International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2014): 5.611

On The Ternary Cubic Diophantine Equation $5(x^2 + y^2) – 6xy + 4(x + y) + 4 = 40z^3$

Dr. G. Janaki[1], P. Saranya[2]

[1]Assistant Professor, Department of Mathematics, Cauvery College for Women, Trichy – 18, India

[2]Assistant Professor, Department of Mathematics, Cauvery College for Women, Trichy – 18, India

Abstract: ***The ternary cubic Diophantine equation*** $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$ ***is analyzed for its non-zero distinct integer solutions. Five different patterns of integral solutions are obtained. Few interesting relations among the solutions and some special polygonal numbers are presented.***

Keywords: Ternary cubic, Diophantine equations, Integral solutions

Mathematical Classification: 11D25

1. Introduction

Ternary cubic equations are rich in variety. For more detailed understanding one can see [1-7]. For the non-trivial integral solutions of ternary cubic Diophantine equations [8-9] has been studied. [10-12] has been referred for various ternary cubic Diophantine equations.

In this communication, we consider yet another interesting ternary cubic equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$ and obtain different patterns of non-trivial integral solutions. Also, a few interesting relations among the solutions and special polygonal, Centered, Gnomonic, Rhombic, Pronic numbers are presented.

2. Notations

$T_{m,n}$ = Triangular Number of rank n.

P_m^n = Pyramidal Number of rank n.

$P(n)$ = Pronic Number of rank n.

$Star_n$ = Star Number of rank n.

CH_n = Centered Hexagonal Number of rank n.

RD_n = Rhombic Dodecagonal Number of rank n.

Gno_n = Gnomonic Number of rank n.

3. Method of Analysis

Consider the equation

$$5(x^2+y^2)-6xy+4(x+y)+4=40z^3 \tag{1}$$

The substitution of the linear transformation

$$x=u+v;\ y=u-v \tag{2}$$

in (1) give $(2u+2)^2+(4v)^2=40z^3$ (3)

We present below different methods of solving (3) and thus obtain different choices of integer solutions of **(1)**

Pattern-1:

Let $z=a^2+b^2$ (4)

Write $40=(6+2i)(6-2i)$ (5)

Using (4) and (5) in (3) and using factorization method we have,

$$(2u+2+i4v)(2u+2-i4v)=(6+2i)(6-2i)[(a+ib)(a-ib)]^3$$

Equating real and imaginary we get

$$u=\frac{1}{2}[6a^3+2b^3-18ab^2-6a^2b\ -2]$$

$$v=\frac{1}{2}[a^3-3b^3-3ab^2+9a^2b\ ]$$

Since our aim is to get integer solutions, we take a=2A, b=2B.

Employing the values of u and v in (2) we get

$$x=28A^3-4B^3-84AB^2+12A^2B-1$$

$$y=20A^3+20B^3-60AB^2-60A^2B-1$$

$$z=4A^2+4B^2$$

Observations

1. $A[z(A,A)],-7x(A,A)$ and $-9y(A,A)$ are cubic integers
2. $6[P_A^{19}+P_A^{13}+T_{10,A}+T_{6,A}-29Gno_A-x(A,1)\equiv 0 \pmod{34}$
3. $z(A,A)-y(A,A)+15T_{6,A}+Pr(A)-[18P_A^{19}+60P_A^5+14Gno_A]\equiv 0 \pmod{14}$
4. $z(A,A)-x(A,A)+T_{6,A}-[6P_A^{19}+12P_A^{18}+Pr(A)+19Gno_A]\equiv 0 \pmod{19}$
5. $y(A,1)+64Pr(A)-[6P_A^{19}+2P_A^{11}+10Gno_A]\equiv 0 \pmod{29}$

Pattern-2:

Write (3) as $(2u+2)^2+(4v)^2=40z^3.1$ (6)

and $1=\dfrac{(6+8i)(6-8i)}{10^2}$ (7)

Using (4),(5) and (7) in (6) and proceeding as in pattern 1 we get the non-zero distinct integer solutions of (1) to be

$$x=20000A^3+20000B^3-60000AB^2-60000A^2B-1$$

$$y=-4000A^3+28000B^3+12000AB^2-84000A^2B-1$$

$$z=400A^2+400B^2$$

Observations:

1. $10000[CC_A - CH_A - 6Gno_A] - x(A,1) \equiv 0 \pmod{20001}$
2. $y(1,B) - 1000[7RD_B + 3CH_B - 8Gno_B] \equiv 0 \pmod{1999}$
3. $1000[12P_B^{16} + 2CH_B - 28GnO_B] - [y(B,1)] \equiv 0 \pmod{34001}$
4. The expression $12z(A,A)$ represents a nasty number
5. $100[120P_B^{16} + 32T_{6,B} - 294Gno_B - [y(B,1) + z(B,1)] \equiv 0 \pmod{33001}$

Pattern-3:

$$\text{Write } 1 = \frac{(4+3i)(4-3i)}{5^2} \tag{8}$$

Using (4),(5) and (8) in (6) and proceeding as in pattern 2 we get the non-zero distinct integer solutions of (1) to be

$$x = 3100A^3 + 1700B^3 - 9300AB^2 - 5100A^2B - 1$$
$$y = 500A^3 + 3500B^3 - 1500AB^2 - 10500A^2B - 1$$
$$z = 100A^2 + 100B^2$$

Observations:

1. $100[6P_A^{19} + 6P_A^{16} - 2T_{19,A} - T_{82,A}] - 6100Gno_A - x(A,1) \equiv 0 \pmod{7799}$
2. $y(A,1) - 100[2P_A^{17} + T_{26,A} - 118\Pr(A)] + 5900Gno_A \equiv 0 \pmod{9399}$
3. $z(A,1) + 50Gno_A - 100\Pr(A) \equiv 0 \pmod{50}$
4. $100[12P_A^{15} + 24T_{6,A} - 17Gno_A] + y(A,1) - x(A,1) \equiv 0 \pmod{6898}$
5. $100[6(P_A^{19} + P_A^{16}) - 28T_{6,A} - 48Gno_A] - [x(A,1) + z(A,1)] \equiv 0 \pmod{6399}$

Pattern-4:

$$\text{Write } 1 = \frac{(11+60i)(11-60i)}{61^2} \tag{9}$$

Using (4),(5) and (9) in (6) and proceeding as in pattern3 we get the non-zero distinct integer solutions of (1) to be

$$x = 2039108A^3 + 6087556B^3 - 18262668A^2B - 6117324AB^2 - 1$$
$$y = -3646580A^3 + 5283820B^3 - 15851460A^2B + 10939740AB^2 - 1$$
$$z = 14884A^2 + 14884B^2$$

Observations:

1. $12z(A,A)$ represents a nasty number
2. $-[x((A,A) + y(A,A)] \equiv 0 \pmod{30}$
3. $y(A,A) - x(A,A) \equiv 0 \pmod{32}$
4. $z(A,A) - x(A,A) \equiv 0 \pmod{9}$

Pattern-5:

$$\text{Write } 40 = \frac{(8+4i)(8-4i)}{2} \tag{10}$$

$$1 = \frac{(1+i)(1-i)}{2} \tag{11}$$

Using (4),(10) and (11) in (6) and proceeding as in pattern 4 we get the non-zero distinct integer solutions of (1) to be

$$x = 20A^3 + 20B^3 - 60AB^2 - 60A^2B - 1$$
$$y = -4A^3 + 28B^3 + 12AB^2 - 84A^2B - 1$$
$$z = 4A^2 + 4B^2$$

Observations:

1. $x(A,A).y(A,A)$ is a square number
2. $A[z(A,A)], -9x(A,A)$ and $-7y(A,A)$ are cubic integers
3. $z(A,A) - y(A,A) + T_{6,A} - [6P_A^{19} + 12P_A^{18} + \Pr(A) + 19Gno_A] \equiv 0 \pmod{19}$
4. $z(A,A) - x(A,A) + 15T_{6,A} + \Pr(A) - [18P_A^{19} + 60P_A^{5} + 14Gno_A] \equiv 0 \pmod{14}$
5. $x(A,1) + 64\Pr(A) - [6P_A^{19} + 2P_A^{11} + 10Gno_A] \equiv 0 \pmod{29}$

4. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary cubic Diophantine equation represented by

$$5(x^2 + y^2) - 6xy + 4(x+y) + 4 = 40z^3$$

One can also search for other patterns of solutions for the above equation.

References

[1] Batta.B and Singh.A.N, History of Hindu Mathematics, Asia Publishing House 1938.

[2] Carmichael, R.D., "The Theory of Numbers and Diophantine Analysis", Dover Publications, New York ,1959.

[3] Dickson,.L.E., "History of the theory of numbers", Chelsia Publishing Co., Vol.II,New York, 1952.

[4] Mollin.R.A., "All solutions of the Diophantine equation $x^2 - Dy^2 = n$, ", For East J.Math. Sci., Social Volume, 1998, Part III, pages-257-293.

[5] Mordell.L.J., "Diophantine Equations", Academic Press, London 1969.

[6] Telang.S.G., "Number Theory", Tata McGraw-Hill Publishing Company, New Delhi 1996.

[7] Gopalan.M.A., Manju Somanath, and Vanitha.N., "On Ternary cubic equation $x^2 - y^2 = z^3$", Acta Ciencia Indica, Volume 33M;number 3, pages 705-707

[8] Gopalan MA, Vidhyalakshmi S, Kavitha.S On Ternary cubicequation $x^2 + y^2 - xy = z^3$ Antartica J Math 10(5):453-460

[9] Gopalan.M.A, Geetha.K, On Ternary cubic equation $2xz = y^2(x+z)^3$,Bessel'sJ.Math3(2)119-123

[10] Gopalan MA, Vidhyalakshmi S, Kavitha.S On Ternary cubicequation $x^2 + y^2 + 7xy = 12z^3$ Antartica J Math 10(5):453-460

[11] Gopalan MA, Vidhyalakshmi S, Mallika S,Lattice points on non homogeneous cubic equation $x^3 + y^3 + z^3 - (x+y+z) = 0$, Impact J.Sci Tech vol7,No.1,51-55

[12] Manju Somanath,Sangeetha.V,Gopalan MA, Bhuvaneshwari.M,On the Ternary Cubic Equation $3(x^2 + y^2) - 2xy + 4(x+y) + 4 = 51z^3$

[13] Gopalan MA,and Sivakami.B,2012,Integral solutions of ternary cubic equation $4x^3 - 4xy + 6y^2 = [(k+1)^2 + 5]w^3$, Impact J.Sci Tech,vol6,No.1,15-22

[14] Gopalan MA, Vidhyalakshmi S, Kavitha.S On Ternary cubic equation

$5(x^2+y^2)-9xy+x+y+1=23z^3$
IJIRR,1,(10),99-101.

[15] Gopalan MA, Vidhyalakshmi S, Sumathi.G On Ternary cubic Diophantine equation $x^3+y^3=z(x^2+y^2-20)=4(x+y)^2z$ IJIRR,1,(10),99-101.Impact J.Sci Tech,vol7(2),01-06

Author Profile

Dr. G. Janaki has completed her under graduation and post graduation in Seethalakshmi Ramasamy college Tiruchirapalli, Tamilnadu followed by M. Phil in St. Josephs college Tiruchirapalli Tamilnadu. She has completed her Doctorate in Bharathidasan university, Tiruchirapalli, Tamilnadu. Currently working as Assistant Professor in Cauvery College for Women.

P. Saranya has completed her under graduation and post graduation in Cauvery College for Women, affiliated to Bharathidasan University Tiruchirapalli. Done M.Phil in Vinayaga Missions University Salem. Pursuing PhD in Cauvery College for Women, affiliated to Bharathidasan University Tiruchirapalli. also working as Assistant Professor in Cauvery College for Women.