

# Solving Neutron Transport Equation in the Reactor using the Intergral Average Derivative Method

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**Abstract:** Space and time dependent neutron diffusion equations with many energy groups and taking into account delayed neutrons are nonlinear partial differential equations. IAD method and finite difference method for simplified equations to partial differential equations often, then be rewritten in matrix form. General solution of differential equations contain the exponential matrix of matrix coefficients. Specific methods applied IAD (average method integral derivative) as an example to determine the flux reactor radius spherical R reflector layer  $R \notin R$ .

**Keywords:** The Neutron field in Reactor, Boltzman transport equation

## 1. Introduction

Neutron diffusion equation describes the density of neutrons in a nuclear reactor core. Many calculation methods have been proposed to solve diffuse neutron mostly based on the method of finite difference applied to the domain geometric shape such as in the case of two-dimensional X-Y or R-Z. On the other hand with complex geometries using finite element method. In addition, the method of analysis [1], [2], the Monte Carlo method [4] the Boltzmann equation in the nuclear reactor has been developed with the complex calculation program. However, there are many difficulties in the process depending on how the calculation method applicable to each specific problem, they relate to the timing and geometry of the system should take into account. During the last few decades, some authors use the hybrid method between a method and a method of this group other groups to take advantage of these methods simultaneously overcoming the disadvantages and shortcomings of each method.

IAD method (average method integral derivative) by Phan Van Hap proposal [12] which we developed and perfected applied to solve the nuclear reactor in combination with traditional methods can solve the equation under a consistent algorithm including singularity where there is no common derivative.

The IAD coordinated approach with some other methods

$$\frac{1}{v_g} \frac{\partial}{\partial t} \Phi_g(\mathbf{r}, t) = \nabla \cdot D_g(\mathbf{r}) \nabla \Phi_g(\mathbf{r}, t) - \Sigma_{a_g} \Phi_g(\mathbf{r}, t) - \sum_{g' > g}^G \Sigma_{s_{g,g'}} \Phi_{g'}(\mathbf{r}, t) + \sum_{g'=1}^G \left( \chi_g v \Sigma_{f_{g'}} (1 - \beta) + \Sigma_{s_{g,g'}} \right) \Phi_{g'}(\mathbf{r}, t) + \sum_{i=1}^I \chi_{g,i} \lambda_i C_i(\mathbf{r}, t); g = 1, 2, \dots, G \quad (1)$$

$$\frac{\partial}{\partial t} C_i(\mathbf{r}, t) = \beta_i \sum_{g=1}^G v \Sigma_{f_g} \Phi_g(\mathbf{r}, t) - \lambda_i C_i(\mathbf{r}, t) \quad (2)$$

where  $\mathbf{r}$  is the position vector (cm),  $t$  is the time (s),  $F_g(\mathbf{r}, t)$  is the scalar neutron flux ( $\text{cm}^{-2} \text{s}^{-1}$ ) in group  $g$ ,  $C_i(\mathbf{r}, t)$  is the concentration of delayed neutron precursors

such as multi-step method, the method rather singular integrals with finite total was some authors to use and have good results in many problems - especially crack theory of mechanics of the continuous environment. Author J.J. Golecki has focused on the application method "phan van hap method" in a series of articles published in scientific journals from 1979 to 2007 [11].

In recent 6 decades, the calculation and design of nuclear reactors was a significant step forward, along with the development of the computer, we built a computational model based reactor facilities of nuclear reactors and review the method of calculation, the successes and limitations of previous methods, the proposed method of applying the average integral derivative (IAD method) to the solution of transport equation neutrons to describe the physical processes taking place in the nuclear reactor core. The result after the problems of nuclear reactors given matrix equation calculated neutron flux for each energy group.

## 2. Diffusion Equation for Neutron and the IAD Method

Based on the kinetic equation for the nuclear reactor neutron energy groups [5] after processing the input data is written

( $\text{cm}^3$ ) in group  $i$ ,  $v_g$  is the mean velocity of the neutron ( $\text{cm s}^{-1}$ ) in group  $g$ ,  $D_g(\mathbf{r})$  is the diffusion coefficient ( $\text{cm}$ ) in group  $g$ ,  $\Sigma_{a_g}(\mathbf{r}, t)$  is the absorption cross-section ( $\text{cm}^{-1}$ ) in group  $g$ ,  $\Sigma_{f_g}(\mathbf{r})$  is the fission cross-section ( $\text{cm}^{-1}$ ) in group  $g$ ,  $\Sigma_{s_{g,g'}}(\mathbf{r})$  is the scattering cross-section

$(cm^{-1})$  from group  $g'$  to group  $g$  such that  $(S_{s,g,g'}(r) = 0 \text{ khi } g \neq g')$ ,  $n$  is the mean number of fission neutrons,  $c_g$  is the spectrum of prompt neutrons in group  $g$ ,  $c_{g,i}$  is the spectrum of  $i$ -group delayed neutrons in group  $g$ ,  $\lambda_i$  is the decay constant ( $s^{-1}$ ) of group  $i$  precursors,  $\beta_i$  is the fraction of delayed neutrons in group  $i$ , and  $\beta = \sum_{i=1}^I \beta_i$  is the total fraction of delayed neutrons.

Using the finite difference method and the IAD method [13], equation (1) and (2) leads to the following matrix:

$$\frac{d\Psi(t)}{dt} = \mathbf{A}\Psi(t) + \mathbf{B} \quad (3)$$

with

$$\Psi(t) = [\bar{\Phi}_1 \quad \bar{\Phi}_2 \quad \dots \quad \bar{\Phi}_G \quad C_1 \quad C_2 \quad \dots \quad C_I]^T \quad (4)$$

Here  $\bar{\Phi}_1, \bar{\Phi}_2, \dots, \bar{\Phi}_G$  is the averaged neutron flux in equations (1) and (2) using the IAD method:

$$\lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_{-\delta}^{\delta} \frac{\Phi_i(x_0 + th) - \Phi_i(x_0)}{t} dt = \bar{\Phi}_i(x_0, h) \equiv \bar{\Phi}_i \quad (5)$$

and

$$\mathbf{B} = \begin{bmatrix} s_g & s_{g,g'} & L & s_{g'} & 0 & 0 & L & 0 \end{bmatrix}^T \quad (6)$$

Here,  $s_g, s_{g,g'}, s_{g,g'}, f_{g,i}$  and  $p_{i,g}$  was defined in reference [1]

Assume that, the matrices  $\mathbf{A}$  and  $\mathbf{B}$  are constant during the interval time  $t_m$  and  $t_{m+1} = t_m + h$  considering the length of the time interval is small. Then, the general analytical of equation (3) takes the following form:

$$\Psi(t_{m+1}) = \sum_{k=0}^K (-1)^k \frac{\exp(hw_k) Q_{K-k,j}}{\prod_{i=0}^K (w_i - w_j)} \Psi(t_m) + \sum_{k=0}^K (-1)^k \frac{\exp(hw_k) Q_{K-k,j}}{\prod_{i=0}^K (w_i - w_j)} \mathbf{A}^k \mathbf{I} \mathbf{A}^{-1} \mathbf{B} \quad (11)$$

This equation is used in the numerical technique for solving the space-time neutron diffusion equations with multi-group of delayed neutrons which based on analytical method for the exponential function of matrix  $\mathbf{A}$ .

### 3. Application to 2 Groups of Spherical Reactors

We apply the IAD method to the specific problem of spherical reactor with two energy groups with a container region of radius  $R$ - $R$  (which acts as a reflect region) and a core of radius  $R_c$ . The diffusion equations for each groups can be written as:

$$\begin{aligned} \text{fast group} \quad & -D_1^c \nabla^2 \Phi_1^c(r) + \Sigma_s^c \Phi_1^c(r) = \frac{1}{k} \nu \Sigma_f \Phi_2^c(r) \\ \text{thermal group} \quad & -D_2^c \nabla^2 \Phi_2^c(r) + \Sigma_a^c \Phi_2^c(r) = \Sigma_s^c \Phi_1^c(r) \end{aligned} \quad (12)$$

We need to find both neutron group fluxes and the unknown

$$\Psi(t_{m+1}) = \exp(h\mathbf{A})\Psi(t_m) + (\exp(h\mathbf{A}) - \mathbf{I})\mathbf{A}^{-1}\mathbf{B} \quad (7)$$

Expanding the exponential matrix  $\exp(h\mathbf{A})$  using Taylor expansion:

$$\exp(h\mathbf{A}) = \sum_{k=0}^K a_k (h\mathbf{A})^k \quad (8)$$

Where  $a_0, a_1, \dots, a_K$  are Taylor expansion coefficients and  $K = G + I - 1$  is the dimension of the matrix  $\mathbf{A}$ .

The coefficient  $a_k$  can be obtained by solving the system of equation:

$$\begin{bmatrix} \exp(hw_0) & (hw_0) & (hw_0)^2 & L & (hw_0)^K \\ \exp(hw_1) & (hw_1) & (hw_1)^2 & L & (hw_1)^K \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \exp(hw_K) & (hw_K) & (hw_K)^2 & L & (hw_K)^K \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (9)$$

where  $w_0, w_1, \dots, w_K$  are the eigenvalues of the matrix  $\mathbf{A}$ . Gauss elimination method is used to solve the equation (9) which yields

$$a_k = (-1)^k \sum_{j=0}^K \frac{\exp(hw_j) Q_{K-k,j}}{\prod_{i=0, i \neq j}^K (w_i - w_j)}; \quad k = 0, 1, \dots, K \quad (10)$$

The coefficient  $a_k$  can be expanded for two groups of neutron energies and for the mean group of delayed neutron. One gets:

constant  $k$  by solving the eigenvalue problem. The quantity  $\Sigma_s^c$  is the cross section for scattering of the fast neutron to thermal neutrons and also plays the role of thermal neutron source;  $\Sigma_a^c$  is the absorption cross section. The fluxes in the reflect region are described by

$$\begin{aligned} \text{fast group} \quad & -D_1^R \nabla^2 \Phi_1^R(r) + \Sigma_s^R \Phi_1^R(r) = 0 \\ \text{thermal group} \quad & -D_2^R \nabla^2 \Phi_2^R(r) + \Sigma_a^R \Phi_2^R(r) = \Sigma_s^R \Phi_1^R(r) \end{aligned} \quad (13)$$

Here,  $\Sigma_s^R$  is the scattering cross section from fast neutrons to thermal neutrons. Because there are no source of neutrons at the reflect region, therefore the right hand side of the first equation in (12) is zero. For the equation in the core region, the fluxes are the linear combination of different functions:

$$\Phi_1^c(r) \equiv A_1 Z(r) + C_1 W(r); \quad \Phi_2^c(r) \equiv A_2 Z(r) + C_2 W(r) \quad (14)$$

The functions  $Z$  and  $W$  are determined from:

$$\nabla^2 Z(r) + B_1^2 Z(r) = 0; \quad \nabla^2 Z(r) + B_2^2 Z(r) = 0 \quad (15)$$

where  $B_1$  and  $B_2$  are chosen such that the expressions (14) for  $\Phi_1^C$  and  $\Phi_2^C$  are solutions to equations (12) and (14) correspondingly. Using Eq. (15), one obtains

$$\left[ A_1 D_1^C B_1^2 + A_1 \Sigma_s^C - \frac{1}{k} \nu A_2 \Sigma_f \right] Z(r) + \left[ C_1 D_1^C B_2^2 + C_1 \Sigma_s^C - \frac{1}{k} \nu C_2 \Sigma_f \right] W(r) = 0 \quad (25)$$

$$\left[ A_2 D_2^C B_1^2 + A_2 \Sigma_s^C - A_1 \Sigma_s^C \right] Z(r) + \left[ C_2 D_2^C B_2^2 + C_2 \Sigma_a^C - A_2 \Sigma_s^C \right] W(r) = 0 \quad (16)$$

Where  $B_1^2$  and  $B_2^2$  are solutions to:

$$(B^2 + \kappa_1^2)(B^2 + \kappa_2^2) = \frac{k_\infty}{k} \kappa_1^2 \kappa_2^2 \quad (17)$$

with the coefficients

$$\kappa_1^2 \equiv \frac{\Sigma_s^C}{D_1}; \quad \kappa_2^2 \equiv \frac{\Sigma_a^C}{D_2}; \quad k_\infty \equiv \frac{\nu \Sigma_f}{\Sigma_a^C} \quad (18)$$

The last expression is denoted by  $k_\infty$  because it represents the coefficient for the case of infinite medium. If expression (14) is a solution to the core equation (13) with  $Z$  and  $W$  given by (15). The quantities  $B_1$  and  $B_2$  are found from the equation (17):

$$B^2 = -\frac{\kappa_1^2 + \kappa_2^2}{2} \pm \frac{1}{2} \sqrt{(\kappa_1^2 + \kappa_2^2)^2 + 4\kappa_1^2 \kappa_2^2 \left( \frac{k_\infty}{k} - 1 \right)} \quad (19)$$

For a reactor, we have

$$\frac{\sqrt{(\kappa_1^2 + \kappa_2^2)^2 + 4\kappa_1^2 \kappa_2^2 \left( \frac{k_\infty}{k} - 1 \right)}}{\kappa_1^2 + \kappa_2^2} > 1 \quad (20)$$

Assume  $\mu^2$  and  $\lambda^2$  are positive real parameters the solutions of this quadratic equation are

$$B_1^2 \equiv \mu^2; \quad B_2^2 \equiv -\lambda^2 \quad (21)$$

Equation (15) becomes

$$\nabla^2 Z(r) + \mu^2 Z(r) = 0; \quad \nabla^2 Z(r) - \lambda^2 Z(r) = 0 \quad (22)$$

In the spherical coordinate system, the solutions are

$$Z(r) = \frac{\sin \mu r}{r}; \quad W(r) = \frac{\sinh \lambda r}{r} \quad (23)$$

This set of equations has general solutions is the distribution of neutron fluxes in the reactor. We convert those equations to the equation in the reflect region by introducing the quantities  $\kappa_3^2$  and  $\kappa_4^2$

$$\kappa_3^2 \equiv \Sigma_s^R / D_1^R; \quad \kappa_4^2 \equiv \Sigma_a^R / D_2^R \quad (24)$$

As a result, Eq. (13) becomes

$$\nabla^2 \Phi_1^R(r) - \kappa_3^2 \Phi_1^R(r) = 0; \quad \nabla^2 \Phi_2^R(r) - \kappa_4^2 \Phi_2^R(r) = -\frac{\Sigma_s^R \Phi_1^R(r)}{D_2^R} \quad (25)$$

The solution of Eq. (25) vanishes at the extrapolated reflect surface. Denote the radius of this extrapolated reflect surface

$$\Phi_1^R(\tilde{R}') = \Phi_2^R(\tilde{R}') = 0 \quad (26)$$

$$(Z) A_1 + (W) C_1 + (-U) S_1 = 0$$

$$(-D_1^C Z') A_1 + (-D_1^C W') C_1 + (D_1^R U') S_1 = 0 \quad (27)$$

$$(a_1 Z) A_1 + (a_2 W) C_1 + (a_3 U) S_1 + (-V) T_1 = 0$$

$$(-a_1 D_2^C Z') A_1 + (-a_2 D_2^C W') C_1 + (-a_3 D_2^C U') S_1 + (-D_2^R V') T_1 = 0$$

The symbols  $Z, W, U, V$  and  $Z', W', U', V'$  are functions and their derivatives evaluated at  $r = R$ . Rewrite Eq. (27) in the matrix form

$$\begin{bmatrix} Z & W & -U & 0 \\ -D_1^C Z' & -D_1^C W' & D_1^R U' & 0 \\ a_1 Z & a_2 W & a_3 U & -V \\ -a_1 D_2^C Z' & -a_2 D_2^C W' & -a_3 D_2^C U' & D_2^R V' \end{bmatrix} \begin{bmatrix} A_1 \\ C_1 \\ T_1 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (28)$$

This system of linear equations (28) has non-trivial solution if and only if its determinant of the coefficients are zero

$$\Delta(k) \equiv \begin{bmatrix} Z & W & -U & 0 \\ -D_1^C Z' & -D_1^C W' & D_1^R U' & 0 \\ a_1 Z & a_2 W & a_3 U & -V \\ -a_1 D_2^C Z' & -a_2 D_2^C W' & -a_3 D_2^C U' & D_2^R V' \end{bmatrix} \equiv 0 \quad (29)$$

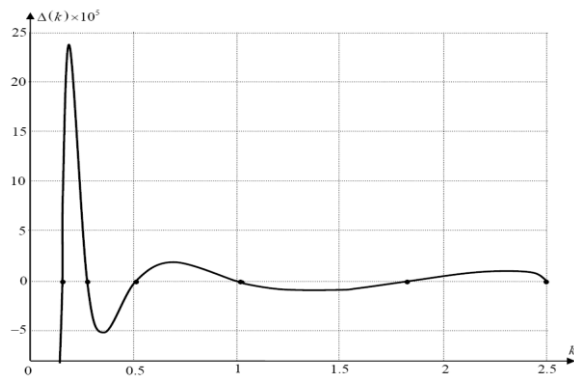
This expression can be considered the threshold condition for the reactor in the two-group approximation because it is related to the parameters of the groups and the dimensions of the reactors. Determining the concentration of the nuclear fuel or the dimensions of the reactor allows one to determine other parameters of the reactor.

## 4. Numerical Results

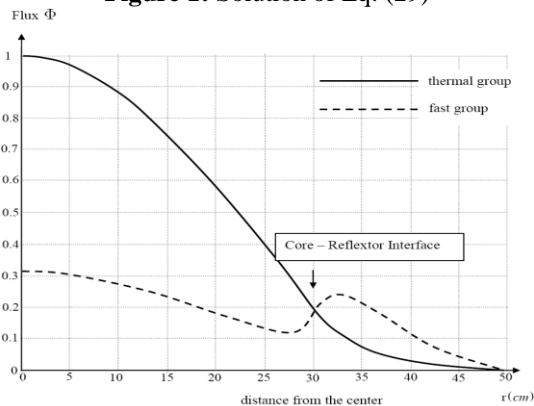
To demonstrate the two-group model of the reactor, we solve the reactor equations numerically. The neutron multiplication factor of the model is given, and we adjust the fuel concentrations to satisfy the threshold condition (29). First, we calculate the radius of the reactor spherical core  $R$ , the radius of the extrapolated reflect surface, and point out some important characteristics of the reactor in our model:

Core region	$D_1^C = 1.44 \text{ cm}$	$D_2^C = 0.229 \text{ cm}$	$\Sigma_a^C = 0.1 \text{ cm}^{-1}$	$\Sigma_f = 0.1 \text{ cm}^{-1}$
Reflect region	$D_1^R = 1.85 \text{ cm}$	$D_2^R = 0.204 \text{ cm}$	$\Sigma_a^R = 0.01226 \text{ cm}^{-1}$	

Using IAD method, and substitute the above parameters to Eq. (29), we obtain an algebraic equation for  $k$ . Solving this equation for the numerical value of the intersection of the horizontal axis with  $\Delta(k)$  for  $0 \leq k \leq k_\infty = 2.5$ .



**Figure 1:** Solution of Eq. (29)



**Figure 2:** Flux of neutron in fast group and thermal group

The general approaches to one-group model can be easily expanded to the two-group model. The consideration of two-group model with a reflect region shows the flux of neutron in the reflect region which are not available in the one-group model. At the same time, the creation of a reflect region allows one to reduce the size of the reactor core as well as its mass. They provide valuable information when designing nuclear reactor as well as devices using nuclear energy.

## 5. Conclusion

Numerical calculation based on finite difference method for reactor with standard geometry is considered. We use the IAD method to solve the neutron diffusion equation for complicated geometries. Combining with the finite difference method, we solve the diffusion equation for two energy groups and one group for delayed neutron. The solutions are expanded for all points in the 3D mesh of the reactor geometry. They are compared to the traditional methods. It comes to our conclusion that the IAD method gives higher precision than other method because IAD method allows one to overcome the region or point of singularity.

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