Lattice Points on the Homogeneous Cone $x^2 + y^2 = 26z^2$

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Abstract: The ternary quadratic homogeneous equation representing cone $x^2 + y^2 = 26z^2$ is considered and analyzed for finding its non zero distinct integral solutions. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns are presented.

Keywords: Homogeneous cone, Ternary Quadratic, Integral solutions, special numbers.

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Notations Used

T$_{m,n}$ - Polygonal number of rank n with size m - P$_m^n$ - Pyramidal number of rank n with size m - g -Gnomonic number of rank n- Pr$_n$ - Pronic number of rank n - Ct$_{16n}$ - Centered hexadecagonal pyramidal number of rank n- OH$_n$ - Octahedral number of rank n- SO$_n$ - Stella octagonal number of rank n- SO$_n$ - carol number - P$_n$ - Polygonal number of rank n with size m - C$_n$-Star number- w$_n$ - woodall number- S$_n$-Star number- C$_n$- Cullen number.

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. This communication concerns with another interesting ternary quadratic equation representing $x^2 + y^2 = 26z^2$ a cone for determining its infinitely many non-zero integral points. Also a few interesting properties among the solutions are presented.

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solution is

$$x^2 + y^2 = 26z^2$$

(1)

Assume $z (a, b) = a^2 + b^2$, where $a, b > 0$

(2)

We illustrate below five different patterns of non-zero distinct integer solutions to (1)

2.1 Pattern: I

Write 26 = $(5 + i) (5 - i)$

Substituting (2) & (3) in (1), and applying the method of factorization, define

$$x + iy (x - iy) = (5 + i) (5 - i) (a + ib)^2 (a - ib)^2$$

Equating real and imaginary parts, we get

x = x (a, b) = $5a^2 - 5b^2 - 2ab$

y = y (a, b) = $a^2 - b^2 + 10ab$

Properties

1. x (A, A) + 2T$_{4A}$ = 0
2. y (A, A + 1) – Ct$_{16A}$ – 2T$_{4A}$ = 0 (mod 2)

2.2 Pattern: II

Instead of (3), Write 26 as

26 = $(1 + 5i) (1 - 5i)$

(6)

Following the procedure presented as in pattern : 1, the corresponding values of x and y are

x = x (a, b) = $a^2 - b^2 - 10ab$

y = y (a, b) = $5a^2 - 5b^2 + 2ab$

Properties:-

1. 5x (A, A) – y (A, A) + 52T$_{4A}$ = 0
2. y (1, 1) – P$^2_2$ $\equiv$ 0 (mod 2)
4. y (1, 1) – x(1,1) – OH$_1$ = Caroll number.
5. y (A, 1) – 5x(A, 1) – 5z(A, 1) + 5T$_{4A}$ - G$_{6A}$ + P$^2_a$ = 0 (mod 2)
6. y (2A$^2$ – 1, A) – 5x(2A$^2$ – 1, A) –52SO$_n$ = 0
7. x (A$^2$,A+1)–PT$_{4A}$+22T$_{4A}$ + 24OH$_n$ = caroll number
8. x (A, A) + 8y (A, A) is a nasty number
9. x (1, 2A + 1) + Ct$_{16A}$-S$_4$ + 20T$_{4A}$ – 8T$_{4A}$ = 0
10. 6x (A, A) is a nasty number

2.3 Pattern: III

Equation (1) is written in the form of ratio as

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\[
\frac{x + z}{5z + y} = \frac{5z - y}{x - z} = \frac{A}{B} = 0,
\]
which is equivalent to the system of equations

\[
\begin{align*}
B(x + z) - A(5z + y) &= 0 \quad (9) \\
B(5z - y) - A(x - z) &= 0 \quad (10)
\end{align*}
\]

Applying the method of cross multiplication the integer solutions of (1) are given by

\[
\begin{align*}
x(a, b) &= A^2 - B^2 + 10AB \quad (11) \\
y(a, b) &= A^2 - 5B^2 - AB \quad (12) \\
z(a, b) &= A^2 + B^2 \quad (13)
\end{align*}
\]

Properties:
1. \(x[A(2A^2 - 1)] - 5z[A(2A^2 - 1)] - 10SO_{A^2} - P_t^2 = 0\)
2. \(y(2A, 2A) - 5x(2A, 2A) + 220T_{4, A} = 0\)
3. \(x(A, A) + 10(A, A) - 11T_{4, A} = 0\)
4. \(z(1, 1) - W = 0 \quad (mod 5)\)
5. \(x(2A, 1) - y(2A, 1) - 20T_{4, A} = 0\)
6. \(x(A, A + 1) + z(A, A + 1) - 6T_{4, A^2} + 2T_{4, A} = 0\)
7. \(y[A(A + 1), 1] - z[A(A + 1), 1] - 22T_{4, A} = 0 \quad (mod 2)\)
8. \(y(2A - 1, A) - x(2A - 1, A) + 4T_{4, A} + 11T_{4, A} = 0\)
9. \(3x(A, A) + 2y(A, A) + 2z(A, A) = 0\)
10. \(3z(A, A) = 0\)

2.4 Pattern: IV

Equation (1) is written as

\[
26z^2 - y^2 = x^2 + 1
\]

Assume \(x(a, b) = 26a^2 - b^2\)

Write (1) as

\[
(\sqrt{26}z + y)^2 = (\sqrt{26} a + b)^2 (\sqrt{26} - 5)
\]

Equating the rational and irrational factors we get

\[
x = x(A, B) = 26a^2 - b^2
\]

\[
y = y(A, B) = 130a^2 + 5b^2 - 52ab
\]

\[
z = z(A, B) = 26a^2 + b^2 - 10ab
\]

Properties:
1. \(y(A, A) - 5z(A, A) + 102T_{4, A} = 0\)
2. \(z[A(2A^2 - 1)] + 10SO_{A^2} = 0\)
3. \(z(A, A + 1) - x(2A^2 - 1, 1) - 5T_{4, A} + 10P_{A, A} = 0 \quad (mod 2)\)
4. \(y[A(2A^2 - 1)] - 5z[A(2A^2 - 1)] + 5z[A(2A^2 - 1)] = 20T_{4, A} = 0 \quad (mod 2)\)
5. \(y(1, 1) - 5T_{4, A} + 10P_{A, A} = 0 \quad (mod 5)\)
6. \(5x(2A^2 - 1, 1) - y(2A^2 - 1, 1) + 156OH_{A^2} = 0 \quad (mod 2)\)
7. \(x(2A, 2A) = 0\)

8. \(y(A, A) - 5x(A, A) = \text{a cubic integer} \quad (21)\)

9. \(y(1, 1) - 5x(1, 1) + 4x - C_{4, A} = 0\)
10. \(6x(1, 1) + z(1, 1) - 4T_{4, A} = 0 \quad (mod 2)\)

2.5 Pattern: V

Assume \(x(a, b) = 26a^2 - b^2\)

Substituting (19) in (1) and applying the method of factorization, define

\[
(\sqrt{26} a + b)^2 = (\sqrt{26} z + y)
\]

Equating rational and irrational, we get

\[
x(a, b) = (26a^2 - b^2) \quad (20)
\]

Properties:
1. \(z[A(A + 1)] - 2P_{A, A} = 0\)
2. \(x[A + 1(A, A + 2)] - 2T_{4, A} - G_{A, A} + P_{4, A} = 0 \quad (mod 2)\)
3. \(y[A + 1(A, A + 2)] - 2T_{4, A} - G_{A, A} = 0 \quad (mod 2)\)
4. \(x[A, 2A^2 - 1] - 2SO_{A^2} = 0\)
5. \(x[A(A + 1) + 2A^2] + 52T_{4, A} = 0\)
6. \(z[A, 4A - 1] - 4T_{4, A} = 0\)
7. \(z[A, 4A - 1] - 2T_{4, A} = 0\)
8. \(12z(1, 1) + 1 = \text{a Nasty number}\)
9. \(x(A, A) - y(A, A) + 2T_{4, A} = 0\)
10. \(z(A, A + 1) - 4T_{4, A} = 0\)

3. Conclusion

In this work, the ternary quadratic Diophantine equations referring a conies are analysed for its non-zero distinct integral points. A few interesting properties between the solutions and special numbers are presented. To conclude, one may search for other patterns of solutions and their corresponding properties for the cone under consideration.

References


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