The Problem of Wave Diffraction in Elastic Medium is Solved using the Integral Averaged Differential Method

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Abstract. ***Wave diffraction in an elastic medium with cavity with hard wall is a physic problem with many potential applications in physics, in fluid dynamics, water irrigation. The boundary problem is transformed into a system of integral equation with singularity. After that, the integral averaged differential (IAD) method is used to calculate the integral and solve the equation. Our results are given for the case the opening angle is $\pi/2$ and $\pi/3$. Comparision to the results of Refences [4] shows the advantage of IAD method.***

Keywords: wave diffraction of elastic medium with gaping corner, integral equations singularity.

1. Introduction

The problem of wave diffraction in an elastic media with an opening angle is considered based on the wave equations with different hard wall boundary. Different methods have been proposed to olve this problem such as the partial differential equation method, finite difference method, The case, the opening angle $\beta = 0$ has already solved. For the opening angle $0 < \beta < \pi$ has many potential applications in physics such as explorative detonations, designs for water dam, ocean dam, water irrigation, and are actively studied. Method of solution using integral equation system with singularity is a new direction for these applied physics problem. It is based on the integral averaged differential method proposed by Phan Van Hap [5, 9]

Let us start from the equation:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mathrm{rot}\,\mathrm{rot}\,\mathbf{u} \qquad (1)$$

Where $\mathbf{u}$ is the displacement vector; λ, μ are elastic constants, ρ is the mass density of the medium.

Express vector $\mathbf{u}$ through the scalar potential φ and the vector potential ψ we obtain:

$$\mathbf{u} = grad\varphi + rot\boldsymbol{\psi} \text{ with } div\boldsymbol{\psi} = 0 \qquad (2)$$

Substitute (2) into (1) we get:

$$grad\left[\rho\frac{\partial^2 \varphi}{\partial t^2} - (\lambda + 2\mu)\Delta\varphi\right] + rot\left[\rho\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} - \mu\Delta\boldsymbol{\psi}\right] = 0 \qquad (3)$$

From this:

$$\frac{1}{a^2}\frac{\partial^2 \varphi}{\partial t^2} = \Delta\varphi, \quad a = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$ is the speed of longitudinal wave,

$$\frac{1}{b^2}\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} = \Delta\boldsymbol{\psi}, \quad b = \sqrt{\frac{\mu}{\rho}}$$ is the speed of transverse wave.

Satisfy (3). For a planar geometry, $(u_z = 0)$, we have:

$$\begin{cases} \bar{u}_r = \dfrac{\partial \varphi}{\partial r} + \dfrac{1}{r}\dfrac{\partial \psi_z}{\partial \theta} \\ \bar{u}_\theta = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} - \dfrac{\partial \psi_z}{\partial r} \end{cases}$$

This leads to the equations:

$$\begin{aligned} \frac{1}{a^2}\frac{\partial^2 \varphi}{\partial t^2} &= \frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial}{\partial}\frac{\varphi}{r} + \frac{\partial 1}{\partial r^2}\frac{{}^2\varphi}{\theta^2} \\ \frac{1}{b^2}\frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial}{\partial}\frac{\psi}{r} + \frac{\partial 1}{\partial r^2}\frac{{}^2\psi}{\theta^2} \end{aligned} \qquad (4)$$

In the region $0 \le r < \infty, -\alpha \le \theta \le \alpha$ $(\beta = 2\alpha)$ a planar wave is propagated:

$$\varphi_0 = f\left[at - r\cos(\theta - \alpha - \varepsilon_1)\right] \qquad (5)$$

where f is a known function of real argument; a is speed of wave propagation. The wave front (5) at time $t = 0$ goes through the origin, and for $t > 0$ is the NMM_1N' in Fig. 1.

Suppose the longitudinal wave speed is a, transverse wave speed is b. the system of equation (4) has the boundary condition

$$\begin{cases} \dfrac{\partial \varphi}{\partial r} + \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = 0 \\ \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} - \dfrac{\partial \psi}{\partial r} = 0 \end{cases}$$ with $\theta = \pm\alpha$, (zero at the boundary for hard wall) (*)

Consider

$$f(x) = \begin{cases} 1 \text{ with } x \ge 0 \\ 0 \text{ with } x < 0 \end{cases}$$

We consider the class of solution:

$$u = \mathrm{Re}\left[F(x + iy)\right]$$

For example:

$$\varphi = \varphi(\Omega_1)$$

where $\Omega_1 = \Omega_1(r, \varphi, t)$ and φ are arbitrary analytic function. Substitute into Eq. (4) we get:

$$\left(\frac{\partial \Omega}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \Omega}{\partial \theta}\right)^2 - \frac{1}{a^2}\left(\frac{\partial \Omega}{\partial t}\right)^2 = 0 \qquad (6)$$

$$\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Omega}{\partial \theta^2} - \frac{1}{a^2}\frac{\partial^2 \Omega}{\partial t^2} = 0 \qquad (7)$$

It is easily seen that this system of equation is satisfied for $\Omega_1 = \theta \pm \chi_1$, $\chi_1 = \arccos\frac{at}{r}$.

Thus any analytic function of Ω_1 is a solution of (4). (X. L. Xobolev called such solution class functional invariant solution).

For $r < at$, χ_1 is imaginary, therefore a real solution has form $\varphi = \operatorname{Re} f(\theta + \chi_1)$, $r < at$. With this, Eq. (4) become $\frac{1}{r^2}\left(\frac{\partial^2 \varphi}{\partial \theta^2} - \frac{\partial^2 \varphi}{\partial \chi_1^2}\right) = 0$.

With $r \geq at$, χ_1 is real, therefore Eq. (4) is hyperbolic.

Similarly, for $\psi = \operatorname{Re}\Psi(\theta + \chi_2)$, $r < \beta t$ and $\psi = \Psi(\theta + \chi_2)$, $r \geq \beta t$

The boundary condition at $(\theta = \pm\alpha)$ is

$$\begin{cases} \varphi_r + \frac{1}{r}\psi_\theta = 0 \\ \frac{1}{r}\varphi_\theta - \psi_r = 0 \end{cases}$$

At the wave front, the values of φ and ψ has the same limits when crossing the boundary from outside to inside. Therefore

$$\varphi = \begin{cases} 1 \text{ on } B'_1O_1B_1 \\ 1 + A \text{ on } A_1B_1 \\ 1 + A' \text{on } A'_1B'_1 \end{cases} ; \psi = \begin{cases} 0 \text{ on } B'_2O_2B_2 \\ B \text{ on } B_2A_2 \\ B' \text{on } B'_2A'_2 \end{cases} \qquad (8)$$

2. Convert the Problem to the System of Integral Equation with Singularity

Peforming the transformation $z_1 = \theta + \chi_1$, $z_2 = \theta + \chi_2$ through the variables

$$w_1 = \frac{1}{\sin\frac{\pi}{2\alpha}z_1}, \quad w_2 = \frac{1}{\sin\frac{\pi}{2\alpha}z_2}$$

Using a conformal map, the regions $OC_2A_1B_1O_1B'_1A'_1C'_2O$ and $OC_2A_2B_2O_2B'_2A'_2C'_2O$ are transformed to two lower halves of the planes:

$-\infty$	$-\frac{1}{p_1}$	-1	$-\frac{1}{k_1}$		$\frac{1}{k_1}$			∞
O_1	B'_1	A'_1	C'_2	O	C_2	A_1	B_1	O_1
$-\infty$			-1		1			∞
O_2	B'_2	A'_2	C'_2	O	C_2	A_2	B_2	O_2

We get

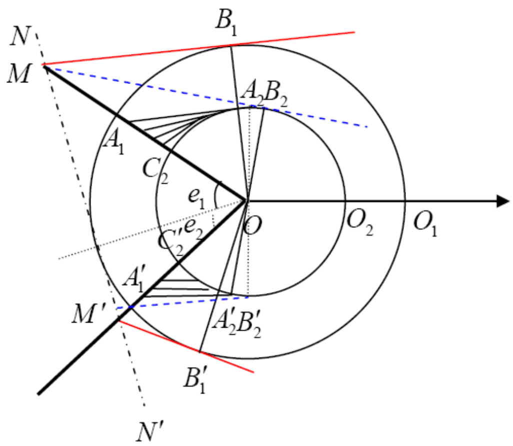

Figure 1: Shematic model of wave diffraction in elastic medium with opening angle $\beta = \varepsilon_1 + \varepsilon_2$ with hard wall.

$$\frac{dw_i}{dz_1} = -\frac{\pi}{2\alpha}w_i\sqrt{w_i^2 - 1}, \quad (i = 1, 2) \qquad (9)$$

Therefore, (*) becomes

$$\begin{cases} \operatorname{Re}\left[\operatorname{cotg}\chi_1 - \sigma_1\sqrt{\sigma_1^2 - 1}\,\varphi'(\sigma_1)\right] + \operatorname{Re}\left[\sigma_2\sqrt{\sigma_2^2 - 1}\,\psi'(\sigma_2)\right] = 0 \\ \operatorname{Re}\left[\sigma_1\sqrt{\sigma_1^2 - 1}\,\varphi'(\sigma_1)\right] - \operatorname{Re}\left[\operatorname{cotg}\chi_2 - \sigma_2\sqrt{\sigma_2^2 - 1}\,\psi'(\sigma_2)\right] = 0 \end{cases} \qquad (10)$$

where $\frac{\sigma_1}{\sigma_2} = \frac{\cos\frac{\pi}{2\alpha}\chi_2}{\cos\frac{\pi}{2\alpha}\chi_1}$ và $\frac{\cos\chi_2}{\cos\chi_1} = \frac{\beta}{a}$, σ_1, σ_2 are boundary values of w_1, w_2,

$$\sigma_1 = \frac{1}{\sin\frac{\pi}{2\alpha}(-\alpha + \chi_1)} = -\frac{1}{\cos\frac{\pi}{2\alpha}\chi_1}; \sigma_2 = \frac{1}{\sin\frac{\pi}{2\alpha}(-\alpha + \chi_2)} = -\frac{1}{\cos\frac{\pi}{2\alpha}\chi_2}$$

From (8) we get:

$$\mathrm{Re}\,\varphi'(s_1) = A\delta\left(s_1 - \frac{1}{p_1}\right) - A'\delta\left(s_1 + \frac{1}{p_1'}\right) \text{ with } \begin{cases} -\infty < s_1 \le -1 \\ 1 \le s_1 < +\infty \end{cases}$$

$$\mathrm{Re}\,\psi'(s_2) = B\delta\left(s_2 - \frac{1}{p_2}\right) - B'\delta\left(s_2 + \frac{1}{p_2'}\right) \text{ with } \begin{cases} -\infty < s_2 \le -\frac{1}{k_2} \\ \frac{1}{k_2} \le s_2 < +\infty \end{cases} \quad (11)$$

We seek the solution to Eq. (9), (10) in the form:

$$\varphi'(w_1) = \bar{\varphi}'(w_1) + F_1(w_1);$$

$$iF_1(w_1) = \frac{1}{\pi}\left(\frac{A}{w_1 - \frac{1}{p_1}} - \frac{A'}{w_1 + \frac{1}{p_1'}}\right)$$

$$\psi'(w_2) = \bar{\psi}'(w_2) + F_2(w_2);$$

$$iF_2(w_2) = \frac{1}{\pi}\left(\frac{B}{w_2 - \frac{1}{p_2}} - \frac{B'}{w_2 + \frac{1}{p_2'}}\right)$$

Set

$$\Phi_1(w_1) = w_1\sqrt{w_1^2 - 1}\,\bar{\varphi}_1'(w_1);\quad \Phi_2(w_2) = w_2\sqrt{w_2^2 - 1}\,\bar{\psi}_1'(w_2)$$

one get the system of equations:

$$\begin{cases} i\,\mathrm{cotg}\,\chi_1\,\mathrm{Im}\Phi_1(s_1) + \mathrm{Re}\Phi_2(s_2) = -s_2\sqrt{1 - s_2^2}\,iF_2(s_2) \\ \mathrm{Re}\Phi_1(s_1) - \mathrm{cotg}\,\chi_2 \begin{cases} i\,\mathrm{Im}\Phi_2(s_2) \\ \mathrm{Re}\Phi_2(s_2) \end{cases} = s_1\sqrt{1 - s_1^2}\,iF_1(s_1) \end{cases} \quad (12)$$

Note that $\mathrm{Re}\Phi_1(s_1) = 0$ for $\begin{cases} -\infty \le s_1 \le -1 \\ 1 \le s_1 \le +\infty \end{cases}$;

$$\mathrm{Re}\Phi_2(s_2) = 0 \text{ for } \begin{cases} -\infty < s_2 \le -\frac{1}{k_2} \\ \frac{1}{k_2} \le s_2 < +\infty \end{cases}$$

Denote $u_2 = \mathrm{Re}\Phi_1, u_1 = \mathrm{Re}\Phi_2$. One gets the following system of intergral equation with singularity:

$$\begin{cases} \frac{1}{\pi} i\,\mathrm{cotg}\,\chi_1 \int_{-1}^{1} \frac{u_1(x_1)}{s_1 - x_1}dx_1 + u_2(x_1) = -s_2\sqrt{1 - s_2^2}\,iF_2(s_2) \\ u(s_1) - \begin{cases} \frac{1}{\pi} i\,\mathrm{cotg}\,\chi_2 \int_{-\frac{1}{k_2}}^{\frac{1}{k_1}} \frac{u_2(x_2)}{s_2 - x_2}dx_2 \\ \mathrm{cotg}\,\chi_2 u_2(x_2) \end{cases} = s_1\sqrt{1 - s_1^2}\,iF_1(s_1) \end{cases}$$

In the second equation, the upper part in the brackets is for the case $s_2 \in [-1, 1]$, and for the lower part $s_2 \in \left[\frac{1}{k_2}, -1\right] \cup \left[1, \frac{1}{k_2}\right]$.

Intergral average differential (IAD) method for solving (12)

Denote $u_2(x_2) = u_2[\alpha(x)] = v(x), s_1 = s, s_2 = \alpha(s)$; $\alpha(s) = \dfrac{\cos\frac{\pi}{2\alpha}\chi_1}{\cos\frac{\pi}{2\alpha}\chi_2}$ and $\dfrac{\cos\chi_2}{\cos\chi_1} = \dfrac{\beta}{\alpha}$, from which

$$\alpha(s) = \cos\frac{\pi}{2\alpha}\left\{\arccos\left(\frac{\beta}{\alpha}\cos\left(\frac{2\alpha}{\pi}\arccos s\right)\right)\right\} \text{ and}$$

$$\frac{1}{\pi}\int_{-\frac{1}{k_1}}^{\frac{1}{k_2}} \frac{u_2(x_2)}{s_2 - x_2}dx_2 = \frac{1}{\pi}\int_c^d \frac{v(x)}{s - x}dx + \frac{1}{\pi}\int_c^d \frac{v(x)}{s - x}\left[\frac{(s - x)\alpha'(x)}{\alpha(s) - \alpha(x)} - 1\right]dx$$

where $d = \alpha^{-1}\left(\frac{1}{k_2}\right), c = \alpha^{-1}\left(-\frac{1}{k_2}\right)$.

System (12) becomes:

$$\begin{cases} \frac{1}{\pi} i\,\mathrm{cotg}\,\chi_1 \int_{-1}^{1} \frac{u(x)}{s - x}dx + v(s) = f_1(s) \\ u(s) - \frac{1}{\pi} i\,\mathrm{cotg}\,\chi_2 \int_c^d \frac{v(x)}{s - x}dx - \frac{1}{\pi} i\,\mathrm{cotg}\,\chi_2 \int_c^d k(s,x)v(x)dx = f_2(s) \end{cases} \quad (13)$$

with

$$f_1(s) = -\alpha(s)\sqrt{1 - \alpha^2(s)}\,iF_2(\alpha(x));\ f_2(s) = -s\sqrt{1 - s^2}\,iF_1(s)$$

$$k(s,x) = \frac{1}{s - x}\left[\frac{(s - x)\alpha'(x)}{\alpha(s) - \alpha(x)} - 1\right] \quad (14)$$

One can apply directly the method for solving system of integral equation with singularity as presented in Reference [8] to solve (13). Below, we use the IAD method [5]. Set

$$\int_{-1}^{1}\frac{u(x)}{s-x}dx = S_1u;\quad \int_{c}^{d}\frac{v(x)}{s-x}dx = S_2v$$

From the first equation in (13) one gets $v(s)=f_1(s)-\frac{1}{\pi}i\,\mathrm{cotg}\,\chi_1 S_1u$. Substitute into the second equation we have:

$$u(s)-\frac{1}{\pi}i\,\mathrm{cotg}\,\chi_2 S_2v-\frac{1}{\pi}i\,\mathrm{cotg}\,\chi_2\int_{c}^{d}k(s,x)\left[f_1(s)-\frac{1}{\pi}i\,\mathrm{cotg}\,\chi_1 S_1u\right]dx = f_2(s)$$

where $k(s,x)$ is given in Eq. (14) $S_2v = S_2\left[f_1(s)-\frac{1}{\pi}i\cot\chi_1 S_1u\right]$. Using the approximation method of singularity integral of Phan Van Hap [6] for the case $\beta=\frac{\pi}{2},\ \beta=\frac{\pi}{3}$; $\lambda = 1, \mu = 0{,}05, \rho = 2$, we obtain the following results:

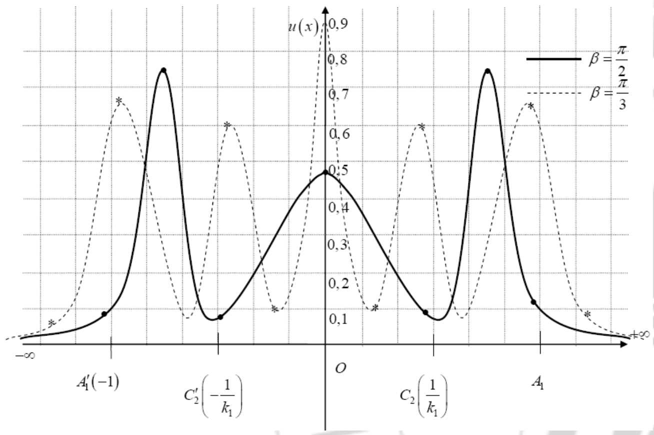

Figure 2: Diffraction function $u(x)$ calculated for the opening $\beta=\frac{\pi}{2}$ and $\beta=\frac{\pi}{3}$ as shown in Fig.1.

Compared to a recent paper [4]. We can see that our IAD method is much simpler than the method of finite difference. We also show that the equation (1.20) is simple analytically. Using this method, the obtained solutions agree well with the phenomenon of wave diffraction at the singular points considered. If one uses the method of reference [4] the numerical procedure is more complicated.

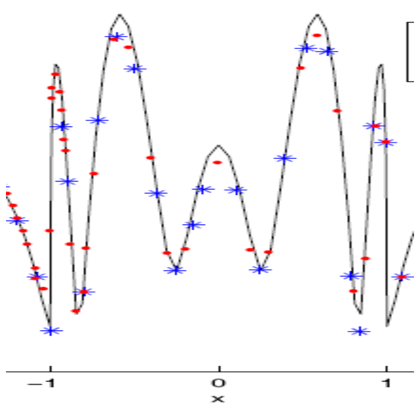

Figure 3: Comparison with results of reference [4]

3. Conclusion

Using the IAD method, we overcome the difficulties of working with singular points by solving the system of intergral equations with singularity using a conformal mapping of complex functions. Current theory can be expanded to study wave diffraction problems without resorting to finite difference method. The IAD method has demonstrated its advantages in solving different problems in physics and mechanics.

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