On The Ternary Cubic Diophantine Equation \(5(x^2 + y^2) - 6xy + 4(x + y) + 4 = 40z^3\)

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Abstract: The ternary cubic Diophantine equation \(5(x^2 + y^2) - 6xy + 4(x + y) + 4 = 40z^3\) is analyzed for its non-zero distinct integer solutions. Five different patterns of integral solutions are obtained. Few interesting relations among the solutions and some special polygonal numbers are presented.

Keywords: Ternary cubic, Diophantine equations, Integral solutions

Mathematical Classification: 11D25

1. Introduction

Ternary cubic equations are rich in variety. For more detailed understanding one can see [1-7]. For the non-trivial integral solutions of ternary cubic Diophantine equations [8-9] has been studied. [10-12] has been referred for various ternary cubic Diophantine equations.

In this communication, we consider yet another interesting ternary cubic equation \(5(x^2 + y^2) - 6xy + 4(x + y) + 4 = 40z^3\) and obtain different patterns of non-trivial integral solutions. Also, a few interesting relations among the solutions and special polygonal, Centered, Gnomonic, Rhombic, Pronic numbers are presented.

2. Notations

- \(T_{m,n}\) = Triangular Number of rank \(n\).
- \(P_m\) = Pyramidal Number of rank \(n\).
- \(P(n)\) = Pronic Number of rank \(n\).
- \(Star_n\) = Star Number of rank \(n\).
- \(CH_n\) = Centered Hexagonal Number of rank \(n\).
- \(RD_n\) = Rhombic Dodecagonal Number of rank \(n\).
- \(Gno_n\) = Gnomonic Number of rank \(n\).

3. Method of Analysis

Consider the equation
\[5(x^2 + y^2) - 6xy + 4(x + y) + 4 = 40z^3\]

The substitution of the linear transformation
\[x = u + v; y = u - v\]
in (1) give \((2u + 2)^2 + (4v)^2 = 40z^3\)

We present below different methods of solving (3) and thus obtain different choices of integer solutions of (1)

Pattern-1:
Let \(z = a^2 + b^2\) (4)
Write \(40 = (6 + 2i)(6 - 2i)\) (5)
Using (4) and (5) in (3) and using factorization method we have,
\((2u+2+i4v)(2u-2-i4v) = (6+2i)(6-2i)((a+ib)(a-ib))^3\)
Equating real and imaginary we get
\[u = \frac{1}{2}[6a^3+2b^3-18ab^2-6a^2b-2]\]
\[v = \frac{1}{2}[a^3-3b^3-3ab^2+9a^2b]\]
Since our aim is to get integer solutions, we take \(a=2A, b=2B\).

Observations
1. \(A[x(A, A), -7x(A, A)]\) and \(-9y(A, A)\) are cubic integers
2. \(6P_A^{19} + P_A^{13} + T_{10.4} + T_{6.4} - 29Gno_A - x(A, I) \equiv 0 (mod 34)\)
3. \(x(A, A) + 15T_{5.4} + Pr(A) - [18P_A^{13} + 60P_A^{12} + 14Gno_A] = 0 (mod 14)\)
4. \(x(A, A) + 15T_{5.4} + Pr(A) - [6P_A^{19} + 12P_A^{18} + Pr(A) + 19Gno_A] = 0 (mod 19)\)
5. \(y(A, I) + 64Pr(A) - [6P_A^{19} + 2P_A^{11} + 10Gno_A] \equiv 0 (mod 29)\)

Pattern-2:
Write (3) as \((2u + 2)^2 + (4v)^2 = 40z^3 . 1\)
\[and \quad 1 = \frac{(6 + 8i)(6 - 8i)}{10^2}\]
Using (4),(5) and (7) in (6) and proceeding as in pattern 1 we get the non-zero distinct integer solutions of (1) to be
\[x = 20000A^2 + 20000B^2 - 60000AB^2 - 60000A^2B - 1\]
\[y = -4000A^3 + 28000B^3 + 12000A^2B - 8400A^2B - 1\]
\[z = 400A^2 + 400B^2\]
Observations:

1. $1000(4C_A - CH_A - 6Gno_A) - x(A, A) \equiv 0 (\text{mod } 2000)$
2. $y(B, B) - 100(77D_B + 3CH_B - 8Gno_B) \equiv 0 (\text{mod } 9999)$
3. $100(12P_B^{16} + 2CH_B - 28Gno_B) - [y(B, B)] \equiv 0 (\text{mod } 34001)$
4. The expression $12(z, A)$ represents a non-zero distinct integer number 5.

Pattern-3:

Write $1 = \frac{(4 + 3i)(4 - 3i)}{5^2}$

Using (4), (5) and (8) in (6) and proceeding as in pattern 2 we get the non-zero distinct integer solutions of (1) to be

$x = 3100A^3 + 1700B^3 - 9300AB^2 - 5100A^2B - 1$

$y = 500A^3 + 3500B^3 - 1500AB^2 - 1050A^2B - 1$

$z = 100A^2 + 100B^2$\n
Observations:

$100(6P_A^2 + 6P_{4,4} - 2T_{19,4} - T_{32,1} - 6100Gno_A - x(A, A) \equiv 0 (\text{mod } 7799)$

$2. y(A, A) - 100(2P_{B,4} - T_{6,6} - 118P(A)) + 5900Gno_A \equiv 0 (\text{mod } 9399)$

$3. z(A, A) + 50Gno_A - 100Pr(A) \equiv 0 (\text{mod } 50)$

$4. 100(12P_B^{16} + 24T_{a,4} - 17Gno_A) + y(A, A) - 100(A) \equiv 0 (\text{mod } 6898)$

$5. 100(6P_A^2 + P_{4,4}) - 28T_{a,4} - 48Gno_A) - [x(A, A) + z(B, B)] \equiv 0 (\text{mod } 6399)$

Pattern-4:

Write $1 = \frac{(11 + 60i)(11 - 60i)}{61^2}$

Using (4), (5) and (9) in (6) and proceeding as in pattern 3 we get the non-zero distinct integer solutions of (1) to be

$x = 2039108^3 + 6087556B^3 - 182626684B^2 - 6117324AB^2 - 1$

$y = -3646580^3 + 5283820B^3 - 15851460B^2 + 10939740B^2 - 1$

$z = 14884A^2 + 14884B^2$\n
Observations:

$1. 12z(A, A)$ represents a non-zero distinct integer number

2. $-[x((A, A) + y(A, A)) \equiv 0 (\text{mod } 30)$

3. $y(A, A) - x(A, A) \equiv 0 (\text{mod } 32)$

4. $z(A, A) - x(A, A) \equiv 0 (\text{mod } 9)$

Pattern-5:

Write $40 = \frac{(8 + 4i)(8 - 4i)}{2}$

$1 = \frac{(1 + i)(1 - i)}{2}$

Using (4), (10) and (11) in (6) and proceeding as in pattern 4 we get the non-zero distinct integer solutions of (1) to be

$x = 20A^3 + 20B^3 - 60AB^2 - 60A^2B - 1$

$y = -4A^3 + 28B^3 + 12AB^2 - 84A^2B - 1$

$z = 4A^2 + 4B^2$\n
Observations:

1. $(x(A, A), y(A, A)$ is a square number

2. $A[z(A, A), -9x(A, A)$ and $-7y(A, A)$ are cubic integers

3. $x(A, A) - y(A, A) + T_{19,4} - [6P_B^{16} + 12P_{2,8} + Pr(A) + 19Gno_A] \equiv 0 (\text{mod } 19)$

4. $x(A, A) - x(A, A) + 15T_{a,4} + Pr(A) \equiv [18P_B^{16} + 60P_{2,8} + 14Gno_A] \equiv 0 (\text{mod } 14)$

5. $(x, A) + 64Pr(A) - [6P_B^{16} + 2P_{2,8} + 10Gno_A] \equiv 0 (\text{mod } 29)$

4. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary cubic Diophantine equation represented by

$5(x^3 + y^3) - 6xy + 4(x + y) + 4 = 40z^3$

One can also search for other patterns of solutions for the above equation.

References

5(x^2 + y^2) - 9xy + x + y + 1 = 23z^3
IJIRR,1,(10),99-101.

x^3 + y^3 = z(x^2 + y^2 - 20) = 4(x + y)^2 z

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