International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2014): 5.611

Existence of Mild Solutions for Fractional Neutral Functional Differential Equations

K. Geetha[1], Dr. K. Kadambavanam[2]

[1]Department of Mathematics, K. S. Rangasamy College of Technology, Tamilnadu, India

[2]Department of Mathematics, Sri Vasavi College, Tamilnadu, India

Abstract: ***In several research works, controllability is one of the major concepts in mathematical control theory, which plays an important role in control systems. The controllability of nonlinear systems provided by evolution equations and qualitative theory of fractional differential equations has been developed. In this paper, we consider the initial value problems of fractional neutral functional differential equations. By applying fixed-point theorem, fractional calculus and controllability theory, a new set of acceptable conditions for approximate controllability of semi linear fractional differential equations are formulated and verified. Finally approximate controllability of semi linear fractional control differential systems is implied under the assumptions that the corresponding linear system is approximately controllable.***

Keywords: Fractional Differential Equations, Fixed-point Method, Semi Linear Fractional Control Differential Systems

1. Introduction

The approximate controllability of fractional evolution equations related to Caputo fractional derivative method was developed [1]. The initial value problem is explained in [2] for a class of fractional neutral functional differential equations and the basis on existence are attained. The controllability of semilinear differential equations was developed and inclusions through the semigroup theory in Banach spaces [3]. All results are attained through fixed point theorems both for single and multivalued mappings.

The finite-dimensional dynamical control systems depicted in semilinear ordinary differential state equations [4]. The multiple point time-variable delays in control are carried out. The constrained local relative controllability are created and verified by generalized open mapping theorem. Then, explained the constant multiple point delays. In [5] existence of mild solutions to the Cauchy Problem for the fractional differential equation with nonlocal conditions: $D^q x(t)=Ax(t)+t^n f(t,x(t),Bx(t))$, $t\in[0,T]$, $n\in\mathbb{Z}^+$, $x(0)+g(x)=x_0$, where $0<q<1$, A refers to the infinitesimal generator of a C_0-semigroup of bounded linear operators on a Banach space X. In [6] briefly explained about the basic theory of fractional differentiation, fractional-order differential equations, methods of their solutions and applications. Gamma and beta functions are most significant role in the fractional differentiation equations. The basic theory of initial value problems was developed for fractional functional differential equations and expanding the corresponding theory of ordinary functional differential equations [7].

The fractional power of operators and some fixed point theorems are introduced in [8]. The groups of fractional neutral evolution equations are obtained with nonlocal conditions and find different condition on the existence according to uniqueness of mild solutions. Nonlocal Cauchy problem were developed [9] for the fractional evolution equations in an arbitrary Banach space. In addition the different criteria on the existence and uniqueness of mild solutions are acquired. This paper [10] addresses the existence of mild solutions for a group of fractional evolution equations by compact analytic semi group. The nonlinear part identifies some local growth conditions in fractional power spaces by verifying existence of mild solutions. In [11] briefly explained the existence of mild solutions for semilinear fractional evolution equations and optimal controls in the αα-norm. An appropriate αα-mild solution of the semilinear fractional evolution equations is established. The existence and uniqueness of αα-mild solutions are verified by involving fractional calculus, singular version Gronwall inequality and Leray–Schauder fixed point theorem. In [12] explained about theory of the fractional differential equations including a generalization of the classical Frobenius method. This work provide [13], if $0<\alpha\leqslant 1$, study the Cauchy problem in a Banach space E fo fractional evolution equations of the form $\dfrac{d^\alpha u}{dt^\alpha}=Au(t)+B(t)u(t)$ Where A refers to closed linear operator provided on a dense set in E into E, which produces a semigroup and $\{B(t):t\geqslant 0\}$ is a family of a closed linear operators identified on a dense set in E into E. The existence and uniqueness of the solution present Cauchy problem is calculated for a wide class of the family of operators $\{B(t):t\geqslant 0\}$. The function is provided that used for theory of integro-partial differential equations of fractional orders.

In this paper, we consider the initial value problems (IVP for short) of fractional neutral functional differential equations with bounded delay of the form

$$\begin{gathered} {}^cD^\alpha[x(t)-g(t,x_t)]=f(t,x_t),\ t\in(t_0,\infty),t_0\geq 0 \\ x_{t_0}=\varphi \end{gathered} \tag{1}$$

Where ${}^cD^\alpha$ is the standard caputo's fractional derivative of order $0<\alpha<1$, $f,g:[t_0,\infty)\times C([t_0-r,t_0+a],R^n)$ are given functions satisfying some assumptions that will be specified later, $\alpha>0$ and $\varphi\in C([-r,0],R^n)$. If

$x \in C([t_0 - r, t_0 + a], R^n)$, then for any $t \in [t_0, t_0 + a]$ define x_t by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r, 0]$.

2. Problem Formulation And Preliminaries:

Throughout this paper, unless otherwise specified, the following notations will be used. We assume that X is a Hilbert space with norm $\|.\| = \sqrt{<.,.>}$. Let $C([0,T],X)$ be the Banach space of continuous functions from [0, T] into X with the norm $\|x\| = \sup_{t\in[0,T]} \|x(t)\|$, here $x \in C([0,T],X)$. In this paper, we also assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup S(t), t >0, of uniformly bounded linear operations in X, that is, there exists M > 1 such that $\|S(t)\| \le M$ for all $t \ge 0$. Without loss of generality, let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. then for any $\alpha > 0$ we can define $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of X. Hence we can define $A^{\alpha} = (A^{-\alpha})^{-1}$, which is a closed bijective linear operator in X. It can be shown that each A^{α} has dense domain and that $D(A^{\beta}) \subset D(A^{\alpha})$ for $0 \le \alpha \le \beta$. Moreover, $A^{\alpha+\beta}x = A^{\alpha}A^{\beta}x = A^{\beta}A^{\alpha}x$ for every $\alpha, \beta \in R$ and $x \in D(A^{\mu})$ with $\mu = \max(\alpha, \beta, \alpha+\beta)$ where $A^0 = I$, I is the identity in X. (for proofs of these facts we refer to the literature [3,4,7]).

We denote by X_α the Hilbert space of $D(A^\alpha)$ equipped with norm $\|x\|_\alpha = \|A^\alpha x\| = \sqrt{< A^\alpha x, A^\alpha x >}$ for $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α.

Then we have $X_\beta \to X_\alpha$, for $0 \le \alpha \le \beta$ (with $X_0 = X$) and the embedding is continuous. Moreover, A^α has the following basic properties

2.1 Lemma

A^α and S(t) have the following properties

i) $S(t) : X \to X_\alpha$ for each t > 0 and $\alpha \ge 0$.

ii) $A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \ge 0$.

iii) For every t > 0, $A^\alpha S(t)$ is bounded in X and there exists $M_\alpha > 0$ such that $\|A^\alpha S(t)\| \le M_\alpha t^{-\alpha}$

iv) $A^{-\alpha}$ is a bounded linear operator for $0 \le \alpha \le 1$, there exists $C_\alpha > 0$ such that $\|A^{-\alpha}\| \le C_\alpha$.

Let us recall the following known definitions in fractional calculus. For more details, see [6, 12]

2.2 Definition

The fractional integral of order α with the lower limit to for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ t > t_0, \ \alpha > 0$$

Provided the right-hand side is pointwise defined on $[t_0, \infty]$, where Γ is the gamma function.

2.3 Definition

The Caputo's derivative of order $\alpha > 0$ with the lower limit to for a function $f : [t_0, \infty) \to R$ can be written as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > t_0,$$

$0 \le n-1 < \alpha < n$

The Caputo derivative of a constant is equal to zero.

According to Definition 2 and 3, it is suitable to rewrite the problem (1) in the equivalent integral equation.

$$x(t) = \varphi(0) - g(t_0, \varphi) + g(t, x_t) + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x_s) ds,$$

$t \in (t_0, t_0 + \delta]$ (2)

Provided that the integral in (2) exists. Applying the Laplace transform to (2), we get

$$x(t) = \int_0^\infty \xi_q(\theta) S(t^q \theta)[\varphi(0) - g(t_0, \varphi) + g(t, x_t)] d\theta$$
$$+ q \int_{t_0}^t \int_0^\infty \theta (t-s)^{q-1} \xi_q(\theta) S((t-s)^q \theta)[f(s, x_s)] d\theta \, ds$$

Where $\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q(\theta^{-\frac{1}{q}}) \ge 0$;

$$\overline{\omega}_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q),$$

$\theta \in (0, \infty)$

Here ξ_q is a probability density function defined on $(0, \infty)$ that is $\xi_q(\theta) \ge 0, \ \theta \in (0, \infty)$ and $\int_0^\infty \xi_q(\theta) d\theta = 1$.

For $x \in X$ and $0 < q < 1$. We define the families $\{U(t): t \geq 0\}$ and $\{V(t): t \geq 0\}$ of operators by

$$U(t) = \int_0^\infty \xi_q(\theta)\, S(t^q\theta)\, d\theta : X \to X_\alpha,$$

$$V(t) = q\int_0^\infty \theta\xi_q(\theta)\, S(t^q\theta)\, d\theta : X \to X_\alpha$$

The following lemma follows from the results in [8, 9, 11].

2.4 Lemma

The operators U and V have the following properties:

i) For any fixed $t \geq 0$,and any $x \in X_\alpha$, we have the operators U(t) and V(t) are linear and bounded operators. i.e. for any $x \in X_\alpha$, $\|U(t)x\|_\alpha \leq M\|x\|_\alpha$ and $\|V(t)x\|_\alpha \leq \dfrac{M}{\Gamma(q)}\|x\|_\alpha$.

ii) The operator U(t) and V(t) are strongly continuous for all $t \geq 0$.

iii) U(t) and V(t) are strongly continuous in X for t > 0.

iv) U(t) and V(t) are compact operators in X for t > 0.

v) For every t > 0 the restriction of U(t) to X_α and the restriction of V(t) to X_α are compact operators in X_α.

vi) $A^\alpha U(t)x = U(t)A^\alpha x$, $A^\alpha V(t)x = V(t)A^\alpha x$ for each $x \in X_\alpha$ and $t \geq 0$.

Due to the argument above, we give the following definition of the mild solution of (1)

2.5 Definition

A solution $x(\cdot; x_0, u) \in C([t_0, t_0 + \delta], X_\alpha)$ is said to be a mild solution of (1) if for any $u \in L_2([t_0, t_0 + \delta], U)$ the integral equation

$$x(t) = U(t)[\varphi(0) - g(t_0, \varphi) + g(t, x_t)] + \int_{t_0}^{t} (t-s)^{q-1} V(t-s) f(s, x_s)\, ds, \quad (3)$$

is satisfied.

2.6 Lemma(Schauder's fixed point theorem)

If K is a closed bounded and convex subset of a Banach space X and $F: K \to K$ is completely continuous then F has a fixed point in K.

3. Main Result

In this section, we formulate and prove conditions for the approximate controllability of semilinear fractional control differential systems. To do this, we first prove the existence of a fixed point of the operator F_ε defined below by using Schauder's fixed point theorem. Secondly, in theorem 13, we show that under certain assumptions the approximate controllability of fractional systems (1) is implied by the approximate controllability of the corresponding linear system (4).

Let $x(T : x_0, u)$ be the state value of (1) at terminal time T corresponding to the control u and the initial value x_0. Introduce the set $R(T, x_0) = \{x(T : x_0, u) : u \in L_2([0,T], U)\}$ which is called the reachable set of system (1) at terminal time T, its closure to X_α is denoted by $\overline{\Re(T, x_0)}$.

3.1 Definition

The system (1) is said to be approximately controllable on [0,T] if $\overline{\Re(T, x_0)} = X_\alpha$ that is given an arbitrary $\varepsilon > 0$ it is possible to steer from the point x_0 to within a distance ε from all points in the state space X_α at time T.

Consider the following linear fractional differential system

$$^{c}D_t^q x(t) = Ax(t) + Bu(t), \quad t \in [0, T],$$
$$x(0) = x_0 \quad (4)$$

The approximate controllability for linear fractional system (4) is a natural generalization of approximate controllability of linear first order control system. It is convenient at this point to introduce the controllability and resolvent operators associated with (4) as

$$\Gamma_0^T = \int_0^T (T-s)^{q-1} V(t-S) B B^* V^*(T-s) ds \;:\; X_\alpha \to X_\alpha$$

$$R(\varepsilon, \Gamma_0^T) = (\varepsilon I + \Gamma_0^T)^{-1} \quad : \quad X_\alpha \to X_\alpha,\ \varepsilon > 0$$

Respectively, where B^* denotes the adjoint of B and $V^*(t)$ is the adjoint of V(t). it is straightforward that the operator Γ_0^T is a linear bounded operator.

3.2 Theorem

Let Z be a seperable reflexive Banach space and let Z^* stands for its dual space. Assume that $\Gamma : Z^* \to Z$ is symmetric, Then the following two conditions are equivalent:

i) $\Gamma : Z^* \to Z$ is positive, that is $(z^*, \Gamma z^*) > 0$ for all non zero $z^* \in Z^*$.

ii) For all $h \in Z,\ z_\varepsilon(h) = \varepsilon(\varepsilon I + \Gamma J)^{-1}(h)$ strongly converges to zero as $\varepsilon \to 0^+$. Here J is the duality mapping of Z into Z^*.

3.3 Lemma

The linear fractional control system (4) is approximately controllable on [0, T] if and only if $\varepsilon R(\varepsilon, \Gamma_0^T) \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology.

3.4 Remark

Before proving the main result, we impose the following conditions on data of the proble

(H1) S(t), t > 0 is a compact analytic semigroup in X.

(H2) there exists a constant $\beta \in [\alpha, 1]$ such that the function $f:[t_0, t_0+\eta]\times X_\alpha \to X_\beta$ satisfy the following conditions:

(a) For each $x \in X_\alpha$ the function $f(\bullet, x)$ is measurable.

(b) For each $t \in [t_0, t_0+\eta]$ the function $f(t,\cdot): X_\alpha \to X_\beta$ is continuous.

(c) For any r > 0 there exists a function $g_r \in L^\infty([t_0, t_0+\eta],[0,\infty])$ such that $\sup\{\|f(t,x)\|_\beta : \|x\|_\alpha \le r, \|y\|_\alpha \le k^* T_r\}$ $\le g_r(t), t\in[t_0, t_0+\eta]$ and there is a constant $\gamma > 0$ such that

$$\liminf_{r\to+\infty} \frac{1}{r}\int_0^t \frac{g_r(s)}{(t-s)^{1-q}}ds \le \gamma < \infty$$

Here $k^* = \max\{k(t,s) : (t,s)\in\Delta\}$

For any $\varepsilon > 0$ and $h \in X_\alpha$ define a control function g(t, x) as follows

$$g(t,x) = V^*(T-t)R(\varepsilon, \Gamma_0^T)[h - U(t)\,\varphi(0)$$
$$-\int_{T_0}^T (T-s)^{q-1}V(T-s)\,f(s,x_s)ds$$

3.5 Lemma

Under the assumption (H1) & (H2) we have $\|g(t,x)\|_\alpha \le \frac{1}{\varepsilon}L_u$,

$$\int_{t_0}^t (t-s)^{q-1}\|A^\alpha\, g(s,x)\|ds \le \frac{1}{\varepsilon}\frac{(t-t_0)^q}{q}L_u, \quad \|x\|_\alpha \le r$$

Here $L_u = L_B M\left[C_\alpha\|A^\alpha h\| + MC_\alpha\|A^\alpha\varphi(0)\|\right]$

$$+\frac{M\,C_\beta}{\Gamma(q)}\int_{T_0}^T (T-s)^{q-1}g_r(s)\,ds$$

$$L_B = \sup_{0\le t\le T}\|V^*(T-t)\|$$

Proof:

The proof of the lemma is straight forward. Indeed by Lemma 4 we have

$$\|g(t,x)\|_\alpha \le \frac{1}{\varepsilon}\|A^\alpha V^*(T-t)\|\left[\|A^{-\alpha}A^\alpha h\| + \|A^{-\alpha}A^\alpha U(T)\varphi(0)\|\right]$$
$$+\left\|\int_{T_0}^T (T-s)^{q-1}A^{-\beta}V(T-s)A^\beta f(s,x_s)ds\right\|$$
$$\le \frac{1}{\varepsilon}\sup_{0\le t\le T}\|V^*(T-t)\|_\alpha\left[C_\alpha\|A^\alpha h\| + MC_\alpha\|A^\alpha\varphi(0)\|\right]$$
$$+\frac{MC_\beta}{\Gamma(q)}\int_{T_0}^T (T-s)^{q-1}g_r(s)ds$$
$$\le \frac{1}{\varepsilon}L_u$$

&

$$\int_{t_0}^t (t-s)^{q-1}\|A^\alpha\, g(s,x)\|ds = \int_{t_0}^t (t-s)^{q-1}\|\,g(s,x)\|_\alpha\, ds$$
$$\le \frac{1}{\varepsilon}\frac{(t-t_0)^q}{q}L_u$$

3.6 Theorem

Assume that there exist $\delta > 0$ and $\gamma \in (0,\infty)$ such that (H1), (H2) are satisfied and

$$\frac{M\,C_{\beta-\alpha}}{\Gamma(q)}\gamma < 1 \qquad (5)$$

Then the IVP (1) has a mild solution on $[t_0, t_0+\eta]$.

Proof:

In the Banach space $C([t_0, t_0+\eta], X_\alpha)$ we consider a set

$$B_r = \{x \in C([t_0, t_0+\eta], X_\alpha)\,/\,x(0) = x_0,\ \|x\|_\alpha \le r\}$$

Where r is a positive constant, for $\varepsilon > 0$, we define the operator $F_\varepsilon : C([t_0, t_0+\eta], X_\alpha) \to C([t_0, t_0+\eta], X_\alpha)$ as follows $(F_\varepsilon x)(t) = z_\varepsilon(t)$

Where

$$z_\varepsilon(t) = U(t)[\varphi(0) - g(t_0,\varphi) + g(t,x_t)]$$
$$+\int_{t_0}^t (t-s)^{q-1}V(t-s)\,f(s,x_s)\,ds$$

It will be shown that for all $\varepsilon > 0$ the operator $F_\varepsilon : C([t_0, t_0+\eta], X_\alpha) \to C([t_0, t_0+\eta], X_\alpha)$ has a fixed point. The proof of the claim is long and technical. Therefore it is convenient to divide it into several steps.

Step 1

For an arbitrary $\varepsilon > 0$, there is a positive constant $r = r(\varepsilon)$ such that $F_\varepsilon(B_{r(\varepsilon)}) \subset B_{r(\varepsilon)}$

If this were not the case, then for each r > 0, there would exist, $x \in B_{r(\varepsilon)}$ and $t_r \in [t_0, t_0+\eta]$ such that $\|(F_\varepsilon x)(t_r)\|_\alpha > r$, from Lemma 4 and assumption (H1)(c), we see that

$$r < \|Z_\varepsilon(t_r)\|_\alpha$$
$$\le \left\|U(t_r)[\varphi(0) - g(t_0,\varphi) + g(t_r, x_{t_r})]\right\|_\alpha$$

$$+\left\| \int_{t_0}^{t_r} (t_r - s)^{q-1} V(t_r - s) f(s, x_s)\, ds \right\|_\alpha$$

$$\leq M \left\| \varphi(0) - g(t_0, \varphi) + g(t_r, x_{t_r}) \right\|_\alpha$$

$$+\int_{t_0}^{t_r} (t_r - s)^{q-1} \left\| A^{\alpha-\beta} A^\beta V(t_r - s) f(s, x_s) \right\| ds$$

$$\leq M \left\| \varphi(0) - g(t_0, \varphi) + g(t_r, x_{t_r}) \right\|_\alpha$$

$$+\frac{M\, C_{\beta-\alpha}}{\Gamma(q)} \int_{t_0}^{t_r} (t_r - s)^{q-1} \left\| f(s, x_s) \right\|_\beta ds$$

$$\leq M \left\| \varphi(0) - g(t_0, \varphi) + g(t_r, x_{t_r}) \right\|_\alpha$$

$$+\frac{M\, C_{\beta-\alpha}}{\Gamma(q)} \int_{t_0}^{t_r} (t_r - s)^{q-1} g_r(s)\, ds$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we have

$$1 \leq \frac{M\, C_{\beta-\alpha}}{\Gamma(q)} \gamma$$

Which is a contradiction. Hence $F_\varepsilon(B_{r(\varepsilon)}) \subset B_{r(\varepsilon)}$ for some $r(\varepsilon) > 0$.

Step 2

$F_\varepsilon : B_r \to B_r$ is continuous.

Let $\{x_n\} \subset B_r$ with $x_n - x \in B_r$ as $n \to \infty$ from the assumption (H2)(b), for each $s \in [t_0, t_0 + \eta]$ we have

$f(s, x_n(s)) \to f(s, x(s))$ and

$g(s, x_n(s)) \to g(s, x(s))$ as $n \to \infty$. Using the following estimations,

$\left\| f(s, x_n(s)) - f(s, x(s)) \right\|_\beta \leq 2 g_r(s)$ and

$\left\| g(s, x_n(s)) - g(s, x(s)) \right\|_\alpha \leq \frac{2}{\varepsilon} L_u$

and the Lebesque Dominated convergence theorem, for each $s \in [t_0, t_0 + \eta]$, we get

$$\left\| (F_\varepsilon x)(t) - (F_\varepsilon x_n)(t) \right\|_\alpha \leq \left\| U(t)[g(s, x_n(s)) - g(s, x(s))] \right\|_\alpha$$

$$+\int_{t_0}^{t} (t-s)^{q-1} \left\| A^\alpha A^{-\beta} V(t-s) A^\beta (f(s, x_n(s)) - f(s, x(s))) \right\| ds$$

$$\leq M \left\| g(s, x_n(s)) - g(s, x(s)) \right\|_\alpha$$

$$+\frac{M\, C_{\beta-\alpha}}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \left\| f(s, x_n(s)) - f(s, x(s)) \right\|_\beta ds$$

$$\leq \frac{2M}{\varepsilon} L_u + \frac{M\, C_{\beta-\alpha}}{\Gamma(q)} 2 g_r(s) \frac{(t-t_0)^q}{q} \to 0$$

Which implies that $F_\varepsilon : B_r \to B_r$ is continuous.

Step 3

For each $\varepsilon > 0$, the set

$V(t) = \{(F_\varepsilon x)(t) : x \in B_r\}$, $r = r(\varepsilon)$ is relatively compact in X_α.

The case t = 0 is trivial. Clearly

$V(0) = \{(F_\varepsilon x)(0) : x(\cdot) \in B_r\} = \{x_0\}$ is compact in X_α.

So let $t \in [t_0, t_0 + \eta]$ be a fixed real number, and let τ be a given real number satisfying $0 < \tau < t$ for and $\delta > 0$ define

$V_\tau(t) = \{(F_\varepsilon^{\tau,\delta} x)(t) : x \in B_r\}$

Where,

$$(F_\varepsilon^{\tau,\delta} x)(t) = \int_\delta^\infty \xi_q(\theta)\, s(t^q \theta)[\varphi(0) - g(t_0, \varphi) + g(t, x_t)] d\theta$$

$$+ q \int_{t_0}^{t} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) s[(t-s)^q \theta] f(s, x_s) d\theta\, ds$$

$$= s(\tau^q \delta) \int_\delta^\infty \xi_q(\theta)[s(t^q \theta - \tau^q \delta)[\varphi(0) - g(t_0, \varphi) + g(t, x_t)] d\theta$$

$$+ s(\tau^q \delta) q \int_{t_0}^{t} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) s[(t-s)^q \theta - \tau^q \delta] f(s, x_s) d\theta\, ds$$

$$= s(\tau^q \delta)\, y(t, \tau)$$

Since $s(\tau^q \delta)$ is compact in X_α and $y(t, \tau)$ is bounded on B_r, the set $V_\tau(t)$ is relatively compact set in X_α. On the other hand,

$$\left\| (F_\varepsilon x)(t) - (F_\varepsilon^{\tau,\delta} x)(t) \right\|_\alpha$$

$$\leq \left\| \int_0^\delta \xi_q(\theta)\, s(t^q \theta)\, d\theta [\varphi(0) - g(t_0, \varphi) + g(t, x_t)] \right\|_\alpha$$

$$+ q \left\| \int_{t_0}^{t-\tau\delta} \int_0 \theta (t-s)^{q-1} \xi_q(\theta) s[(t-s)^q \theta] f(s, x_s) d\theta\, ds \right\|_\alpha$$

$$+ q \left\| \int_{t-\tau\delta}^{t} \int^\infty \theta (t-s)^{q-1} \xi_q(\theta) s[(t-s)^q \theta] f(s, x_s) d\theta\, ds \right\|_\alpha$$

$$\leq M[\|\varphi(0)-g(t_0,\varphi)\|_\alpha+\frac{1}{\varepsilon}L_u]\int_0^\delta \xi_q(\theta)\,d\theta$$

$$+qM\,C_{\beta-\alpha}\|g_r\|_{L^\infty}\int_{t_0}^{t-\tau}(t-s)^{q-1}\int_0^\delta \theta\xi_q(\theta)d\theta\,ds$$

$$+qM\,C_{\beta-\alpha}\|g_r\|_{L^\infty}\int_{t-\tau}^{t}(t-s)^{q-1}\int_0^\delta \theta\xi_q(\theta)d\theta\,ds$$

This implies that there are relatively compact sets arbitrarily close to the set V(t) for each $t\in[t_0,\,t_0+\eta]$. Hence $V(t),\ \ t\in[t_0,\,t_0+\eta]$ is relatively compact in X_α. Since it is compact at t=0, we have the relatively compactness of V(t) for all $t\in[t_0,\,t_0+\eta]$.

Step 4

$V=\{F_\varepsilon x\in C([t_0,\,t_0+\eta],X_\alpha)/\,x\in B_r\}$ is an equicontinuous family of functions on $[t_0,\,t_0+\eta]$ for $t_0<t_1<t_2<t$.

$$\|z(t_2)-z(t_1)\|_\alpha$$

$$\leq\|U(t_2)[\varphi(0)-g(t_0,\varphi)+g(t_2,x_{t_2})]-U(t_1)[\varphi(0)-g(t_0,\varphi)+g(t_1,x_{t_1})]\|_\alpha$$

$$+\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}V(t_2-s)f(s,x_s)\,ds\right\|_\alpha$$

$$+\left\|\int_{t_0}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]V(t_2-s)f(s,x_s)\,ds\right\|_\alpha$$

$$+\left\|\int_{t_0}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]f(s,x_s)\,ds\right\|_\alpha$$

$$=I_1+I_2+I_3+I_4\quad(7)$$

By using Holder's inequality & Assumption (H2) we obtain

$$I_2=\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}V(t_2-s)f(s,x_s)\,ds\right\|_\alpha$$

$$\leq\frac{M}{\Gamma(q)}\frac{C_{\beta-\alpha}\|g_r\|_{L^\infty}(t_2-t_1)^q}{q}$$

$$I_3=\left\|\int_{t_0}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]V(t_2-s)f(s,x_s)\,ds\right\|_\alpha$$

$$\leq\frac{MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q)}\left[\frac{(t_2-t_0)^q-(t_1-t_0)^q-(t_2-t_1)^q}{q}\right]$$

$$I_4=\left\|\int_{t_0}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]f(s,x_s)\,ds\right\|_\alpha$$

$$\leq\int_{t_0}^{t_1-\eta}(t_1-s)^{q-1}\,\|V(t_2-s)-V(t_1-s)\|\|f(s,x_s)\|_\alpha\,ds$$

$$+\int_{t_1-\eta}^{t_1}(t_1-s)^{q-1}\,\|V(t_2-s)-V(t_1-s)\|\|f(s,x_s)\|_\alpha\,ds$$

$$\leq\frac{C_{\beta-\alpha}\|g_r\|_{L^\infty}}{q}[(t_1-t_0)^q-\eta^q]\sup_{s\in[t_0,t_1-\eta]}\|V(t_2-s)-V(t_1-s)\|$$

$$+\frac{2MC_{\beta-\alpha}\|g_r\|_{L^\infty}}{\Gamma(q)}\frac{\eta^q}{q}$$

Since the assumption (H1) and lemma 4 imply the continuity of V(t), t > 0 in t in the uniform operator topology. It can be easily seen that I_4 tends to zero independently of $x\in B_r$ as $t_2-t_1\to 0,\,\eta\to 0$. It is clear that $I_1,I_2,I_3\to 0$ as $t_2-t_1\to 0$. Thus the righthand side of (7) does not depend on particular choices of x and tends to zero as $t_2-t_1\to 0$, which means that $\{(F_\varepsilon x),\,x\in B_r\}$ is equicontinuous. Thus $F_\varepsilon(B_r)$ is equicontinuous and also bounded. By the Ascoli – Arzeda theorem $F_\varepsilon(B_r)$ is relatively compact in $C([t_0,\,t_0+\eta],X_\alpha)$.

4. Conclusion

In this work, approximate controllability of semi linear fractional control differential systems is investigated. A new set of conditions for approximate controllability of semi linear fractional differential equations are verified using fixed-point theorem, fractional calculus and controllability theory. Hence for all $\varepsilon>0$, F_ε is completely continuous operator on $C([t_0,\,t_0+\eta],X_\alpha)$. Thus from the Schauder's fixed point theorem, F_ε has a fixed point. Therefore, the IVP of fractional neutral functional differential equations obtained a mild solution on $[t_0,\,t_0+\eta]$.

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