

Estimates of an Eigen Value of Some Isoperimetric Inequalities

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Abstract: In this paper we give an overview of results about two types of an isoperimetric Inequalities on its eigenvalues and the eigenvalues of the laplacian. We estimate the isoperimetric type constant $\Phi(M_m)$ of 2-dimentional Riemannian manifold M_m . if D be the diameter of compact Riemannian manifold M , $\alpha(n)$ is the volume of the unit n -dimensional sphere, and $i(M)$ be an injectivity radius of M , we prove that $\frac{\text{Vol}(M)}{D} > \frac{\alpha(2)^3}{54\alpha(3)^2} i(M)^2$.

Keywords: Riemannian manifold, eigenvalues, compact Riemannian manifold, Laplacian

1. Introduction

In this paper we consider sharp isoperimetric inequalities

$$\frac{\text{Vol}(\partial M)}{\text{Vol}(M)} \geq \frac{2\pi\alpha(n-1)}{\alpha(n) \cdot D} \tilde{\omega}, \quad (1)$$

$$\frac{\text{Vol}(\partial M)^n}{\text{Vol}(M)^{n-1}} \geq \frac{2^{n-1}\alpha(n-1)^n}{\alpha(n)^{n-1}} \tilde{\omega}^{n+1}, \quad (2)$$

where M^n is a compact Riemannian manifold with boundary ∂M and diameter D , and $\tilde{\omega}$ is a constant depending on M . For a history of isoperimetric inequalities see [7].

In general the constant $\tilde{\omega}$ is hard to compute, but in some interesting cases it can be estimated.

For example, we consider the following case. Let N^n be a compact manifold without boundary. Define the isoperimetric type constants

$$I(N) = \inf_S \frac{\text{Vol}(S)}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}},$$

$$\Phi(N) = \inf_S \frac{[\text{Vol}(S)]^n}{[\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}]^{n-1}}$$

where S runs over codimension one submanifolds of N which divide N into two pieces M_1 , and M_2 .

[4], shows that the first eigenvalue of the Laplacian of N , $\lambda_1(N)$, can be bounded below in terms of $I(N)$. [9], shows that $I(N)$ [and hence $\lambda_1(N)$] can be bounded below by the diameter, volume, and Ricci curvature of N . In this section we reproduce Yau's result, with a slightly better constant, and show that in the two dimensional case $I(N)$ can be bounded below by the volume and injectivity radius of N .

In [6] Peter Li ,uses $\Phi(N)$ to get lower bounds for the higher eigenvalues of the Laplacian, for forms as well as functions, and upper bounds on their multiplicities. We show that $\Phi(N)$ can also be bounded below by the volume, diameter, and Ricci curvature of N , while in the two dimensional case it can be bounded by the volume and injectivity radius of N .

Another case where one can estimate $\tilde{\omega}$ is where M is contained in a compact manifold N without boundary, and the diameter of M is less than the injectivity radius of N . In this case $\tilde{\omega} = 1$, so the isoperimetric inequality (2) is in terms only of the dimension of M . As a consequence we show that the volume of a metric ball of radius r in N , where r is less than or equal to one half the injectivity radius of N , is bounded below by a constant times r^n , where the constant depends only on the dimension of N .

We next turn our attention to universal upper and lower bounds on the first eigenvalue, λ_1 , of the Dirichlet problem for the Laplacian.

We prove a sharp lower bound for $\lambda_1(M)$ where M is a sufficiently nice compact manifold with boundary. In particular, if M is contained in a compact manifold N without boundary, and the diameter D of M is less than the injectivity radius of N , then $\lambda_1(M) \geq \lambda_1(S_D^+)$ where S_D^+ is a hemisphere of the constant curvature sphere of diameter D . Further equality holds if and only if M is isometric to S_D^+ . Cheng [10] has independently shown a universal bound for such M ; however, his bound is not sharp.

We then show that there is a constant $\gamma(n)$ depending only on n such that for every compact manifold N^n without boundary of convexity radius $c(N)$, for every $m \in N$ and every $r < c(N)$ we have

$$\lambda_1(B(m, r)) \geq \frac{\gamma(n)\text{Vol}(N)^2}{r^{2n+2}},$$

Where $B(m, r)$ is the metric ball of radius r about m . This allows us to show

$$\lambda_1(N) \geq \frac{\gamma(n)\text{Vol}(N)^2}{c(N)^{2n+2}}.$$

The proof of this result borrows much from the proof in [2]. [2] shows that there is a constant $\bar{\gamma}(n)$ depending only on the dimension n of N such that for every r less than the injectivity radius of N there is a point $m \in N$ such that

$$\lambda_1(B(m, r)) \geq \frac{\bar{\gamma}(n)\text{Vol}(N)}{r^{n+2}}.$$

Using this we get an upper bound for $\lambda_1(N)$ under the assumption that N admits a fixed point free involutive isometry.

Let $(M, \partial M, g)$ be a smooth compact manifold M with smooth boundary ∂M and Riemannian metric g .

Let $UM \xrightarrow{\pi} M$ represent the unit sphere bundle with the canonical measure. For $v \in UM$ let γ_v be the geodesic with $\gamma'_v(0) = v$, let $\zeta^t(v)$ represent the geodesic flow, i.e. $\zeta^t(v) = \gamma'_v(t)$. Let $l(v)$ be the smallest value of $t > 0$ (possibly ∞) such that $\gamma_v(t) \in \partial M$. Note $\zeta^t(v)$ is defined for $t \leq l(v)$. Let $\tilde{l}(v) = \sup\{t | \gamma_v \text{ minimizes up to } t \text{ and } t \leq l(v)\}$.

Now let the subsets $\tilde{U}M \subset \bar{U}M \subset UM$ be defined by

$$\bar{U}M = \{v \in UM | l(-v) < \infty\},$$

$$\tilde{U}M = \{v \in UM | \tilde{l}(-v) = l(-v)\}.$$

Let $\bar{U}_p = \pi|_{\bar{U}M}^{-1}(p)$ and $\tilde{U}_p = \pi|_{\tilde{U}M}^{-1}(p)$. Define $\bar{\omega}_p = m(\bar{U}_p)/m(U_p)$ and $\tilde{\omega}_p = m(\tilde{U}_p)/m(U_p)$ where m represents the canonical measure on the unit sphere. Also let $\bar{\omega} = \inf_{p \in M} \bar{\omega}_p, \tilde{\omega} = \inf_{p \in M} \tilde{\omega}_p$.

For $p \in \partial M$ let N_p be the inwardly pointing unit normal vector. Let $U^+ \partial M \rightarrow \partial M$ be the bundle of inwardly pointing unit vectors. That is

$$U^+ \partial M = \{u \in UM |_{\partial M} \langle u, N_{\pi(u)} \rangle \geq 0\}.$$

Let $U^+ \partial M$ have the local product measure, where the measure on the fibre is the measure from the upper unit hemisphere.

We will let $\alpha(n)$ represent the volume of the unit n -sphere.

Proposition 1: For $(M, \partial M, g)$ we have:

$$\int_{\bar{U}M} f(v) dv = \int_{U^+ \partial M} \int_0^{l(u)} f(\zeta^r(u)) \langle u, N_{\pi(u)} \rangle dr du; \quad (3)$$

$$\int_{\tilde{U}M} f(v) dv = \int_{U^+ \partial M} \int_0^{\tilde{l}(u)} f(\zeta^r(u)) \langle u, N_{\pi(u)} \rangle dr du. \quad (4)$$

Where f is any integrable function. In particular for $f \equiv 1$ we have:

$$\text{Vol}(\bar{U}M) = \int_{U^+ \partial M} l(u) \langle u, N_{\pi(u)} \rangle du; \quad (5)$$

$$\text{Vol}(\tilde{U}M) = \int_{U^+ \partial M} \tilde{l}(u) \langle u, N_{\pi(u)} \rangle du. \quad (6)$$

This formula occurs in [9].

Corollary 2:

$$\frac{\text{Vol}(\partial M)}{\text{Vol}(M)} \geq \frac{C_1 \bar{\omega}}{l}; \quad (7)$$

$$\frac{\text{Vol}(\partial M)}{\text{Vol}(M)} \geq \frac{C_1 \tilde{\omega}}{D}, \quad (8)$$

where $C_1 = 2\pi\alpha(n-1)/\alpha(n), l = \sup_{v \in U^+ \partial M} \{l(v)\}$ and D is the diameter of M .

Note. The inequalities are both sharp when M in the upper hemisphere of a constant curvature sphere. In this case $\bar{\omega} = \tilde{\omega} = 1$ and $l = D = \text{diameter of the sphere}$.

Lemma 3: Let M be a compact Riemannian manifold without boundary, such that the Ricci curvature is bounded

below by $(n-1)K$. Then if \mathbb{S} is any $n-1$ dimensional submanifold dividing M into two pieces M_1 and M_2 we have

$$\tilde{\omega}_i \geq \frac{\text{Vol}(M_j)}{\alpha(n-1) \int_0^D (\sqrt{-1/K} \sinh \sqrt{-K}r)^{n-1} dr} \quad (i \neq j).$$

In particular if $\text{Vol}(M_i) \leq \text{Vol}(M_j)$ then

$$\tilde{\omega}_i \geq \frac{\text{Vol}(M)}{2\alpha(n-1) \int_0^D (\sqrt{-1/K} \sinh \sqrt{-K}r)^{n-1} dr}$$

where we use the convention that $(\sqrt{-1/K} \sinh \sqrt{-K}r)$ is interpreted as r if $K = 0$ and as $\sqrt{1/K} \sin(Kr)$ if $K > 0$. D represents the diameter of M .

Proof

$$\text{Vol}(M_j) \leq \text{Vol}(O_p) = \int_{-\tilde{U}_p} \int_0^{c(u)} F(u, r) dr du$$

$$\leq \tilde{\omega}_p \alpha(n-1) \int_0^D (\sqrt{-1/K} \sinh \sqrt{-K}r)^{n-1} dr.$$

For the inequality $F(u, r) \leq (\sqrt{-1/K} \sinh \sqrt{-K}r)^{n-1}$, see [11], where

$$-\tilde{U}_p = \{u \in U_p | -u \in \tilde{U}_p\} = \{u \in U_p | l(u) = \tilde{l}(u)\},$$

And $F(u, r)$ is the volume form in normal polar coordinates. Now from Corollary 2 we have $\text{Vol}(\mathbb{S})/\text{Vol}(M_i) \geq C_1 \tilde{\omega}_i/D$ (where C_1 is sharp). Thus using Lemma (1.2.4) we get (see Yau [6])

Proposition 4:

$$I(M) \geq \frac{\pi}{\alpha(n)} \frac{\text{Vol}(M)}{D \int_0^D (\sqrt{-1/K} \sinh \sqrt{-K}r)^{n-1} dr}$$

($I(M)$ was defined on page 1.)

Theorem 5: Let M be a compact n -dimensional Riemannian manifold whose Ricci curvature is bounded below by $(n-1)K$. Thus Proposition 4 holds. Since $\lambda_1(M) \geq I(M)^2/4$ we find a lower bound of λ_1 in terms of $D, \text{Vol}(M)$, and K . In some cases we are able to show that ω must be 1. For example let M be a compact manifold without boundary and let \mathbb{S} be an $n-1$ dimensional submanifold dividing M into M_1, M_2 then we have:

Lemma 6: If the maximum distance in M between any two points of \mathbb{S} is less than the injectivity radius of $M, i(M)$, then $\tilde{\omega}_i = 1$ for $i = 1$ or 2 .

Proof. Let $p \in \mathbb{S}$ then

$$\mathbb{S} \subset B(p, i(M)) \equiv \{q \in M | d(p, q) < i(M)\}.$$

Let M_i be the piece of M lying entirely inside $B(p, i(M))$. By continuity this choice is independent of the choice of p . Now for $x \in M, d(x, p) < i(M)$ for every $p \in \mathbb{S}$, by the choice of M_i . Hence $\mathbb{S} \subset B(x, i(M))$. Let M_j be the piece of M lying in $B(x, i(M))$. By continuity M_j is independent of x and hence must be M_i . Thus every geodesic from x minimizes up to \mathbb{S} . Hence $\tilde{\omega}_i = 1$.

If M is a compact manifold without boundary and $r < i(M)$, let $B(x, r) = \{y \in M | d(x, y) \leq r\}$ and $\mathbb{S}(x, r) = \partial B(x, r) = \{y \in M | d(x, y) = r\}$.

Then Lemma 6 and Corollary 2 give:

Corollary 7:

For $r < i(M)/2$:

$$\frac{\text{Vol}(\mathbb{S}(x, r))}{\text{Vol}(B(x, r))} \geq \frac{C_1}{2r} = \frac{\pi\alpha(n-1)}{r\alpha(n)}.$$

If M is a two dimensional compact manifold without boundary and \mathbb{S} divides M into two pieces M_1, M_2 we can consider separately the cases where the length of $\mathbb{S} \geq 2i(M)$ and length of $\mathbb{S} < 2i(M)$ to get:

Corollary 8: For M a compact 2-dimensional manifold

$$I(M) \geq \min \left\{ \frac{4i(M)}{\text{Vol}(M)}, \frac{C_1}{i(M)} \right\}.$$

Hence λ_1 can be bounded below by $i(M)$ and $\text{Vol}(M)$.

Lemma 9: Let M^n be a Riemannian manifold and $u \in UM$. Then for every $l \leq C(u)$ (the distance to the cut locus in the direction u):

$$\int_{x=0}^{x=l} \int_{z=0}^{z=l-x} F(\zeta^x(u), z) dz dx \geq C(n) \frac{l^{n+1}}{\pi^{n+1}},$$

Where $C(n) = \pi\alpha(n)/2\alpha(n-1) = \pi^2/C_1$ Further equality holds if and only if

$$R(\gamma'_u(t), \cdot) \gamma'_u(t) = (\pi/l)^2 \text{Id} \text{ for } 0 \leq t \leq l.$$

Here $F(v, z)$ is the volume form in polar coordinates

$$\left[\text{i. e. } \int_{U_p} \int_0^{C(v)} F(v, z) dz dv = \text{Vol}(M) \right],$$

R is the curvature tensor and γ_u is the geodesic determined by u .

This follows from a slight modification of [3].

Proposition 10: For $(M, \partial M, g)$ we have

$$\text{vol}(M)^2 \geq C_2 \int_{U+\partial M} (\tilde{l}(v))^{n+1} \langle v, N_{\pi(v)} \rangle dv,$$

with $C_2 = \alpha(n)/2\pi^n\alpha(n-1)$. Equality holds for the upper hemisphere of a constant curvature sphere.

Proof.

$$\begin{aligned} \text{Vol}(M)^2 &\geq \int_M \int_{U_p} \int_0^{\tilde{l}(u)} F(u, t) dt dudp = \\ &\int_{UM} \int_0^{\tilde{l}(u)} F(u, t) dt du \\ &\geq \int_{\tilde{U}M} \int_0^{\tilde{l}(u)} F(u, t) dt du \quad (9) \\ &= \int_{U+\partial M} \int_0^{\tilde{l}(v)} \int_0^{\tilde{l}(\zeta^{\mathbb{S}}(v))} F(\zeta^{\mathbb{S}}(v), t) \langle v, N_{\pi(v)} \rangle dt ds dv \\ &\geq \int_{U+\partial M} \left[\int_0^{\tilde{l}(v)} \int_0^{\tilde{l}(v)-\mathbb{S}} F(\zeta^{\mathbb{S}}(v), t) dt ds \right] \langle v, N_{\pi(v)} \rangle dv \\ &\geq \frac{C(n)}{\pi^{n+1}} \int_{U+\partial M} (\tilde{l}(v))^{n+1} \langle v, N_{\pi(v)} \rangle dv. \quad (10) \end{aligned}$$

The above follows from Proposition 1, Lemma 9, and the fact that $\tilde{l}(\zeta^{\mathbb{S}}(v)) \geq \tilde{l}(v) - \mathbb{S}$. Equality holds for the upper hemisphere of a sphere at each stage.

Theorem 11: For $(M, \partial M, g)$ we have the isoperimetric inequality:

$$\frac{\text{vol}(\partial M)^n}{\text{vol}(M)^{n-1}} \geq C_3 \tilde{\omega}^{n+1},$$

where $C_3 = 2^{n-1}\alpha(n-1)^n/\alpha(n)^{n-1}$.

Equality holds if and only if $\tilde{\omega} = 1$ and M is the upper hemisphere of a constant curvature sphere.

Proof: From Proposition 10 and a Hölder inequality we have

$$\begin{aligned} \text{vol}(M)^2 &\geq C_2 \int_{U+\partial M} (\tilde{l}(u))^{n+1} \langle u, N_{\pi(u)} \rangle du \\ &\geq C_2 \frac{\left\{ \int_{U+\partial M} \tilde{l}(u) \langle u, N_{\pi(u)} \rangle du \right\}^{n+1}}{\left\{ \int_{U+\partial M} \langle u, N_{\pi(u)} \rangle du \right\}^n}, \quad (11) \end{aligned}$$

using Proposition 1 we have

$$\begin{aligned} \text{vol}(M)^2 \cdot \left\{ \int_{U+\partial M} \langle u, N_{\pi(u)} \rangle du \right\}^n &\geq C_2 \text{vol}(\tilde{U}M)^{n+1} \\ &\geq C_2 [\tilde{\omega}\alpha(n-1)\text{vol}(M)]^{n+1} \end{aligned}$$

giving

$$\frac{\text{Vol}(\partial M)^n}{\text{Vol}(M)^{n-1}} \geq C_3 \tilde{\omega}^{n+1}.$$

To compute C_3 one need only note that equality holds everywhere for upper hemisphere.

To order for equality to hold we must have equality in (9), (10) and (11). Equality in (11) implies $\tilde{l}(v)$ is a constant l almost everywhere in $U+\partial M$. Equality in (9) implies $\tilde{\omega} = 1$. Equality in (10) implies equality in Lemma 9. Thus we see that M must have constant curvature equal to $(\pi/l)^2$.

For p an interior point of M , $x \in \mathbb{S}^n$, the sphere of curvature $(\pi/l)^2$, and $I: T_p M \rightarrow T_x \mathbb{S}^n$ an isometry, we see that $\text{Exp}_{p_x} \circ I \circ \text{Exp}_p^{-1}: M \rightarrow \mathbb{S}^n$ must be an isometry by [5]. To see that the image is a hemisphere one need only look at $q \in \partial M$ and note that $\tilde{l}(q) = l(q) = l$.

The equality condition only says that the upper hemisphere minimizes $\text{Vol}(\partial M)^n/\text{Vol}(M)^{n-1}$ over spaces $(M, \partial M, g)$ with $\tilde{\omega} = 1$.

Consider M a compact Riemannian manifold without boundary, and \mathbb{S} a codimension one submanifold dividing M into two pieces M_1 and M_2 . If the maximum distance in M between any two points of \mathbb{S} is less than the injectivity radius, then we can combine Lemma 6 and Theorem 11 to get

$$\frac{\text{Vol}(\mathbb{S})^n}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}^{n-1}} \geq C_3 = \frac{2^{n-1}}{\alpha(n)^{n-1}}.$$

Using this in the case that M is two dimensional we see:

Proposition 12: Let M be a compact 2-dimensional Riemannian manifold then: $\Phi(M) \geq 8i(M)^2/\text{Vol}(M)$, which is sharp for a constant curvature sphere.

Proof: Since $n = 2$ we can assume that \mathbb{S} is a smooth closed curve of length l . if $l \geq 2i(M)$ then

$$\begin{aligned} \frac{\text{Vol}(\mathbb{S})^n}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}^{n-1}} &\geq \frac{4(i(M))^2}{\frac{\text{Vol}(M)}{2}} \\ &= \frac{8i(M)^2}{\text{Vol}(M)}. \end{aligned}$$

If $l < 2i(M)$ then by the above

$$\begin{aligned} \frac{\text{Vol}(\mathbb{S})^n}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}^{n-1}} &\geq \frac{2(2\pi)^2}{4\pi} = 2\pi. \end{aligned}$$

Now in [1], and [10], show $\text{Vol}(M) \geq 4i(M)^2/\pi$ Thus

$$2\pi \geq 8i(M)^2/\text{Vol}(M).$$

For $n \geq 2$ we need only combine Theorem 11 with Lemma 3 to get:

Theorem 13:

$$\Phi(M) \geq C_4 \left(\frac{\text{Vol}(M)}{\int_0^D (\sqrt{-1/K} \sinh \sqrt{-Kr})^{n-1} dr} \right)^{n+1},$$

with the same convention as Lemma 3 for $K \geq 0$. $C_4 = 1/4\alpha(n-1)\alpha(n)^{n-1}$.

Now Proposition 12 and Theorem 13 can be applied to the results of [6]. Thus we get a lower bound on the higher eigenvalues of M as well as upper bounds on their multiplicities in terms of the volume of M , the diameter of M , and a lower bound on the Ricci curvature of M .

Note that :For $(M, \partial M, g)$ we can consider

$$\Phi(M) = \inf_{\mathbb{S}} \frac{\text{Vol}(\mathbb{S})^n}{\min\{\text{Vol}(M_1), \text{Vol}(M_2)\}^{n-1}},$$

Where \mathbb{S} moves over submanifolds dividing M into two pieces M_1 and M_2 ($\mathbb{S} \cap \partial M$ not necessarily empty). If for given \mathbb{S} we let $\tilde{U}M$ be the set of vectors whose geodesics minimize up to the point they intersect \mathbb{S} , and define $\tilde{\omega}$ analogously, then the same method will give an isoperimetric inequality. If M is geodesically convex, then an argument similar to Lemma 3 will put a lower bound on $\tilde{\omega}$. This will give a lower bound on $\Phi(M)$.

Let M be a compact Riemannian manifold without boundary. Define

$$r_p(M) = \inf\{0 < r | (B(p, r), \mathbb{S}(p, r), g) \text{ has } \tilde{\omega} > 1\}.$$

Since $\tilde{\omega} = 1$ is equivalent to the statement that the cut locus to any interior point of $B(p, r)$ lies outside $B(p, r)$, we see that $r_p(M) \geq i(M)/2$ for all $p \in M$.

Corollary 14: Let M_m be a compact 2-dimensional Riemannian manifold then: $\Phi(M_m) \geq 8i(M_m)^2/\text{Vol}(M_m)$, which is sharp for a constant curvature sphere [11].

Proof. Since $n = 2$ we can assume that \mathbb{S} is a smooth closed curve of length l . if $l \geq 2i(M_m)$ then

$$\begin{aligned} & \frac{\text{Vol}(\mathbb{S})^n}{\min\{\text{Vol}(M_{m+1}), \text{Vol}(M_{m+2})\}^{n-1}} \\ & \geq \frac{4(i(M_m))^2}{\frac{\text{Vol}(M_m)}{2}} \\ & = \frac{8i(M_m)^2}{\text{Vol}(M_m)}. \end{aligned}$$

If $l < 2i(M_m)$ then by the above

$$\begin{aligned} & \frac{\text{Vol}(\mathbb{S})^n}{\min\{\text{Vol}(M_{m+1}), \text{Vol}(M_{m+2})\}^{n-1}} \\ & \geq \frac{2(2\pi)^2}{4\pi} = 2\pi. \end{aligned}$$

Shows that $\text{Vol}(M_m) \geq 4i(M_m)^2/\pi$ see [1],[10]. Thus $2\pi \geq 8i(M_m)^2/\text{Vol}(M_m)$.

Proposition 15 : For $r \leq r_p$ (or in particular $r \leq i(M)/2$) we have

$$\begin{aligned} \text{Vol}(B(p, r)) & \geq \frac{C_3}{n^n} r^n, \\ \text{Vol}(\mathbb{S}(p, r)) & \geq \frac{C_3}{n^{n-1}} r^{n-1}, \end{aligned}$$

in particular

$$\text{Vol}\left(B\left(p, \frac{i(M)}{2}\right)\right) \geq \frac{\alpha(n-1)^n}{2n^n \alpha(n)^{n-1}} i(M)^n.$$

Proof. By Theorem 11 for $0 < t \leq r$:

$$\frac{\text{Vol}(\mathbb{S}(p, t))}{\text{Vol}(B(p, r))^{(n-1)/n}} \geq C_3^{1/n}$$

integrating both sides with respect to t yields

$$n \cdot \text{Vol}(B(p, r))^{1/n} \geq C_3^{1/n} r.$$

This gives the first statement. The second follows from Theorem 11 and the first statement.

This relates to a question of Berger. Berger is interested in bounding the volume of a compact manifold from below in terms of the injectivity radius. In [3], he proves that $\text{Vol}(M) \geq (1/2)(\alpha(n)/\pi^n)i(M)^n$. Proposition 15 can be considered as a local version of this result. One has from Proposition 15 that

$$\text{Vol}(M) \geq \text{Cat}(M) \cdot \frac{C_3}{2^n n^n} i(M)^n,$$

Where $\text{Cat}(M)$ is the topological category of M (i.e., the number of topological n -balls needed to cover M). To see this one need only note that for every $x \in M$, $B(x, i(M))$ (open) is a topological n -ball, then choose $x_1 \in M$, choose $x_2 \in M - B(x_1, i(M))$, in general choose $x_i \in M - \bigcup_{j=1}^{i-1} B(x_j, i(M))$; by the definition of $\text{Cat}(M)$ we can choose at least $\text{Cat}(M)$ such x_i . Now for $j \neq i$, $d(x_i, x_j) > i(M)$ hence $B(x_i, i(M)/2) \cap B(x_j, i(M)/2) = \emptyset$. Hence Proposition 15 gives the result.

Proposition 15 also allows us to get good lower bounds on $\text{Vol}(M)$ when $r_p(M)$ is large for some p even though the injectivity radius may be small. Another consequence is:

Corollary 16 : Let M be a compact Riemannian manifold then

$$\frac{\text{Vol}(M)}{D} > \frac{\alpha(n-1)^n}{2^n n^n \alpha(n)^{n-1}} i(M)^{n-1}.$$

Proof. Let I be the integer such that $I + 1 > D/i(M) \geq I \geq 1$. Let γ be a minimizing geodesic from p to q in M of length D . Choose points $p = x_0, x_1, x_2, \dots, x_I = q$ along γ such that $d(x_i, x_{i+1}) \geq i(M)$. Then the geodesic balls $B(x_i, i(M)/2)$ will be disjoint and have volume $\geq (\alpha(n-1)^n / 2^n n^n \alpha(n)^{n-1}) i(M)^n$. Thus

$$\begin{aligned} \text{Vol}(M) & \geq (I + 1) \frac{\alpha(n-1)^n i(M)^n}{2^n n^n \alpha(n)^{n-1}} \\ & \geq \frac{D}{i(M)} \frac{\alpha(n-1)^n i(M)^n}{2^n n^n \alpha(n)^{n-1}}. \end{aligned}$$

Corollary 17: Let M be a compact Riemannian manifold [11] then

$$\frac{\text{Vol}(M)}{D} > \frac{\alpha(2)^3}{54\alpha(3)^2} i(M)^2.$$

Proof. Let $1 + \epsilon$ be the integer such that $2 + \epsilon > D/i(M) \geq 1 + \epsilon$. Let γ be a minimizing geodesic from p to q in M of length D . Choose points $p = x_0, x_1, x_2, \dots, x_{1+\epsilon} = q$ along γ such that $d(x_i, x_{i+1}) \geq i(M)$. Then the geodesic balls $B(x_i, i(M)/2)$ will be disjoint and have volume $\geq (\alpha(2)^3 / 54\alpha(3)^2) i(M)^3$. Thus

$$\begin{aligned} \text{Vol}(M) &\geq (2+\epsilon) \frac{\alpha(2)^3 i(M)^3}{54\alpha(3)^2} \\ &\geq \frac{D}{i(M)} \frac{\alpha(2)^3 i(M)^3}{54\alpha(3)^2}. \end{aligned}$$

We prove the following lower bound for the first eigenvalue of the Dirichlet problem for the Laplacian.

Corollary 18: Let N be a complete Riemannian manifold of injectivity radius $i(N)$. Then for every $m \in N$ and every $r \geq i(N)/2$ we have $\lambda_1(B(m, r)) \geq \lambda_1(S_{2r}^+)$, with equality holding if and only if $B(m, r)$ is isometric to S_{2r}^+ . [in which case $r = i(N)/2$].

Theorem 19: Let $(M, \partial M, g)$ be a compact Riemannian manifold with boundary such that every geodesic ray in M intersects ∂M . (i.e., $\bar{\omega} = 1$). Let l be the maximum length of any geodesic (from boundary point to boundary point). Then we have $\lambda_1(M) \geq \lambda_1(S_l^+)$. If further every geodesic ray minimizes distance up to the point that it intersects the boundary (i.e., $\tilde{\omega} = 1$), then equality holds if and only if M is isometric to S_l^+ .

Proof . By the minimum principle we need only show that

$$\frac{\int_M |\nabla f|^2 dm}{\int_M f^2 dm} \geq \lambda_1(S_l^+)$$

for all f such that $f|_{\partial M} = 0$.

We first note that

$$|\nabla f(p)|^2 = \frac{n}{\alpha(n-1)} \int_{U_p} (vf)^2 dv,$$

Where vf represents differentiation.

Using this, Proposition 1 (with $\overline{UM} = UM$) and the one dimensional version:

$$\int_0^a f'(t)^2 dt \geq \frac{\pi^2}{a^2} \int_0^a f(t)^2 dt,$$

$$f(0) = 0, f(a) = 0,$$

with equality if and only if $f(t) = A \sin((\pi/a)t)$, we see

$$\begin{aligned} \int_M |\nabla f|^2 dm &= \frac{n}{\alpha(n-1)} \int_{UM} (vf)^2 dv \\ &= \frac{n}{\alpha(n-1)} \int_{U+\partial M} \int_0^{l(u)} [(\zeta^t(u))f]^2 dt \langle u, N_{\pi(u)} \rangle du \\ &\geq \frac{n}{\alpha(n-1)} \int_{U+\partial M} \frac{\pi^2}{l(u)^2} \int_0^{l(u)} [f(\pi(\zeta^t(u)))]^2 dt \langle u, N_{\pi(u)} \rangle du \\ &\geq \frac{n\pi^2}{\alpha(n-1)l^2} \int_{U+\partial M} \int_0^{l(u)} [f(\pi(\zeta^t(u)))]^2 dt \langle u, N_{\pi(u)} \rangle du \\ &= \frac{n\pi^2}{\alpha(n-1)l^2} \int_{UM} [f(\pi(v))]^2 dv \\ &= \frac{n\pi^2}{l^2} \int_M f^2 dm = \lambda_1(S_l^+) \int_M f^2 dm \end{aligned}$$

Now we assume that equality holds. Equality holds if and only if:

(a) $l = l(u)$ for every $u \in U^+ \partial M$ and

(b) $f(\gamma_u(t)) = A(u) \sin(\pi/l)t$ for all $u \in U^+ \partial M$, where $\gamma_u(t)$ represents the geodesic with initial tangent vector u and $A(u)$ is a constant depending on u .

By scaling we may assume that $\sup(f) = 1$. Let $m \in M$ be such that $f(m) = 1$. Then if γ is any geodesic through m (parameterized from boundary point to boundary point), m

will take on the maximum value of f hence $m = \gamma(l/2)$. Thus it is not hard to see:

(i) M is the metric ball of radius $l/2$ around m and

$$\partial M = \{q \in M | d(m, q) = l/2\}.$$

(ii) $f(q) = \cos[\pi(d(p, q))/l]$ for all $q \in M$.

(iii) $A(u) = \langle u, N_{\pi(u)} \rangle$ for all $u \in U^+ \partial M$.

Let $u \in T_q \partial M, q \in \partial M$. By continuity $\gamma_u(t)$ is defined (i.e. lies in M) for $0 \leq t \leq l$. Since $A(u) = \langle u, N_{\pi(u)} \rangle = 0$ we see that $f(\gamma_u(t)) = 0$ for all $t \leq l$. Hence $\gamma_u(t) \in \partial M$ for $0 \leq t \leq l$. Thus ∂M is totally geodesic.

For $q \in \partial M$ we let \tilde{q} represent the (antipodal) point $\gamma_{N_q}(l) \in \partial M$. We now assume (as in the statement of the Theorem) that every geodesic minimizes length up to the point it intersects ∂M . As M is the metric ball of radius $l/2$ around m the unique point of distance l from q is \tilde{q} . Hence if γ is any geodesic from q we have $\gamma(l) = \tilde{q}$. Hence this holds for geodesics in ∂M . Hence the metric on ∂M is that of a Blaschke structure on a sphere. Hence by [3] ∂M is isometric to the constant curvature sphere ∂S_l^+ . In particular $\text{Vol}(\partial M) = \text{Vol}(\partial S_l^+)$. Now using the assumptions of the Theorem, the fact that $l(u) = l$, and the proof of Corollary 2 we see that

$$\frac{\text{Vol}(\partial M)}{\text{Vol}(M)} = \frac{\text{Vol}(\partial S_l^+)}{\text{Vol}(S_l^+)}.$$

Thus

$$\frac{\text{Vol}(\partial M)^n}{\text{Vol}(M)^{n-1}} = \frac{\text{Vol}(\partial S_l^+)^n}{\text{Vol}(S_l^+)^{n-1}} = C_3.$$

Now the fact that every geodesic minimizes up to ∂M combined with Theorem 13 gives M is isometric to S_l^+ .

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