$ is a Picard exceptional value of $F$.
Hence by Nevanlinna’s Second Fundamental Theorem we have
$T(r, F) \leq N(r, 0; F) + N\left(r, \frac{P}{R}; F\right)$
+ $N\left(r, \infty; F\right) + S(r, F)$
= $N(r, 0; F) + S(r, F)$.

By Lemma 2.2 and (3.8) we have
$T(r, f) \leq N(r, 0; f) + S(r, f)$, which contradicts the condition $\delta(0, f) > \frac{1+1}{2l+1}$.

Subcase 2H \( \equiv 0 \).
By Lemma 2.4 we have
$T(r, F) \leq N(r, 0; F) + N\left(r, \frac{G}{P}; F\right)$
+ $N\left(r, \infty; F\right) + S(r, F)$.
Hence by (2.3) we have
$N\left(r, \frac{G}{P}; F\right) \leq N(r, 0; F)$.

From (3.9) and (3.10) we have
$T(r, F) \leq \frac{l+1}{l} N(r, 0; F) + N(r, 0; G) + S(r, F)$.

Hence by Lemma 2.2 and (3.11) we have
$T(r, F) \leq T(r, P(f)) - T(r, f) + N(r, 0; f)$
+ $\frac{1}{l} N(r, 0; f) + N(r, 0; g) + S(r, f) + S(r, g)$.
Thus
$T(r, f) \leq N(r, 0; f) + \frac{1}{l} N(r, 0; f) + N(r, 0; g) + S(r, f) + S(r, g)$.

Therefore
$T(r, f) \leq \frac{l+1}{l} N(r, 0; f) + N(r, 0; g) + S(r, f) + S(r, g)$.
which contradicts the condition $\delta(0, f) > \frac{1+1}{2l+1}$.

This completes the proof.

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References

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