Integer Solutions of Non-Homogeneous Ternary Cubic Equation $x^2+y^2-xy=103z^2$

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Abstract: ***The non- homogeneous ternary cubic equation given by $x^2+y^2-xy=103z^2$ is analyzed for its non-zero distinct integer solutions. Introducing the linear transformations $x = u + v$, $y = u - v$ ($u \neq v \neq 0$) and applying the methods of factorization, different patterns of integer solutions of the above equations are obtained.***

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Notations Used

$t_{m,n}$ = Polygonal number of rank n with sides m.- p_n^m = Pyramidal number of rank n with size m.

G_n = Gnomonic number

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. The non-homogeneous ternary cubic equation offers an unlimited field for research because of their variety

[1-2].For an extensive review of various problems one may refer [3-12]. This communication concerns with yet another interesting ternary cubic equation $x^2 + y^2 - xy = 103z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions and special numbers are given.

2. Method of Analysis

The ternary cubic equation under consideration

$$x^2 + y^2 - xy = 103z^3 \quad (1)$$

Introducing the linear transformations

$$x = u + v,\ y = u - v,\ u \neq v \neq 0) \quad (2)$$

in (1), it simplifies to $u^2 + 3v^2 = 103z^3$ (3)

Taking $z = z(a, b) = a^2 + 3b^2 = (a + i\sqrt{3}b)(a - i\sqrt{3}b)$ (4)

where a and b non-zero distinct integers, different patterns of solutions of (1) are given below.

2.1 Pattern: I

$$\text{Let } 103 = (10 + i\sqrt{3})(10 - i\sqrt{3}) \quad (5)$$

Substituting (4) and (5) in (3) and employing the method of factorization, we have

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (10 + i\sqrt{3})(10 - i\sqrt{3})(a + i\sqrt{3}b)^3(a - i\sqrt{3}b)^3$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{3}v) = (10 + i\sqrt{3})(a + i\sqrt{3}b)^3 \quad (6)$$

$$(u - i\sqrt{3}v) = (10 - i\sqrt{3})(a - i\sqrt{3}b) \quad (7)$$

Equating the real and imaginary parts in (6) or (7), we have

$u = u(a, b) = 10a^3 - 60ab^2 - 6a^2b + 9b^3$

$v = v(a, b) = a^3 - 6ab^2 + 20a^2b - 30b^3$ In view of (2), the values of x, y are given by

$$x = x(a, b) = 11a^3 - 66ab^2 + 14a^2b - 21b^3 \quad (8)$$

$$y = y(a, b) = 9a^3 - 54ab^2 - 26a^2b + 39b^3 \quad (9)$$

Thus (4), (8) and (9) represent the non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $y(a, a) - x(a, a) - 232P_a^5 + 116t_{4,a} = 0$
2. $z(a, a) - 4t_{4,a} = 0$
3. $x(a, 1) - 22P_a^5 + 3t_{4,a} + G_{33a} + 21 = 0$
4. $x(1, b) + 42P_b^5 + 45t_{4,b} - G_{7b} \equiv 0 \pmod{11}$
5. $y(1, b) - 78P_b^5 + 93t_{4,b} + G_{13b} \equiv 0 \pmod{3}$.
6. $y(2a, 2a) + 532P_b^5 - 266\, t_{4,a} = 0$.
7. $x(3a, 4a) + 7430\, P_b^5 - 3715\, t_{4,a} = 0$
8. $x(b, -1) - 22\, P_b^5 + 25t_{4,b} + G33b \equiv 0 \pmod{11}$

2.2 Pattern: II

$$\text{Let } 103 = \frac{(20 + i2\sqrt{3})(20 - i2\sqrt{3})}{4} \quad (10)$$

Substituting (4) and (10) in (3) and employing the Method of factorization, we have $(u + i\sqrt{3}v)(u - i\sqrt{3}v) = \frac{1}{4}[(20 + i2\sqrt{3})(20 - i2\sqrt{3})(a + i\sqrt{3}b)^3(a - i\sqrt{3}b)^3]$

Equating the positive and negative factors, we get

$$(u + i\sqrt{3}v) = \frac{1}{2}[(20 + i2\sqrt{3})(a + i\sqrt{3}b)^3] \quad (11)$$

$$(u - i\sqrt{3}v) = \frac{1}{2}[(20 - i2\sqrt{3})(a - i\sqrt{3}b)^3] \quad (12)$$

Equating the real and imaginary parts in (11) or (12), we have

$u = u\,(a, b) = \frac{1}{2}\,[20\,a^3 - 180ab^2 - 18a^2b + 18b^3]$

$v = v\,(a, b) = \frac{1}{2}\,[2\,a^3 - 18ab^2 + 60a^2b - 60b^3]$

In view of (2), the values of x, y are given by

$$x = x\,(a, b) = 11a^3 - 99ab^2 - 21a^2b + 21b^3 \qquad (13)$$
$$y = y\,(a, b) = 9a^3 - 81ab^2 - 39a^2b + 39b^3 \qquad (14)$$

Thus (4), (13) and (14) represents the non-zero distinct integral solution of (1) in two parameters.

Properties:-

1. $y\,(a, a) - x\,(a, a) + 32P_a^5 - 16t_{4,\,a} = 0$
2. $9x\,(b, b) - 11y\,(b, b) = 0$
3. $x\,(a, 1) - 22P_a^5 + 99P_a - 109t_{4,a} + 21 = 0$
4. $y\,(1, b) - 78P_b^5 + 81t_{4,b} - 39P_b \equiv 0 \pmod 3$
5. $z\,(a, 2a) - t_{28,a} - G_{6n} - 1 = 0$
6. $y\,(-1, b) - 78\,P_b^5 + 39\,P_b + 81\,t_{4,b} \equiv 0 \pmod 3$
7. $x\,(2a, 1) - 176\,P_b^5 + 4\,t_{4,a} + G_{99b} \equiv 0 \pmod 2$
8. $y\,(4, b) - 78\,P_b^5 + 363\,t_{4,b} + G_{312,b} - 577 = 0.$

2.3 Pattern III

$$\text{Write (3) as } u^2 + 3v^2 = 103z^3 * 1 \qquad (15)$$
$$\text{Write 1 as } 1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \qquad (16)$$

Substituting (4), (5) and (16) in (15) and employing the method of factorization, we have

$$(u + i\sqrt{3}\,v)\,(u - i\sqrt{3}\,v) = \frac{1}{4}\,[(1 + i\sqrt{3})\,(1 - i\sqrt{3})$$
$$(10 + i\sqrt{3})\,(10 - i\sqrt{3})\,(a + i\sqrt{3}\,b)^3\,(a - i\sqrt{3}\,b)^3]$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{3}\,v) = \frac{1}{2}\,[\,(1 + i\sqrt{3})\,(10 + i\sqrt{3})\,(a + i\sqrt{3}\,b)^3\,] \qquad (17)$$
$$(u - i\sqrt{3}\,v) = \frac{1}{2}\,[\,(1 - i\sqrt{3})\,(10 - i\sqrt{3})\,(a - i\sqrt{3}\,b)^3\,] \qquad (18)$$

Equating the real and imaginary parts in (17) or (18), we have

$u = u\,(a, b) = \frac{1}{2}\,[7a^3 - 63ab^2 - 99a^2b + 99b^3]$

$v = v\,(a, b) = \frac{1}{2}\,[11a^3 - 99ab^2 + 21a^2b - 21b^3]$

In view of (2), the values of x, y are given by

$$x = x\,(a, b) = 9a^3 - 81ab^2 - 39a^2b + 39b^3 \qquad (19)$$
$$y = y\,(a, b) = -\,2ab^3 + 18ab^2 - 60a^2b + 60b^3 \qquad (20)$$

Properties:-

1. $y\,(a, a) - x\,(a, a) - 176P_a^5 + 88t_{4,\,a} = 0$
2. $x\,(1, b) + 78P_a^5 + 39\,P_b + 81t_{4,9} \equiv 0 \pmod 3$
3. $x\,(b, b) + 9y(b, b) = 0$
4. $z\,(b, b) - 4t_{4,b} = 0$
5. $y\,(1, b) - 60P_b - 138t_{4,b} \equiv 0 \pmod 2$
6. $x\,(b, 4) - 18\,P_b^5 + 165\,t_{4,b} + G_{648,\,b} - 2497 = 0$
7. $9x\,(b, b) - 2y\,(b, b) + 1232\,P_b^5 - 616\,t_{4,b} = 0$
8. $z\,(4a, 3a) - 43\,t_{4,a} = 0.$

2.4 Pattern IV

Instead of (16),

$$\text{Write 1 as } \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49} \qquad (21)$$

Substituting (4), (10) and (21) in (15) and employing the method of factorization, we have

$$(u + i\sqrt{3}\,v)\,(u - i\sqrt{3}\,v) = \frac{1}{4\times 49}[(1 + i4\sqrt{3})(1-\sqrt{3})$$
$$(20 - i2\sqrt{3})\,(20 + i2\sqrt{3})\,(a + i\sqrt{3}\,b)^3\,(a - i\sqrt{3}\,b)^3]$$

Equating the positive and negative factors, we get

$$(u + i\sqrt{3}\,v) = \frac{1}{14}[(1 + i4\sqrt{3})(20 + i2\sqrt{3})(a + i\sqrt{3}\,b)^3] \quad (22)$$
$$(u - i\sqrt{3}\,v) = \frac{1}{14}\,[(1 - i4\sqrt{3})(20 - i2\sqrt{3})(a - i\sqrt{3}\,b)^3] \quad (23)$$

Equating the real and imaginary parts in (22) or (23), we have

$u = u\,(a, b) = \frac{1}{14}\,[-\,4a^3 - 36ab^2 - 738a^2b + 738b^3]$

$v = v\,(a, b) = \frac{1}{14}\,[82a^3 - 738ab^2 - 12a^2b + 12b^3]$

In view of (2), the values of x, y are given by

$$x = x\,(a, b) = \frac{1}{7}\,[39a^3 - 387ab^2 - 375a^2b + 375b^3] \qquad (24)$$
$$y = y\,(a, b) = \frac{1}{7}\,[-\,48a^3 + 351ab^2 - 363a^2b + 363b^3] \qquad (25)$$

As our interest in on finding integer solutions, replacing a by 7a and b by 7b in (4), (24) and (25), the corresponding integer solutions of (1) in two parameters are given by

$x = x\,(a, b) = -\,1911a^3 + 18963\,ab^2 - 18375a^2b + 18375b^3$

$y = y\,(a, b) = -\,2352a^3 + 17199\,ab^2 - 17787a^2b + 17787b^3$

$z = z\,(a, b) = 49a^2 + 147b^2$

Properties:

1. $x\,(a, a) - y(a, a) + 63798P_a^5 - 31899t_{4,a} = 0$
2. $z\,(a, a) - t_{394,a} - G_{97a} - P_a - 1 + t_{4,a} = 0$
3. $x\,(1, b) - 36750\,P_b^5 + 18375P_b + 18963t_{4,b} + 1911 = 0$
4. $y\,(a, 1) + 4704P_a^5 + 17199P_a + 32634t_{4,a} - 17787 = 0$
5. $z\,(a, 1) - 49t_{4,a} \equiv 0 \pmod 7$
6. $x\,(a, -a) + 34104\,P_b^5 - 17052\,t_{4,a} = 0$
7. $y\,(a, -a) - 29694\,P_b^5 + 14847\,t_{4,a} = 0$
8. $z\,(a, -a) - 196\,t_{4,a} = 0.$

3. Conclusion

In this paper, we have illustrated different methods of obtaining infinitely Many non-zero distinct integer solutions to the cubic equations $x^2 + y^2 - xy = 103z^2$ are rich in variety, one may search for non –zero distinct integer solutions to the other choices of cubic equations with three or more variables along with their corresponding properties.

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