Lattice Points on the Homogeneous Cone $x^2+y^2=26z^2$

Dr. P. Jayakumar[1], R. Venkatraman[2]

[1]Professor of Mathematics, Periyar Maniammai University Vallam Thanjavur-613403, India

[2]Assistant Professor of Mathematics, SRM University Vadapalani Campus, Chennai – 600026, India

Abstract: ***The ternary quadratic homogeneous equation representing cone $x^2+y^2=26z^2$ is considered and analyzed for finding its non zero distinct integral solutions. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns are presented.***

Keywords: Homogeneous cone, Ternary Quadratic, Integral solutions, special numbers.

2010 Mathematics Subject Classification: 11D09

Notations Used

$T_{m,n}$ - Polygonal number of rank n with size m - P_n^r- Pyramidal number of rank n with size m - g -Gnomonic number of rank n- Pr_n - Pronic number of rank n - $Ct_{16,n}$- Centered hexadecagonal pyramidal number of rank n- OH_n - Octahedral number of rank n- SO_n - Stella octangular number of rank n - ky_n - kynea number - $carl_n$ -carol number- w_n - woodall number- S_n-Star number- C_n-Cullen number.

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. This communication concerns with another interesting ternary quadratic equation representing $x^2 + y^2 = 26z^2$ a cone for determining its infinitely many non-zero integral points. Also a few interesting properties among the solutions are presented.

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solution is

$$x^2 + y^2 = 26z^2 \quad (1)$$

Assume $z(a, b) = a^2 + b^2$, where $a, b > 0$ (2)

We illustrate below five different patterns of non-zero distinct integer solutions to (1)

2.1 Pattern: I

Write $26 = (5 + i)(5 – i)$ (3)

Substituting (2) & (3) in (1), and applying the method of factorization, define

$$(x + iy)(x – iy) = (5 + i)(5 – i)(a + ib)^2 (a – ib)^2$$

Equating real and imaginary parts, we get

$$x = x(a, b) = 5a^2 – 5b^2 – 2ab \quad (4)$$

$$y = y(a, b) = a^2 – b^2 + 10ab \quad (5)$$

Properties

1. $x(A,A) + 2T_{4,A} = 0$
2. $y(A, A + 1) – Ct_{16,A} – 2T_{4,A} \equiv 0 \pmod 2$
3. $z[A(A + 1), A(A + 1)] – 2(P_A)^2 = 0$
4. $x[1, A(2A^2 –1)] –5[1, A(2A^2 –1)] + 52SO_A = 0$
5. $y(1, 1) – PT_3 \equiv 0 \pmod 5$
6. $x[(2A^2 +1), A] –5y[(2A^2 + 1),A] +156OH_A = 0$
7. $y(A, 2A^2 +1) + z(A, 2A^2 +1) –2T_{4,A}- 30OH_A = 0$
8. $x(2A^2 –1, A) –3y(2A^2 –1, A) –2z(2A^2 –1, A) +7T_{4,A}+32SO_A= 0$
9. $x[(A + 1),(A + 2)] + CH_A – 2T_{23,A} + 20T_{4,A} \equiv 0 \pmod 2$
10. $3x(A, A)$ is a nasty number.

2.2 Pattern: II

Instead of (3), Write 26 as

$$26 = (1 + 5i)(1 – 5i) \quad (6)$$

Following the procedure presented as in pattern : 1, the corresponding values of x and y are

$$x = x(a, b) = a^2 – b^2 – 10ab \quad (7)$$

$$y = y(a, b) = 5a^2 – 5b^2 +2ab \quad (8)$$

Properties:-

1. $5x(A, A) – y(A, A) + 52T_{4,A} = 0$
2. $y(1, 1) –P_2^5 \equiv 0 \pmod 2$
3. $y[A,(A + 1)(2A + 1)] – x[A, (A + 1)(2A + 1)] –312P_a^4 = 0$
4. $y(1,1) – x(1,1) – OH_3$ = Carol number.
5. $y(A,1) – 5x(A,1) – 5z(A,1) +5T_{4,A} – G_{6A} +P_2^5 \equiv 0 \pmod 2$
6. $y(2A^2 –1, A) –5x(2A^2 –1, A) –52SO_A= 0$
7. $x(A^2,A+1) –PT_A+22T_{4,A}+ 24OH_A$= carol number
8. $x(A,A) + 8y(A, A)$ is a nasty number
9. $x(1, 2A + 1) + Ct_{16,A}–S_A + 20T_{3,A} – 8T_{4,A}= 0$
10. $6x(A, A)$ is a nasty number

2.3 Pattern: III

Equation (1) is written in the form of ratio as

$$\frac{x+z}{5z+y} = \frac{5z-y}{x-z} = \frac{A}{B}, \quad B \neq 0,$$

which is equivalent to the system of equations

$$B(x+z) - A(5z+y) = 0 \quad (9)$$
$$B(5z-y) - A(x-z) = 0 \quad (10)$$

Applying the method of cross multiplication the integer solutions of (1) are given by

$$x(a, b) = A^2 - B^2 + 10AB \quad (11)$$
$$y(a, b) = A^2 - 5B^2 - AB \quad (12)$$
$$z(a, b) = A^2 + B^2 \quad (13)$$

Properties:

1. $x[1, A(2A^2-1)] + z[1, A(2A^2-1)] - 10SO_A - P_1^5 = 0$
2. $y(2A, 2A) - 5x(2A, 2A) + 220T_{4,A} = 0$
3. $x(A,1) + 10(A, 1) - 11T_{4,A} + O_4 + Ky_1 = 0$
4. $z(1,1) - W_2 \equiv 0 \pmod 5$
5. $x(2A, 1) - y(2A, 1) - 20T_{4,A} - G_{11A} = 0$
6. $y(A, A+1) + z(A, A+1) - 6T_{4,A} + 2T_{3,A} = 0$
7. $y[A(A+1), 1] - z[A(A+1), 1] - 22T_{3,A} \equiv 0 \pmod 2$
8. $y(2A-1, A) - x(2A-1, A) + 4T_{4,A} + 11T_{6,A} = 0$
9. $3x(A, A) + 2y(A,A) + 2z(A, A)$ is a Nasty number.
10. $3z(A, A)$ is a Nasty number.

2.4 Pattern: IV

Equation (1) is written as

$$26z^2 - y^2 = x^2 * 1 \quad (14)$$
$$\text{Assume } x(a, b) = 26a^2 - b^2) \quad (15)$$
$$\text{Write (1) as } 1 = (\sqrt{26}+5)(\sqrt{26}-5) \quad (16)$$

Substituting (15) & (16) in (14) and applying the method of factorization, define

$$(\sqrt{26}\, z + y) = (\sqrt{26}\, a + b)^2 (\sqrt{26} - 5)$$

Equating the rational and irrational factors we get

$$x = x(A, B) = 26a^2 - b^2$$
$$y = y(A, B) = 130a^2 + 5b^2 - 52ab \quad (17)$$
$$z = z(A, B) = 26a^2 + b^2 - 10ab \quad (18)$$

Properties

1. $y(A, A) - 5z(A, A) + 102\,T_{4,A} = 0$
2. $z[1, (A(2A^2-1)] + 10SO_A = 0$
3. $z(A, A+1) - 27T_{4,A} - G_A + 10P_A \equiv 0 \pmod 2$
4. $y[(2A^2-1),1] - 5[(2A^2-1),1] + 52(G_A)^2 - P_3^5 \equiv 0 \pmod 2$
5. $y(1, A) - 5T_{4,A} + G_{26A} - P_6^5 \equiv 0 \pmod 5$
6. $5x(2A^2+1, A) - y(2A^2+1) + 156OH_A \equiv 0 \pmod 2$
7. $x(2A, 2A)$ is a Perfect square
8. $\frac{1}{42}[y(A, A) - 5x(A, A)]$ is a Cubic integer
9. $y(1, 1) - 5x(1,1) + z(1, 1) + C_3 = 0$
10. $6x(1, 1) + z(1, 1) - Carl_4 \equiv 0 \pmod 2$

2.5 Pattern: V

$$\text{Assume } x(a, b) = 26a^2 - b^2) \quad (19)$$

Substituting (19) in (1) and applying the method of factorization, define

$$(\sqrt{26}\, a + b)^2 = (\sqrt{26}\, z + y)$$

Equating rational and irrational, we get

$$x(a, b) = (26a^2 - b^2) \quad (20)$$
$$y(a, b) = 26a^2 + b^2 \quad (21)$$
$$z = z(a, b) = 2ab \quad (22)$$

Properties:

1. $z[A(A+1), A(A+1)] - 2(P_A)^2 = 0$
2. $x[(A+1), (A+2)] - 25T_{4,A} - G_{24A} + P_3^6 \equiv 0 \pmod 2$
3. $y(A, A+1) - 27T_{4,A} - G_A \equiv 0 \pmod 2$
4. $.z(A, 2A^2-1) - 2SO_A = 0$
5. $x[A,(A+1)(2A+1)] + 52T_{4,A} - P_A^4 = 0$
6. $z(A, 2A-1) - 2T_{6,A} = 0$
7. $z(A, 8A-7) - 2T_{18,A} = 0$
8. $12z(1, 1)$ is a Nasty number
9. $x(A, A) - y(A, A) + 2T_{4,A} = 0$
10. $z(A, A+1) - 4T_{3,A} = 0$

3. Conclusion

In this work, the ternary quadratic Diophantine equations referring a conies are analysed for its non-zero distinct integral points. A few interesting properties between the solutions and special numbers are presented. To conclude, one may search for other patterns of solutions and their corresponding properties for the cone under consideration.

References

[1] Dickson, L.E., History of theory of numbers, Vol.11Chelsea publishing company, New-York (1952).
[2] Mordell, L.J., Diophantine equation, Academic press, London (1969) Journal of Science and Research, Vol (3), Issue 12, 20-22 (December -14)
[3] Jayakumar. P, Sangeetha, K "Lattice points on the cone $x^2 + 9y^2 = 50z^2$" International Journal of Science and Research, Vol (3), Issue 12, 20-22 (December -2014)
[4] Jayakumar P, Kanaga Dhurga, C," On Quadratic Diopphantine equation $x^2 + 16y^2 = 20z^2$" Galois J. Maths, 1(1) (2014), 17-23.
[5] Jayakumar. P, Kanaga Dhurga. C, "Lattice points on the cone $x^2 + 9y^2 = 50z^2$" Diophantus J.Math,3(2) (2014), 61-71
[6] Jayakumar. P, Prabha. S "On Ternary Quadratic Diophantine equation $x^2+15y^2=14z^2$" Archimedes J. Math., 4(3) (2014), 159-164.
[7] Jayakumar, P, Meena, J "Integral solutions of the Ternary Quadratic Diophantine equation: $x^2 + 7y^2 = 16z^2$" International Journal of Science and Technology, Vol.4, Issue 4, 1-4, Dec 2014.
[8] Jayakumar. P, Shankarakalidoss, G "Lattice points on Homogenous cone $x^2 + 9y^2 = 50z^2$" International journal of Science and Research, Vol (4), Issue 1, 2053-2055, January -2015.
[9] Jayakumar. P, Shankarakalidoss. G "Integral points on The Homogenous cone $x^2 + y^2 = 10z^2$ International Journal for Scienctific Research and Development, Vol (2), Issue 11, 234-235, January -2015
[10] Jayakumar.P, Prapha.S "Integral points on the cone $x^2 + 25y^2 = 17z^2$" International Journal of Science and Research Vol(4), Issue 1, 2050-2052, January-2015.

[11] Jayakumar.P, Prabha. S, "Lattice points on the cone $x^2 + 9y^2 = 26z^2$ "International Journal of Science and Research Vol (4), Issue 1, 2050-2052, January -2015

[12] Jayakumar. P, Sangeetha. K, "Integral solution of the Homogeneous Biquadratic Diophantine equation with six unknowns: $(x^3-y^3)z = W^2 – P^2)R^4$ "International Journal of Science and Research, Vol(3), Issue 12, 1021-1023 (December-2014)

Author Profile

P. Jayakumar received the B. Sc, M.Sc degrees in Mathematics from Madras University in 1980 and 1983 and the M. Phil, Ph.D degrees in Mathematics from Bharathidasan University , Thiruchirappalli in 1988 and 2010.Who is now working as Professor of Mathematics, Periyar Maniammai University, Vallam, Thanajvur-613 403,Tamil Nadu,India.

R. Venkatraman received the B.Sc, M.Sc,and MPhil degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 2002, 2004 and 2006. Who is now working as Assistant Professor of Mathematics, SRM University Vadapalani Campus, Chennai - 600026.