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On The Ternary Cubic Diophantine Equation $5(x^2 + y^2) – 6xy + 4(x + y) + 4 = 40z^3$

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Abstract: ***The ternary cubic Diophantine equation*** $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$ ***is analyzed for its non-zero distinct integer solutions. Five different patterns of integral solutions are obtained. Few interesting relations among the solutions and some special polygonal numbers are presented.***

Keywords: Ternary cubic, Diophantine equations, Integral solutions

Mathematical Classification: 11D25

1. Introduction

Ternary cubic equations are rich in variety. For more detailed understanding one can see [1-7]. For the non-trivial integral solutions of ternary cubic Diophantine equations [8-9] has been studied. [10-12] has been referred for various ternary cubic Diophantine equations.

In this communication, we consider yet another interesting ternary cubic equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$ and obtain different patterns of non-trivial integral solutions. Also, a few interesting relations among the solutions and special polygonal, Centered, Gnomonic, Rhombic, Pronic numbers are presented.

2. Notations

$T_{m,n}$ = Triangular Number of rank n.

P_m^n = Pyramidal Number of rank n.

$P(n)$ = Pronic Number of rank n.

$Star_n$ = Star Number of rank n.

CH_n = Centered Hexagonal Number of rank n.

RD_n = Rhombic Dodecagonal Number of rank n.

Gno_n = Gnomonic Number of rank n.

3. Method of Analysis

Consider the equation

$$5(x^2+y^2)-6xy+4(x+y)+4=40z^3 \quad (1)$$

The substitution of the linear transformation

$$x = u + v;\ y = u - v \quad (2)$$

in (1) give $(2u+2)^2+(4v)^2=40z^3 \quad (3)$

We present below different methods of solving (3) and thus obtain different choices of integer solutions of **(1)**

Pattern-1:

Let $z = a^2 + b^2 \quad (4)$

Write $40 = (6+2i)(6-2i) \quad (5)$

Using (4) and (5) in (3) and using factorization method we have,

$$(2u+2+i4v)(2u+2-i4v) = (6+2i)(6-2i)[(a+ib)(a-ib)]^3$$

Equating real and imaginary we get

$$u = \frac{1}{2}[6a^3 + 2b^3 - 18ab^2 - 6a^2b \ \ -2]$$

$$v = \frac{1}{2}[a^3 - 3b^3 - 3ab^2 + 9a^2b \ \ ]$$

Since our aim is to get integer solutions, we take a=2A, b=2B.

Employing the values of u and v in (2) we get

$$x = 28A^3 - 4B^3 - 84AB^2 + 12A^2B - 1$$

$$y = 20A^3 + 20B^3 - 60AB^2 - 60A^2B - 1$$

$$z = 4A^2 + 4B^2$$

Observations

1. $A[z(A, A)], -7x(A, A)$ and $-9y(A, A)$ are cubic integers
2. $6[P_A^{19} + P_A^{13} + T_{10,A} + T_{6,A} - 29Gno_A - x(A,1) \equiv 0 \pmod{34}$
3. $z(A, A) - y(A, A) + 15T_{6,A} + Pr(A) - [18P_A^{19} + 60P_A^5 + 14Gno_A] \equiv 0 \pmod{14}$
4. $z(A, A) - x(A, A) + T_{6,A} - [6P_A^{19} + 12P_A^{18} + Pr(A) + 19Gno_A] \equiv 0 \pmod{19}$
5. $y(A,1) + 64Pr(A) - [6P_A^{19} + 2P_A^{11} + 10Gno_A] \equiv 0 \pmod{29}$

Pattern-2:

Write (3) as $(2u+2)^2 + (4v)^2 = 40z^3 .1 \quad (6)$

and $1 = \frac{(6+8i)(6-8i)}{10^2} \quad (7)$

Using (4),(5) and (7) in (6) and proceeding as in pattern 1 we get the non-zero distinct integer solutions of (1) to be

$$x = 20000A^3 + 20000B^3 - 60000AB^2 - 60000A^2B - 1$$

$$y = -4000A^3 + 28000B^3 + 12000AB^2 - 84000A^2B - 1$$

$$z = 400A^2 + 400B^2$$

Observations:

1. $10000[CC_A - CH_A - 6Gno_A] - x(A,1) \equiv 0 \pmod{20001}$
2. $y(1,B) - 1000[7RD_B + 3CH_B - 8Gno_B] \equiv 0 \pmod{1999}$
3. $100[12P_B^{16} + 2CH_B - 28GnO_B] - [y(B,1)] \equiv 0 \pmod{34001}$
4. The expression $12z(A,A)$ representsa nasty number
5. $100[120P_B^{16} + 32T_{6,B} - 294Gno_B - [y(B,1) + z(B,1)] \equiv 0 \pmod{33001}$

Pattern-3:

$$\text{Write } 1 = \frac{(4+3i)(4-3i)}{5^2} \tag{8}$$

Using (4),(5) and (8) in (6) and proceeding as in pattern 2 we get the non-zero distinct integer solutions of (1) to be

$$x = 3100A^3 + 1700B^3 - 9300AB^2 - 5100A^2B - 1$$
$$y = 500A^3 + 3500B^3 - 1500AB^2 - 10500A^2B - 1$$
$$z = 100A^2 + 100B^2$$

Observations:

1. $100[6P_A^{19} + 6P_A^{16} - 2T_{19,A} - T_{82,A}] - 6100Gno_A - x(A,1) \equiv 0 \pmod{7799}$
2. $y(A,1) - 100[2P_A^{17} + T_{26,A} - 118\Pr(A)] + 5900Gno_A \equiv 0 \pmod{9399}$
3. $z(A,1) + 50Gno_A - 100\Pr(A) \equiv 0 \pmod{50}$
4. $100[12P_A^{15} + 24T_{6,A} - 17Gno_A] + y(A,1) - x(A,1) \equiv 0 \pmod{6898}$
5. $100[6(P_A^{19} + P_A^{16}) - 28T_{6,A} - 48Gno_A] - [x(A,1) + z(A,1)] \equiv 0 \pmod{6399}$

Pattern-4:

$$\text{Write } 1 = \frac{(11+60i)(11-60i)}{61^2} \tag{9}$$

Using (4),(5) and (9) in (6) and proceeding as in pattern3 we get the non-zero distinct integer solutions of (1) to be

$$x = 2039108A^3 + 6087556B^3 - 18262668A^2B - 6117324AB^2 - 1$$
$$y = -3646580A^3 + 5283820B^3 - 15851460A^2B + 10939740AB^2 - 1$$
$$z = 14884A^2 + 14884B^2$$

Observations:

1. $12z(A,A)$ representsa nasty number
2. $-[x((A,A) + y(A,A)] \equiv 0 \pmod{30}$
3. $y(A,A) - x(A,A) \equiv 0 \pmod{32}$
4. $z(A,A) - x(A,A) \equiv 0 \pmod{9}$

Pattern-5:

$$\text{Write } 40 = \frac{(8+4i)(8-4i)}{2} \tag{10}$$

$$1 = \frac{(1+i)(1-i)}{2} \tag{11}$$

Using (4),(10) and (11) in (6) and proceeding as in pattern 4 we get the non-zero distinct integer solutions of (1) to be

$$x = 20A^3 + 20B^3 - 60AB^2 - 60A^2B - 1$$
$$y = -4A^3 + 28B^3 + 12AB^2 - 84A^2B - 1$$
$$z = 4A^2 + 4B^2$$

Observations:

1. $x(A,A).y(A,A)$ is a square number
2. $A[z(A,A)], -9x(A,A)$ and $-7y(A,A)$ are cubic integers
3. $z(A,A) - y(A,A) + T_{6,A} - [6P_A^{19} + 12P_A^{18} + \Pr(A) + 19Gno_A] \equiv 0 \pmod{19}$
4. $z(A,A) - x(A,A) + 15T_{6,A} + \Pr(A) - [18P_A^{19} + 60P_A^{5} + 14Gno_A] \equiv 0 \pmod{14}$
5. $x(A,1) + 64\Pr(A) - [6P_A^{19} + 2P_A^{11} + 10Gno_A] \equiv 0 \pmod{29}$

4. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary cubic Diophantine equation represented by

$$5(x^2 + y^2) - 6xy + 4(x+y) + 4 = 40z^3$$

One can also search for other patterns of solutions for the above equation.

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