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Subclasses of Meromorphically Multivalent Functions Defined By a Differential Operator

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Abstract: ***In this chapter we have investigated and studied two new Subclasses $\sum_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$ and $\sum^{+}_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$ of Meromorphic multivalent functions in the punctured disk D, utilizing principle of Differential sub ordinates we introduced various inclusion relationship & properties of particular Subclasses those are defined and derived using differential operator. The outcomes which are concerned with subordination properties like Convolution of functions, the radii of starlikeness, coefficient inequalities, and convexity closure theorems etc. are derived & studied.***

Keywords: Multivalent functions, radii of starlikeness, coefficient inequalities, convexity, closure theorems.

1. Introduction

In this chapter we have investigated and studied two new Subclasses $\sum_{\varrho\tau\varsigma\xi\,mp}(\alpha,\beta,\eta)$ and $\sum^{+}_{\varrho\tau\varsigma\xi\,mp}(\alpha,\beta,\eta)$ of Meromorphic multivalent functions in the punctured disk D, The outcomes which are concerned with subordination properties like Convolution of functions, the radii of starlikeness, coefficient inequalities, and convexity closure theorems etc. are derived & studied. We extended the well known concept of $(n,\delta)-$neighborhood functions to the above defined Subclasses of multivalent functions. Let us assume that the class $\varpi \in D$ which is Regular function belonging to D given by $D = \{z \in C: |z| < 1\}$. $(\forall z \text{ in } D)$. Consider

$\Omega = \{w \in \varpi : w \text{ at } z = 0 \text{ is } 0 \text{ \& mod } w(z) < 1,\}$ (1.1)

Be the class of Schwarz functions. For $0 \le \alpha < 1$. Let us assume $q(\propto) = q \in \tilde{A} : (0) = 1$ and $\text{Re}\,p\, q(z) > \alpha, \forall z \text{ in } D.$ (1.2)

Note that $q = q(0)$ is the popular class of Schwarz Caratheodory functions. The popular class of Schwarz's functions & Caratheodary functions plays very much useful role in geometrical function theory of Regular function This is investigated and studied by various researchers. It's very simple to check $\forall$ q in $q(\alpha)$ if & only if the following condition is satisfied

$\frac{p(z)-\propto}{1-\propto} \in P$ (1.3)

Using (1.3) and applying the properties of the functions which are in q, the lemmas given below for the above defined functions in the class $q(\alpha)$ can be obtained.

2. Preliminary Lemmas:

Lemma 1.1 Let us assume $p \in \varpi$. I. e. here after it is to be taken as $p \in P(\alpha) \Leftrightarrow \exists\, w \in \Omega$ satisfying

$q(z) = \frac{1-\langle 4(1-\eta)\propto -1\rangle w(z)}{1-w(z)}$ (1.4)

Lemma 1.2 (Herglotz formula): A function $p \in \varpi$ contained in the class $P(\alpha)$ $\exists$ a probability measure $\mu(y) on\ \partial D$ satisfying the following Equation given by $q(z) = \int_{|y|} \frac{1-\langle 4(1-\eta)\propto -1\rangle y(z)}{1-y(z)} d\mu(y), (\forall z \text{ in } D).$ (1.5)

We know that if $g \prec f$ i. e. here after it is to be taken as g at $z = 0$ is equal to 0 and f $at\ z = 0$ and $g(D)$ is subseset of $f(D)$. Particularly, if f is univalent in disk D we will get the equivalence relation $g(0) = f(0)$ if & only if $g(z) \prec f(z)$ & $g(D) \subset f(D)$. Let us assume that $\sum_1$ represents the classes of all Meromorphic functions g given by

$g(z) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty} b_n z^n \quad (\forall\, b_n \ge 0,\ q \in N).$ (1.6)

the above defined functions are multivalent Analytic in the punctured unit disk $D^* = D\backslash\{0\}$. given by $\sum_q^{+}$ Represent subclass of $\sum_q$ which contains the functions given by

$g(z) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty} b_n z^n \ge 0 \quad (\forall\, z \text{ in } D^*)$ (1.7)

Function $g \in \sum_q$ is said to be Meromorphic univalent starlike of order $\propto$ $(0 \le \propto < 1)$. Where,

$-Rep\left\{\frac{1}{q}\frac{zg'(z)}{g(z)}\right\} > \propto \quad (z \in U).$

The class $\sum_q^{+}(\propto)$ represents such functions that are as given above. If $g \in \sum_q$ is given by (1.6) & $f \in \sum_q$ given below

$f(z) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty} a_n z^n.$

I. e. here after it is to be taken as the Convolution that is the $g * f$ as

$(g * f) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty} b_n a_n z^n \ge 0$

$(\forall\, q \text{ in } N \text{ \& } \forall\, z \text{ in } D^*).$

For all functions$g \in \Sigma_q$, we are defining the differential operator $D^m_{\varrho\tau\varsigma\ \xi q}$ as given as follows

$D^0_{\varrho\tau\varsigma\ \xi q}\, g(z) = g(z)$

$$D^1_{\varrho\tau\varsigma\ \xi q}\, g(z) = D_{\varrho\tau\varsigma\ \xi q}\, g(z) = (\tau-\xi)(\tau-\varsigma)\frac{\left[z^{q+1}g(z)\right]''}{z^{q-1}} + \frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}\frac{\left[z^{q+1}g(z)\right]'}{z^q} + \left[\frac{\varrho-(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}\right] g(z). \quad (1.8)$$

$D^k_{\varrho\tau\varsigma\xi q}\, g(z) = D_{\varrho\tau\varsigma\xi q}\left[D^{n-1}_{\varrho\tau\varsigma\xi q}\, g(z)\right].$ (1.9)

Whereit is obviously for all,

$$\begin{pmatrix} 0 < \varrho \le \frac{1}{2}, 0 \le \xi < 1, \tau \ge 1, 0 \le \varsigma < 1, 0 \le \eta < 1 \\ 0 \le \propto < 1, 0 < \beta \le 1 \text{ and } k \in N \end{pmatrix}.$$

If the function $f \in \Sigma_q$ as given in Equation (1.6) i. e. here after it is to be taken as, from (1.8) and (1.9), we have been obtained that

$$D^k_{\varrho\tau\varsigma\xi q}\, g(z) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty} \Phi_n(\varrho,\tau,\varsigma,\xi,k,q) b_n z^n \quad (1.10)$$

$(k \text{ in } N, q \text{ in } N, z \text{ in } D^*)$,

Here over it is,

$$\Phi_n(\varrho,\tau,\varsigma,\xi,k,q) = \left\{1 + \left[(n+q)\left(\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho} + (n+q+1)(\tau-\xi)(\tau-\varsigma)\right)\right]\right\}^k. \quad (1.11)$$

from (1.10) $\Rightarrow D^k_{\varrho\tau\varsigma\xi q}\, g(z)$ in terms of convolution can be given as follows

$$D^k_{\varrho\tau\varsigma\xi q}\, g(z) = (g * h)(z) \quad (1.12)$$

Where it is,

$$h(z) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty} \Phi_n(\varrho,\tau,\varsigma,\xi,k,q) z^n. \quad (1.13)$$

Note that, the case $\xi = \frac{1}{2}$ & $\tau = \varsigma$ of $D^k_{\varrho\tau\varsigma\xi,q}$ which denotes Differential operator was investigated and studied by researchers Srivastava and Patel [60]. The differential operator tor $D^m_{\varrho\tau\varsigma\xi p} f(z)$ for $p = 1$ was considered in [50]. For the differential operator $D^k_{\varrho\tau\varsigma\xi q}\, g(z)$, we are going to define a new subclass of functions given by Σ_q.

Definition 1.1 A function given by $g \in \Sigma_q$ contained in the class $\Sigma_{\varrho\tau\varsigma\xi mq}(\alpha,\beta,\eta)$, if It will satisfy the inequality given as follows.

$$\left|\frac{\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 2q(1-\eta)}{\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 8q(1-\eta)^2\alpha - 2q(1-\eta)}\right| < \beta \quad (1.14)$$

And $\begin{pmatrix} 0 < \varrho \le \frac{1}{2}, 0 \le \xi < 1, \tau \ge 1, 0 \le \varsigma \le 1, \\ 0 \le \eta < 1, 0 \le \propto < 1, 0 < \beta \le 1 \text{ and } \forall\, k \text{ in } N \end{pmatrix}$.

Recently Mogra et.al.[62] introduced and studied given by $\Sigma_{\varrho\tau\varsigma\xi mq}(\propto,\beta,\eta)$ $\forall$ $q = 1$ *and* $k = 0$ is the class of Meromorphic univalent starlike functions of order $\propto$ & type β. It is simple to verify for $k = 0$ & $\beta = 1$ the class $\Sigma_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$ reduces into the class $\Sigma_q^*(\propto)$. Let us assume another subclass of Σ_q as follows

$$\Sigma^+_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta) = \Sigma^+_{\varrho\tau\varsigma\xi kq}(\propto,\beta) \cap \Sigma_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta). \quad (1.15)$$

the main purpose of this chapter is to study and represent systematically investigation of the classes $\Sigma_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$ and $\Sigma_q^*(\propto,\beta,\eta)$.

4.2 Properties of the Class $\Sigma_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.

We are going to start this part of the chapter for function which contained in given by $\Sigma_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$ With respect required necessary & sufficient conditions in terms of subordinations.

Theorem 2.1 A function $g \in \Sigma_q$ contained in $\Sigma_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$ if & only if that will satisfy the condition as given as follows.

$$\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec \frac{q\langle 4(1-\eta)\propto -1\rangle\beta z - q}{1-\beta z} \quad (\forall\, z \text{ in } D). \quad (2.1)$$

Proof Let $g \in \Sigma_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$. I. e. here after it is to be taken as, from (1.6), we have

$$\left|\frac{-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 2q(\eta-1)}{-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 4k(1-\eta) + 2q(\eta-1)\langle 4(1-\eta)\propto -1\rangle}\right|^2 < \beta^2$$

Or

$$\frac{(1-\beta^2)}{(-\langle 4(1-\eta)\propto -1\rangle)^2 - 1}\left|-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 2q(\eta-1)\right|^2$$

$$-\frac{2[1+\beta^2(-\langle 4(1-\eta)\propto -1\rangle)]}{(-\langle 4(1-\eta)\propto -1\rangle)^2 - 1}\, Rep\left\{-\frac{1}{2q(1-\eta)} \cdot \frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}\right\} < \beta^2$$

If β ≠ 1, we have been obtained the relation (inequality)

$$\left|-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} - 2q(1-\eta)\right|^2$$

$$-2\frac{1+\beta^2\langle 4(\eta-1)\propto +1\rangle}{1-\beta^2}\, Rep\left\{-\frac{1}{2q(1-\eta)} \cdot \frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}\right\}$$

$$+\left[\frac{1+\beta^2\langle 4(\eta-1)\propto +1\rangle)}{1-\beta^2}\right]^2$$

$$< \frac{-1-\beta^2\langle 4(\eta-1)\propto +1\rangle}{1-\beta^2} + \left[\frac{1+\beta^2\langle 4(\eta-1)\propto +1\rangle}{1-\beta^2}\right]^2$$

That is,

$$\frac{\left|-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 2q(\eta-1)\frac{1+\beta^2\langle 4(\eta-1)\propto +1\rangle}{1-\beta^2}\right|}{\frac{\beta[1+\langle 4(\eta-1)\propto +1\rangle]}{1-\beta^2}} < 1 \quad (2.2)$$

the above inequality gives the equality

$$G(z) = -\frac{1}{2q(1-\eta)} \cdot \frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}$$

Which belong to the unit disk centered at $\frac{1+\beta^2\langle 4(\eta-1)\propto +1\rangle}{1-\beta^2}$ and with the radius $\frac{\beta[1+\langle 4(\eta-1)\propto +1\rangle]}{1-\beta^2}$.

Simple to verify and to check

$$G(z) = \frac{1-\langle 4(1-\eta)\propto -1\rangle\beta z}{1-\beta z}$$

Corresponds in the unit disk D onto the disk given by

$$\frac{\left|w - \frac{1+\beta^2\langle 4(\eta-1)\propto +1\rangle}{1-\beta^2}\right|}{\frac{\beta[1+\langle 4(\eta-1)\propto +1\rangle]}{1-\beta^2}} < 1.$$

Since G is Multivalent and $G(0) = F(0)$, $G(D) \subset F(D)$, we have $F(z) \prec F(z)$, s. t.

$$-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec 2q(1-\eta)\frac{1-\langle 4(1-\eta)\propto -1\rangle\beta z}{1-\beta z}$$

Or $\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec \frac{-2q(1-\eta)+2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle\beta z}{1-\beta z\beta z}$.

Conversely, suppose that subordination

$\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec \frac{p\langle 4(1-\eta)\propto -1\rangle\beta z - q}{1-\beta z}$ is true.

I. e. here after it is to be taken as

$$-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} = 2q(1-\eta)\frac{1-\langle 4(1-\eta)\propto -1\rangle\beta w(z)\beta z}{1-\beta w(z)\beta z}. \quad (2.3)$$

Simplifying (2.3), we get the following inequality

$$\left|\frac{\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + 2q(1-\eta)}{\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^q_{\varrho\tau\varsigma\xi q}\, g(z)} + 2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle}\right| < \beta$$

which proves that $g \in \Sigma_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$. If $\beta = 1$, inequality

(1.14) becomes

$$\left|\frac{-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\, f(z)}+2p(\eta-1)}{-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle}\right| < \beta \qquad (2.4)$$

From above it is simple to get the following Subordination condition

$$-\frac{z\left[D^k_{\varrho\tau\varsigma\xi p}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec 2q(1-\eta)\frac{1-\langle 4(1-\eta)\propto-1\rangle z\beta z}{1-z}$$

$$\frac{z\left[D^k_{\varrho\tau\varsigma\xi p}\, g(z)\right]'}{D^m_{\varrho\tau\varsigma\xi q}\, g(z)} \prec \frac{-2q(1-\eta)+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle 2q(1-\eta)z}{1-z\beta z}. \qquad (2.5)$$

Hence the theorem proved.

Remark 2.1 Since

$$Rep\,\frac{1-\langle 4(1-\eta)\propto-1\rangle\beta z}{1-\beta z} > \propto$$

It follows that

$$-Re\left\{\frac{1}{2q(1-\eta)}\cdot\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}\right\} > \propto.$$

$\therefore D^k_{\varrho\tau\varsigma\xi q}\, g(z) \in \sum_p^*(\alpha)$.

Structural formula for the class $\sum_{\varrho\tau\varsigma\xi kq}(\propto, 1, \eta)$:

Theorem 2.2 Function $g \in \sum_q$ said to be in the class $\sum_{\varrho\tau\varsigma\xi kq}(\propto, 1, \eta)$ if & only if ∃ a probability measure given by μ(y) on ∂D such as

$$g(z) = \left[\frac{1}{z^q} + \sum_{n=1-q}^{\infty}\frac{z^n}{\Phi_n(\varrho,\tau\varsigma,,\xi,k,q)}\right]$$
$$*\left\{z^{-q}\cdot \exp\int_{|y|} 2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]\, ln\,(1-$$
$$xzd\mu(y) \qquad (2.6)$$

Here z is in D^*. The mapping between $\sum_{\varrho\tau\varsigma\xi kq}(\propto, 1, \eta)$ and the probability measure μ(y) is injective mapping.

Proof In agreement with the subordination condition (2.5), we can say that

$g \in \int_{\varrho\tau\xi kq}(\propto, 1, \eta)$ If & only if

$$-\frac{1}{2q(1-\eta)}\cdot\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \in q(\alpha).$$

From Lemma 1.2, we obtained

$$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\, f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\, f(z)} = 2q(1-\eta)\int_{|y|}\frac{1-\langle 4(1-\eta)\propto-1\rangle yz}{1-yz}\, d\mu(y)$$

Which is equivalent to

$$\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} = \int_{|x|}\frac{-2q(1-\eta)+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle yz}{1-yz}\, d\mu(y)$$

Integrating both the sides of above inequality, we have obtained

$$z^q D^k_{\varrho\tau\varsigma\xi q}\, g(z) = \exp\int_{|y|} 2q(1-\eta)[1+\langle 4(\eta-1)\propto$$
$$+1ln1-yzd\mu y.$$

Or

$$D^k_{\varrho\tau\varsigma\xi q}\, g(z) =$$
$$z^{-q}\cdot exp\int_{|y|} 2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]\, log(1-$$
$$xzd\mu(y) \qquad (2.7)$$

$$D^k_{\varrho\tau\varsigma\xi q}\, g(z) = (g*h)(z).$$

$$h(z) = \frac{1}{z^q} + \sum_{n=1-q}^{\infty}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)z^n$$

Using (2.10)

$$g(z) = \left[\frac{1}{z^q} + \sum_{n=1-q}^{\infty}\frac{z^n}{\Phi_k(\varrho,\tau,\varsigma,\xi,k,q)}\right] * \{ln(1-yz)\, d\mu(y)\}.$$

Where it is obviously for all $(\forall\, z\ in\ U^*)$.

Theorem 2.3 Let $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto, 1, \eta)$ i. e. here after it is to be taken as

$$z^q D^k_{\varrho\tau\varsigma\xi q}\, g(z) \prec (1-z)^{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}, \quad \forall\, z\ in\ U.$$

Proof Let $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto, 1, \eta)$ i. e. here after it is to be taken as by (2.5) we have

$$\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec \frac{-2q(1-\eta)+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle z}{1-z}$$

Since the function

$$\frac{-2q(1-\eta)+2q(1-\eta)\langle 2(1-\eta)\propto-1\rangle z}{1-z}$$

is convex and in U, In agreement with Goluzin's statement we have obtained

$$\int_0^z\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}\, d\zeta \prec \int_0^z\frac{-2q(1-\eta)+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle\zeta}{\zeta(1-\zeta)}\, d\zeta$$

Or

$$lnD^k_{\varrho\tau\varsigma\xi q}\, g(z) \prec ln\frac{[1-z]^{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}}{(z)^q}$$

Thus ∃ $a\ \ function\, w \in \Omega$ s. t.

$$lnD^k_{\varrho\tau\varsigma\xi q}\, g(z) \prec ln\frac{[1-w(z)]^{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}}{w(z)^q},$$

The above defined relation is equivalent to the subordination condition given below.

$$z^q D^k_{\varrho\tau\varsigma\xi q}\, g(z) \prec [1-z]^{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}.$$

Structural formula for the class $\sum_{\varrho\tau\varsigma\xi kq}(\propto, \beta, \eta)$:

Theorem 2.4 let $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto, \beta, \eta)$. I. e. here after it is to be taken as

$$g(z) = \left[z^{-q} + \sum_{n=1-q}^{\infty}\frac{z^n}{\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}\right] *$$
$$\left[z^{-q}\exp\left(2q(1-\eta)[\beta+\right.\right.$$
$$4\eta-1\beta\propto+\beta 0zw(\zeta)1-\beta w(\zeta)d\zeta. \qquad (2.8)$$

Where $(\forall\, z\ in\ D^*)$ & $(w \in \Omega)$.

Proof Let $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto, \beta, \eta)$ and since we have obtained

$$\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \prec \frac{p\langle 4(1-\eta)\propto-1\rangle\beta z-p}{1-\beta z}, \qquad (\forall\, z\ in\ D),$$

$$\therefore \frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} = \frac{p\langle 4(1-\eta)\propto-1\rangle\beta w(z)-2p(1-\eta)}{1-\beta w(z)}, \qquad (2.9)$$

From above Equation, we have

$$\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} + \frac{2p(1-\eta)}{z} = \frac{2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle\beta w(z)-2q(1-\eta)}{1-\beta w(z)},$$

Integrating above implies that

$$\frac{ln\left[z^q D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]}{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]} = \beta\int_0^z\frac{w(\zeta)}{[1-\beta w(\zeta)]}\, d\zeta. \qquad (2.10)$$

$$D^k_{\varrho\tau\varsigma\xi q}\, g(z) = (g*h)(z),$$

Where it is obviously for all,

$$h(z) = z^{-q} + \sum_{n=1-q}^{\infty}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)z^n.$$

Using (2.10)

$$g(z) = \left[z^{-q} + \sum_{n=1-q}^{\infty}\frac{z^n}{\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}\right] *$$

$\left[z^{-q} \exp\left(2q(1-\eta)[1+\right.\right.$
4η−1∝+1β0zw(ζ)1−βw(ζ)dζ.

Theorem 2.5 If function $g \in \Sigma_q$ is said to be in the class $\Sigma_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$, i. e. here after it is to be taken as

$$D^k_{\varrho\tau\varsigma\xi q}\, g(z) * \left\{ \frac{-2q(1-\eta)z^{-q}+(q+1)z^{1-q}}{(1-z)^2}\left(1-\beta e^{i\theta}\right) + \frac{z^{-q}}{(1-z)}\left[2q(1-\eta)-2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle \beta e^{i\theta}\right] \right\} \neq 0 \quad \forall\, z \text{ in } D^* \;\&\; \theta \in (0,2\pi). \tag{2.11}$$

Proof Let $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$. I. e. here after it is to be taken as, from (2.1) gives the following

$$-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \neq \frac{-2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle \beta e^{i\theta}+q}{1-\beta e^{i\theta}} \tag{2.12}$$

$\forall\, z$ in D, $\theta \in (0,2\pi)$

It is simple to show that the above condition (2.12) is equivalent to the following

$$(1-\beta e^{i\theta})z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]^{re^{i\theta}} D^k_{\varrho\tau\varsigma\xi q}\, g(z) \neq 0 \tag{2.13}$$

$\therefore D^k_{\varrho\tau\varsigma\xi q}\, g(z) = D^k_{\varrho\tau\varsigma\xi q}\, g(z) * \left(z^{-q}+z^{1-q}+\cdots+\frac{1}{z}+1+\right.$
z1−z=Dϱτςξqkg(z)∗z−q1−z
(2.14)

And $z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]' =$
$D^k_{\varrho\tau\varsigma\xi q}\, g(z) * \left(-qz^{-q}+(1-q)z^{1-q} - \cdots - \frac{1}{z} + \frac{z}{(1-z)^2}\right) =$
$D^k_{\varrho\tau\varsigma\xi q}\, g(z) * \frac{-qz^{-q}+(1+p)z^{1-q}}{(1-z)^2}$ (2.15)

$-\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)} \neq \frac{-2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle \beta e^{i\theta}+q}{1-\beta e^{i\theta}}$. Where $\forall\, z$ in D, $\theta \in (0,2\pi)$.

Coefficient Estimates

Theorem 2..6 Let function g be of the form (1.6) contained in the class $\sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$.I. e. here after it is to be taken as, for $m \geq 3 - 2q(1-\eta)$

$$\frac{|a_m|(m+q)\Phi_m(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)[1+\langle 4(\eta-1)\alpha+1\rangle]} \leq \beta. \tag{2.16}$$

here $\Phi_m(\varrho,\tau,\xi,k,q)$ is given by (1.11)

Proof We use the method of Koegh and Clunie [12] to prove the above mentioned Coefficient Estimates given by (2.16). For $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$, we have

$$\frac{\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}+2q(1-\eta)}{\frac{z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]'}{D^k_{\varrho\tau\varsigma\xi q}\, g(z)}+2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle} = zw(z)$$

where w is Analytic function in the unit D & $|w(z)| \leq \beta$ $\forall\, z$ in D. I. e. here after it is to be taken as

$$z\left[D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right]' + qD^k_{\varrho\tau\varsigma\xi q}\, g(z) = zw(z)\left[\begin{matrix} z\left(D^k_{\varrho\tau\varsigma\xi q}\, g(z)\right)' \\ +2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle D^k_{\varrho\tau\varsigma\xi q}\, g(z)\end{matrix}\right]. \tag{2.17}$$

$zw(z) = \sum_{n=1}^{\infty} w_n z^n$,

Using the equations (1.10) & (2.17), here we are going to obtain that

$$\sum_{n=1-q}^{\infty}(n+q)\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n z^{n+q} = \left\{\frac{-q[1+\langle 4(\eta-1)\propto+1\rangle]}{\frac{1}{\sum_{n=1-q}^{\infty}[n+q\langle 4(1-\eta)\propto-1\rangle\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)]a_n z^{n+q}}}\right\}\sum_{n=1}^{\infty} w_n z^n. \tag{2.18}$$

By using above Equation (2.18), we have obtained

$m\Phi_{m-q}(\varrho,\tau,\varsigma,\xi,k,q)a_{m-q}$
$= -2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]w_m,$

For m = 1,2

And

$m\Phi_{m-q}(\varrho,\tau,\varsigma,\xi,k,q)a_{m-q} = -2q(1-\eta)[1+$
4η−1∝+1wm +n=1−qm−1−q n+2q1−η21−η∝−1
Φnanwm−q−n ∀m≥3 (2.19)

From (2.19), we obtain

$$\left\{\begin{matrix} -4q(1-\eta)(1-\alpha)+ \\ \sum_{n=1-q}^{m-1-q}[n+2q(1-\eta)\langle 4(1-\eta)\propto -1\rangle\Phi_n]a_n z^{n+q}\end{matrix}\right\} \times \sum_{n=1}^{\infty} w_n z^n = \sum_{n=1-q}^{m-q}[(n+q)\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)]a_n z^{n+q} + \sum_{n=1-q}^{m-1-q} c_n z^{n+q} \tag{2.20}$$

It is known that, if $h(z) = \sum_{n=1-q}^{m-1-q} h_m z^m$ i. e. here after it is to be taken as for $0 < r < 1$,

$$\sum_{m=0}^{\infty}|h_m|^2 r^{2m} = \frac{1}{2\pi}\int_0^{2\pi}\left|h(re^{ii\theta})\right|^2 d\theta. \tag{2.21}$$

∴ the following

$\left|\sum_{n=1}^{\infty} w_n z^n\right| < \beta|z| < \beta,$

Using (2.20) and (2.21), we have been obtained the relation

$\sum_{n=1-q}^{m-q}\left[(n+\right.$
2q1−η2Φnϱ,τ,ς,ξ,k,q2an2r2(n+q)+

$\sum_{n=m+1-q}^{\infty}|c_n|^2 r^{2(n+q)} \leq \beta^2\{q^2[1+\langle 4(\eta-1)\propto+1\rangle]^2$
$+\sum_{n=1-q}^{m-1-q}[n+$
2q1−η41−η∝−12Φnϱ,τ,ς,ξ,k,q2an2r2n+q

Letting $r \to 1$, we have been obtained the relation

$\sum_{n=1-q}^{m-1-q}[n+2q(1-\eta)]^2\left(\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)\right)^2|a_n|^2$
$\leq 4(1-\eta)^2p^2\beta^2[1+\langle 4(\eta-1)\propto+1\rangle]^2$
$+\sum_{n=1-q}^{m-1-q}[n+$
2q1−η41−η∝−12Φnϱ,τ,ς,ξ,k,q2an2.

The above relation (inequality) implies

$m^2\Phi_{m-q}(\varrho,\tau,\varsigma,\xi,k,q)^2|a_{m-q}|^2$
$\leq 4(1-\eta)^2q^2\beta^2[1+\langle 4(\eta-1)\propto+1\rangle]^2$
$+\sum_{n=1-q}^{m-1-q}[k+2q(1-\eta)]^2\left(\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)\right)^2|a_n|^2$

Finally, replacing m −q by m, we have

$$|a_m| \leq \frac{4q(1-\eta)\beta\left[\frac{1+\langle 4(\eta-1)\alpha+1\rangle}{2}\right]}{(m+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)},$$

Hence above theorem is proved. Now theorem 2.6 given by the class $\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.

Corollary 2.1 If $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$is given by (1.6), i. e. here after it is to be taken as $\forall\, z$: $0 < |z| = r < 1$

$|g(z)| \leq$
$\frac{1}{r^q}+$
$q\beta[1+\langle 4(\eta-1)\propto+1\rangle]r^{1-q}\sum_{n=1-q}^{\infty}\frac{1}{(n+q)\Phi_n(\varrho,\tau\varsigma,\xi,k,q)}.$

$|g(z)| \geq \frac{1}{r^q}+2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]$

$\times r^{1-q} \sum_{n=1-q}^{\infty} \frac{1}{(n+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}$.

And

$|g'(z)| \geq \frac{2q(1-\eta)}{r^{q+1}} - 2q(1-\eta)\beta[1+\langle 4(\eta-1) \propto +1\rangle]$

$\times r^{2-q} \sum_{n=1-q}^{\infty} \frac{1}{(n+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}$.

$|g'(z)| \leq \frac{2q(1-\eta)}{r^{q+1}} + 4q(1-\eta)^2\beta[1+\langle 4(\eta-1) \propto +1\rangle]$

$\times r^{2-q} \sum_{n=1-q}^{\infty} \frac{1}{(n+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}$.

It will contained in the class given as $\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$. which is sufficient condition

Theorem 2.7 Let us assume that $g \in \sum_p$ represented by (1.6). If $\forall \propto (0 \leq \alpha < 1)$ and $\forall \beta (0 < \beta \leq 1)$

$$\sum_{n=1-q}^{\infty} \left\{ \begin{matrix} n(\beta+1)+ \\ 2q(1-\eta)[1+\beta\langle 4(1-\eta) \propto -1\rangle] \end{matrix} \right\} \times \Phi n\varrho,\tau,\varsigma,\xi,k,qan$$

$$\leq q(1-\eta)\beta[1+\langle 4(\eta-1) \propto +1\rangle] \qquad (2.22)$$

Where it is, $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$.

Proof We assume $g(z) = z^{-q} + \sum_{n=1-q}^{\infty} a_n z^n$.

We get

$M = \left|z[D_{\varrho\tau\varsigma\xi q}^k g(z)]' + 2p(1-\eta)D_{\varrho\tau\varsigma\xi q}^k g(z)\right|$

$-\beta\left|z[D_{\varrho\tau\varsigma\xi q}^k g(z)]' + \right.$

$2q1-\eta41-\eta\propto-1D\varrho\tau\varsigma\xi qkg(z)$

$= \left|\sum_{n=1-q}^{\infty}(n+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n z^n\right|$

$-\beta\left|\frac{-2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}{z^q}\right| + \sum_{n=1-q}^{\infty}[n+2q(1-$

$\eta41-\eta\propto-1\Phi n\varrho,\tau,\varsigma,\xi,k,qanzn$

For $0 < |z| = r < 1$, we have been obtained the following inequality

$r^q M \leq \sum_{n=1-q}^{\infty}(n+q)\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n r^{n+q}$

$-\beta\left[\begin{matrix} 2q(\eta-1)(\langle 4(\eta-1)\propto+1\rangle+1) \\ -\sum_{n=1-q}^{\infty}|n+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle|\Phi_n a_n r^{n+q} \end{matrix}\right]$.

Or $r^q M \leq \sum_{n=1-q}^{\infty}\{n(\beta+1)+2q(1-\eta)[1+$

$\beta41-\eta\propto-1$

$\times \Phi_n|a_n|r^{n+q} - 2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]$.

Because of the inequality mentioned above holds

$\forall r\ (0 < r < 1)$, letting $r \to 1$, we have obtained

$M \leq \sum_{n=1-q}^{\infty} n(\beta+1)+2q(1-\eta)[1+\beta\langle 4(1-\eta)\propto$

$-1\times\Phi nan-2q1-\eta\beta1+4\eta-1\propto+1$.

Using the Equation (2.22), we have obtained $M \leq 0$, ∴

$\left|\frac{z[D_{\varrho\tau\varsigma\xi q}^k g(z)]'}{D_{\varrho\tau\varsigma\xi q}^k g(z)} + 2p(1-\eta)\right| < \beta\left|\frac{z[D_{\varrho\tau\varsigma\xi q}^k g(z)]'}{D_{\varrho\tau\varsigma\xi q}^k g(z)} + \right.$

$2q1-\eta41-\eta\propto-1$.

Consequently, $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$.

3. Properties of the class $\sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$:

In this part of the chapter we are proving that $\forall$ functions which are in the class (2.22) $\sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$ the Equation (2.22) is required necessary and sufficient condition.

Theorem 3.1 Let us assume a function $g \in \sum_q^+$, i. e. here after it is to be taken as $\forall\ g$ in the class $\sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$ If & only if

$\sum_{n=1-q}^{\infty}\{n(\beta+1)+2q(1-\eta)[1+\beta\langle 2(1-\eta)\propto-1\rangle]\}$

$\times \Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n \leq 2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto$

$+1$.

Proof In agreement with result obtained as 2.7, we are proving that "only if" part. Consider that

$g(z) = z^{-q} + \sum_{n=1-q}^{\infty} a_n z^n \quad (a_n \geq 0, q\ in\ N)$

is in the class $\sum_{\varrho\tau\varsigma\xi kq}^{+}(\propto,\beta,\eta)$.

$$\therefore \left|\frac{\frac{z[D_{\varrho\tau\varsigma\xi q}^k g(z)]'}{D_{\varrho\tau\varsigma\xi q}^k g(z)}+2q(1-\eta)}{\frac{z[D_{\varrho\tau\varsigma\xi q}^k g(z)]'}{D_{\varrho\tau\varsigma\xi q}^k g(z)}+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle}\right| =$$

$$\left|\frac{\sum_{n=1-q}^{\infty}(k+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n r^{n+q}}{\frac{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}{z^q}-\sum_{n=1-q}^{\infty}[n+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle]\Phi_n a_n z^n}\right| < \beta$$

$\forall\ z\ in\ D.\ \therefore Rep\ z \leq |z|$. It is obvious that

$$\operatorname{Re}\left\{\frac{\sum_{n=1-q}^{\infty}(n+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n r^{n+q}}{\frac{2q(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}{z^q}-\sum_{n=1-q}^{\infty}[n+2q(1-\eta)\langle 4(1-\eta)\propto-1\rangle]\Phi_n a_n z^n}\right\} < \beta. \qquad (3.1)$$

take the values of z on x-axis of z-plane s. t.

$\frac{1}{2q(1-\eta)} \cdot \frac{z[D_{\varrho\tau\varsigma\xi q}^k g(z)]'}{D_{\varrho\tau\varsigma\xi q}^k g(z)}$

If we clear the denominator in the Equation (2.9.1) and consider as $z \to 1$ all positive values i. e. here after it is to be taken as we get

$\sum_{n=1-q}^{\infty}(n+2q(1-\eta))\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)a_n$

$\leq 2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle] - \sum_{n=1-q}^{\infty}\beta\,[n+$

$2q1-\eta41-\eta\propto-1\Phi n\varrho,\tau,\varsigma,\xi,k,q\ an$

Or

$\sum_{n=1-q}^{\infty}\{n(\beta+1)+2q(1-\eta)[1+\beta\langle 4(1-\eta)\propto$

$-1)\Phi n\varrho,\tau,\varsigma,\xi,k,qan$

$\leq q\beta[1+\langle 4(\eta-1)\propto+1\rangle]$.

Corollary 3.1 If a function $g \in \sum_q^+$ defined as in equation (1.7) contained in the class $\sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$ i. e. here after it is to be taken as we say that for $n \geq 1-2q(1-\eta)$.

$$b_n \leq \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{n(\beta+1)+2q(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}, \qquad (3.2)$$

for the functions given below the equality

$$g_n(z) = \frac{1}{z^q} - \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{n(\beta+1)+2q(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)} z^n$$

Corol. 3.1 gives a distortion result for $\sum_{\varrho\tau\varsigma\xi kq}^{+}(\propto,\beta,\eta)$.

Theorem 3.2 If a function $g \in \sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$, i. e. here after it is to be taken as $\forall\ z\ (0 < |z| < 1)$

$$|g(z)| \geq \frac{1}{z^p} - \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} r^{1-q}$$

$$|g(z)| \geq \frac{1}{z^q} + \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} r^{1-q}.$$

Where the inequality holds true for the functions given as

$$g_q(z) = \frac{1}{z^q} + \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\alpha-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} z^{1-q}$$

at $z = ir, r$.

Proof Supposing $g \in \sum_{\varrho\tau\varsigma\xi kq}^{+}(\propto,\beta,\eta)$ and using the inequality given as

$\sum_{n=1-q}^{\infty} b_n \leq$

$$\frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\alpha-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} \quad (3.3)$$

this by theorem 3.1, the proof of the given theorem. In the next part we are proving that the class $\sum_{\varrho\tau\varsigma\xi kq}^{+}(\propto,\beta,\eta)$ is closed under convolution.

Theorem 3.3 Let us assume that function $h(z)$ which is Analytic in U^* and given $h(z) = z^{-q} + \sum_{n=1-q}^{\infty} c_n z^n$ implies that $0 \le c_n \le 1$. If function g given in (1.7) is in the class $\sum_{\varrho\tau\varsigma\xi kq}^{+}(\propto,\beta,\eta)$ i. e. here after it is to be taken as the convolution of functions $g \,\&\, h$ $(g * h)$ contained in $\sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$.

Proof Since $f \in \sum_{\varrho\tau\xi mp}^{+}(\propto,\beta,\eta)$, i. e. here after it is to be taken as by Theorem 3.1,

$\sum_{n=1-q}^{\infty}\{n(\beta+1)+$
2q1−η1+β41−η∝−1Φnϱ,τ,ς,ξ,k,qbk
$\le 2q(1-\eta)\beta[1+$
4η−1∝+1.

In agreement with the inequality given as above relation & the result given below

$$(g*h)(z) = z^{-q} + \sum_{n=1-q}^{\infty} b_n c_n z^n$$

∴

$\sum_{n=1-q}^{\infty}\left\{\begin{matrix} n(\beta+1)+ \\ 2q(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle] \end{matrix}\right\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)b_n\sum_{n=1-q}^{\infty}\{n(\beta+$
1+2q1−η1+β41−η∝−1Φnϱ,τ,ς,ξ,k,qbk
$\le 2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]$.

Theorem 3.4 If the function g defined as $g \in \sum_{\varrho\tau\varsigma\xi kq}^{+}(\propto,\beta,\eta)$, and the integral operator as given below $G_{c,q}(z) = \frac{c}{z^{p+c}}\int_0^z t^{c+2p(1-\eta)+1} f(t)dt$, $c > 0$. Contained in $\sum_{\varrho\tau\varsigma\xi mp}^{+}(\propto,\beta,\eta)$.

Proof it is very simple to understand and simple to verify & check that

$$F_{c,p}(z) = f(z) * \left(z^{-p} + \sum_{k=1-p}^{\infty}\frac{c}{c+2q(1-\eta)+k}z^k\right).$$

$$\therefore 0 < \frac{c}{c+2q(1-\eta)+k} \le 1,$$

Theorem holds true.

4. Neighborhoods and Partial Sums

The well-known $(n,\delta)-$ neighborhoods of Regularfunctions & partial sums was earlier investigated and studied by many authors Aouf [71], Goodman [19], [20], [21], Orhan [46], Ruscheweyh [56], [59], Liu [54], Altina [3], [46], [58], Deniz and Orhan [14] and more recently by Liu and Srivastava [38]. We have defined the $(n,\delta)-$ neighborhoods of Analytic functions $g \in \sum_1$ of the form (2.7.6).

Definition 4.1 ∀, $\delta = \frac{2(\tau-\xi)^2+\frac{1}{\varrho}(\tau-\xi)(1-\varrho^2)}{1+2(\tau-\xi)^2+\frac{1}{\varrho}(\tau-\xi)(1-\varrho^2)} > 0$

& non negative sequence $s = \{s_n\}_{n=1-q}^{\infty}$

Where it is obviously for,

$$s_n = \frac{\{\beta[1-2q(1-\eta)+|\langle 4(1-\eta)\propto-1\rangle|2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]} \quad (4.1)$$

$(n \ge 1-q, q \in N, 0 \le \alpha < 1, 0 < \beta \le 1)$.

The $(n,\delta)-$ neighborhoods of Analytic functions $g \in \sum_q$ given by (1.6) defined as

$$N_\delta(g) = \left\{\begin{matrix} g \in \sum_q : g(z) = z^{-q} + \sum_{n=1-q}^{\infty} b_n z^n \text{ and} \\ \sum_{n=1-q}^{\infty} s_n|b_n - a_n| \le \delta \end{matrix}\right\}.$$

(4.2)

∀ $s_n = n$, definition 1.4 corresponds to the $(n,\delta)-$ neighborhoods of analytic functions considered by Ruschewyh [56]. Using definition 4.1, we are proving results for $(n,\delta)-$ neighborhoods of Analytic functions of the class $\sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$.

Theorem 4.1 let $g \in \sum_{\varrho\tau\varsigma\xi kq}(\propto,\beta,\eta)$ denoted by Equation (1.6). If a function g *satisfy* the following relation

$$[g(z)+\varepsilon z^q](1+\varepsilon)^{-1} \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta) \quad (4.3)$$

$(\varepsilon \in C,\ |\varepsilon| < \delta,\ \delta \ge 0)$

i. e. here after it is to be taken as

$$N_\delta(f) \subset \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta) \quad (4.4)$$

Proof $g \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$ if & only if

$$\frac{z\left[D_{\varrho\tau\varsigma\xi q}^{k}\, g(z)\right]' + 2q(1-\eta)D_{\varrho\tau\varsigma\xi q}^{k}\, g(z)}{\beta z\left[D_{\varrho\tau\varsigma\xi q}^{k}\, g(z)\right]' + \beta\langle 4(1-\eta)\propto-1\rangle 2q(1-\eta)D_{\varrho\tau\varsigma\xi q}^{k}\, g(z)} \ne \sigma \quad (4.5)$$

$(\forall\, z \text{ in } D,\ \sigma \in C,\ |\sigma| = 1)$

Equivalent to the following

$$\frac{(g*h)(z)}{z^{-q}} \ne 0 \quad (\forall\, z \text{ in } D). \quad (4.6)$$

It is convenient for

$h(z) = z^{-q} + \sum_{k=1-q}^{\infty} c_n z^n$

$= z^{-q} +$

$$\sum_{n=1-q}^{\infty}\frac{\{\beta\sigma[n+\langle 4(1-\eta)\propto-1\rangle 2q(1-\eta)]-(n+2q(1-\eta))\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]\sigma}z^n$$

(4.7)

From (4.7) it is oblivious that

$$|c_n| = \left|\frac{\{\beta\sigma[n+\langle 4(1-\eta)\propto-1\rangle 2q(1-\eta)]-(n+2q(1-\eta))\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]\sigma}\right|$$
$$\le \frac{\{\beta\sigma[n+|\langle 4(1-\eta)\propto-1\rangle|2q(1-\eta)]+(n+2q(1-\eta))\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]\sigma}$$

(4.8)

Further in accordance with (4.3), and by using (4.6) we have obtained the following result $\frac{[g(z)+\varepsilon z^q](1+\varepsilon)^{-1}*h(z)}{z^{-q}} \ne 0$.

Or $\frac{(g*h)(z)}{z^{-q}} \ne \varepsilon$, $(\forall\, z \text{ in } D)$. which is equivalent to the result given below

$$\left|\frac{(g*h)(z)}{z^{-q}}\right| \ge \delta \quad (4.9)$$

Let us now consider

$f(z) = z^{-q} + \sum_{n=1-q}^{\infty} a_n z^n \in N_\delta(g)$,

i. e. here after it is to be taken as we have obtained

$$\left|\frac{[g(z)-f(z)]*(z)}{z^{-q}}\right| = \left|\sum_{n=1-q}^{\infty}(b_n - a_n)z^{n+q}\right|$$
$$\le \sum_{n=n-q}^{\infty}\frac{\{\beta[n+|\langle 4(1-\eta)\propto-1\rangle|2q(1-\eta)]+(n+2q(1-\eta))\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}$$
$$|b_n - a_n||z|^{n+q} < \delta$$

$(\forall\, z \text{ in } D, n \ge 1-q, q \in N, \delta > 0)$.

Definition 4.2 ∀, $\delta > 0$ for a non -ve sequence $S = \{s_n\}_{n=1-q}^{\infty}$

Here it is obviously for,

$$s_n = \frac{\{n(\beta+1)+2n(1-\eta)[1+\beta\langle 4(1-\eta)\alpha-1\rangle]\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]}$$

$(n \ge 1-q, q \in N, 0 \le \propto < 1, 0 < \beta \le 1)$

The $(n,\delta)-$ neighborhoods holo morphic functions g s.

t. $g \in \sum_q^+$ as given in (1.7) which is given by

$$N_\delta(g) = \begin{Bmatrix} f \in \sum_q^+ : f(z) = z^{-q} + \sum_{n=1-q}^{\infty} a_n z^n \\ and \ \sum_{n=1-q}^{\infty} s_n |a_n - b_n| \le \delta \end{Bmatrix} \quad (4.10)$$

We have obtained the following result on the (n, δ) – neighborhoods of analytic functions of the class $\sum_{\varrho\tau\varsigma\xi mp}^+(\alpha, \beta, \eta)$.

Theorem 4.2 If the function g as given in Equation (1.7) belonging to the class defined $\sum_{\varrho\tau\varsigma\xi kq}^+(\alpha,\beta,\eta)$,
I. e. here after it is to be taken as

$$N_\delta(g) \subset \sum_{\varrho\tau\varsigma\xi kq}^+(\alpha,\beta,\eta), \quad (4.11)$$

where it is obviously for all,

$$\delta = \frac{2(\tau-\xi)(\tau-\varsigma) + \frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}}{1+2(\tau-\xi)(\tau-\varsigma) + \frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}}.$$

From above it is clear that means δ can't increase.

Proof $\forall\, g \in \sum_{\varrho\tau\varsigma\xi kq}^+(\alpha,\beta,\eta)$ of the form (1.7), theorem 3.1 immediately yields

$$\sum_{n=1-q}^{\infty} \frac{\{n(\beta+1)+2q(1-\eta)[1+\beta\langle 4(1-\eta)\alpha-1\rangle]\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]} a_k \le \frac{1}{\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,1,q)} \quad (4.12)$$

Let $f(z) = z^{-q} + \sum_{n=1-q}^{\infty} a_n z^n \in N_\delta(f)$

Where $$\delta = \frac{2(\tau-\xi)(\tau-\varsigma) + \frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}}{1+2(\tau-\xi)(\tau-\varsigma) + \frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}} > 0.$$

from the condition (4.10) we have obtained

$$\sum_{n=1-q}^{\infty} s_n |a_n - b_n| \le \delta \quad (4.13)$$

Using (4.12) & (4.13), we have obtained

$$\sum_{n=1-q}^{\infty} s_n a_n \le \sum_{n=1-q}^{\infty} s_n b_n + \sum_{n=1-q}^{\infty} s_n |a_n - b_n| \le \frac{1}{\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} + \delta = 1$$

Thus in accordance with theorem 3.1, we get $f \in \sum_{\varrho\tau\varsigma\xi kq}^+(\alpha,\beta,\eta)$.To prove more accuracy of functions $g \in \sum_{\varrho\tau\varsigma\xi kq}^+(\alpha,\beta,\eta)$and $g \in \sum_q^+$ as follows

$$g(z) = z^{-q} + \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\alpha-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} z^{1-q} \quad (4.14)$$

$$g(z) = z^{-q} + \left[\frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\alpha-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)} + \frac{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]}{\{\beta[1-2q(1-\eta)+\langle 4(1-\eta)\alpha-1\rangle 2q(1-\eta)]+1\}\Phi_{1-q}(\varrho,\tau,\varsigma,\xi,k,q)}\right] z^{1-q} \quad (4.15)$$

Hence the function $f \in N_\delta(g)$ but in agreement with Theorem 3.1, $g \in \sum_{\varrho\tau\varsigma\xi kq}^+(\alpha,\beta,\eta)$. Consequently, our theorem is proved. In the next part, we have introduced as given in Equation (1.6)

$$k_k(z) = \begin{cases} z^{-q}, & k = 1,2,\cdots,-q \\ z^{-q} + \sum_{n=1-q}^{k-1} a_n z^n & k = 1-q, 2-q,\cdots \end{cases} \quad (4.16)$$

we have determined lower bounds which are sharp for Rep $\left\{\frac{g(z)}{k_k(z)}\right\}$ and $Re\left\{\frac{k_k(z)}{g(z)}\right\}$.

Theorem 4.3 Let $g \in \sum_q$ is given by (1.6) and let $k_k(z)$ be given by (4.16). Suppose that

$$\sum_{n=1-q}^{\infty} \theta_n |a_n| \le 1 \quad (4.17)$$

Here it is obviously for all,

$$\theta_n = \frac{\{n(\beta+1)+2q(1-\eta)[1+\beta\langle 4(1-\eta)\alpha-1\rangle]\}\Phi_n(\varrho,\tau,\varsigma,\xi,k,q)}{2q(1-\eta)\beta[1+\langle 4(\eta-1)\alpha+1\rangle]}$$

I. e. here after it is to be taken as $\forall\ k \ge 1-q$, we get

$$Rep\left\{\frac{g(z)}{k_k(z)}\right\} > 1 - \frac{1}{\theta_k} \quad (4.18)$$

$$Rep\left\{\frac{k_k(z)}{g(z)}\right\} > \frac{\theta_k}{1+\theta_k} \quad (4.19)$$

the above defined results are sharp for the following function which is extremal ∀ k ≥ 1 − q

$$g(z) = z^{-q} - \frac{1}{\theta_k} z^k \quad (4.20)$$

Proof We can show from the Equation (4.17) that $\theta_{n+1} > \theta_n > 1,\ (n > 1-q)$.
hence, by using the hypothesis of the equation (4.17), we get

$$\sum_{n=1-q}^{k-1} |b_n| + \theta_k \sum_{n=k}^{\infty} |b_n| \le \sum_{n=1-q}^{\infty} \theta_n |b_n| \le 1. \quad (4.21)$$

Let $$w(z) = \theta_k \left[\frac{g(z)}{k_k(z)} - \left(1 - \frac{1}{\theta_k}\right)\right] = 1 + \frac{\theta_k \sum_{n=k}^{\infty} b_n z^{n+q}}{1+\sum_{n=1-q}^{k-1} b_n z^{n+q}} \quad (4.22)$$

Using equations (4.21) & (4.22),we have obtained $\left|\frac{w(z)-1}{w(z)+1}\right| = \left|\frac{\theta_k \sum_{n=k}^{\infty} b_n z^{n+q}}{2+2\sum_{n=1-q}^{k-1} b_n z^{n+q} + \theta_k \sum_{n=k}^{\infty} b_n z^{n+q}}\right|$

$$\le \frac{\theta_k \sum_{n=1-q}^{k-1} |b_n|}{2-2\sum_{n=1-q}^{k-1} |b_n| - \theta_k \sum_{n=1-q}^{k-1} |b_n|} \le 1 \quad \forall\, z\ in\ D. \quad (4.23)$$

∴ Rep w(z) > 0, (∀z in D). By using equation (4.22), we can easily obtain equation (4.18). Now g as in (4.20), here after for more accurate results, we can show that for $z \to 1^-$ $\frac{g(z)}{k_k(z)} = 1 - \frac{1}{\theta_k} z^k \to 1 - \frac{1}{\theta_k}$
this proves the bound in the equation (4.18) is the better possible. Similar to this if Let $\Phi_z = (1 +$
$\theta kkk(z)g(z) - \theta k1 + \theta k = 1 - 1 + \theta kn = k\infty bnzn+q$
$1+n=1-q\infty bnzn+q$ (4.24)
And by making use of equation (4.21), we have obtained the following relation $\left|\frac{\Phi(z)-1}{\Phi(z)+1}\right| = \left|\frac{-(1+\theta_k)\sum_{n=k}^{\infty} b_n z^{n+q}}{2+2\sum_{n=1-q}^{\infty} b_n z^{n+q} - (1+\theta_m)\sum_{k=m}^{\infty} b_n z^{n+q}}\right|$

$$\le \frac{(1+\theta_k)\sum_{n=k}^{\infty} |b_n|}{2-2\sum_{n=1-k}^{\infty} |b_n| + (1+\theta_k)\sum_{n=k}^{\infty} |b_n|} \le 1 \quad (4.25)$$

leads to the Equation given as (4.19). The bound in the Equation (4.19 is more accurate with the extremal function f as shown in Equation (4.20). Our theorem is proved.

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