Abstract: In this chapter we have investigated and studied two new subclasses \( \sum_{\alpha \in \mathbb{C}} (\alpha, \beta, \eta) \) and \( \sum_{\alpha \in \mathbb{C}}^{+} (\alpha, \beta, \eta) \) of meromorphic multivalent functions in the punctured disk \( D \), utilizing the principle of Differential subordinates we introduced various inclusion relationship & properties of particular subclasses those are defined and derived using differential operator. The outcomes which are concerned with subordination properties like Convolution of functions, the radii of starlikeness, coefficient inequalities, and convexity closure theorems etc. are derived & studied.

Keywords: Multivalent functions, radii of starlikeness, coefficient inequalities, convexity, closure theorems.

1. Introduction

In this chapter we have investigated and studied two new subclasses \( \sum_{\alpha \in \mathbb{C}} (\alpha, \beta, \eta) \) and \( \sum_{\alpha \in \mathbb{C}}^{+} (\alpha, \beta, \eta) \) of meromorphic multivalent functions in the punctured disk \( D \). The outcomes which are concerned with subordination properties like Convolution of functions, the radii of starlikeness, coefficient inequalities, and convexity closure theorems etc. are derived & studied. We extended the well known concept of \( (n, \delta) \) - neighborhood functions to the above defined subclasses of multivalent functions. Let us assume that the class \( \mathfrak{m} \subseteq D \) which is a regular function belonging to \( D \) given by \( D = \{ z : |z| < 1 \} \). Consider
\[
\Omega = \{ w \in \mathfrak{m} : w \text{ at } z = 0 \text{ is 0 and } \text{mod } w(z) < 1 \}. \quad (1.1)
\]
Be the class of Schwarz functions. For \( 0 \leq \alpha < 1 \).

Let us assume \( q(\infty) = q \in A : 1 - \text{Re } q(z) > 0, \forall z \in D \).

Note that \( q = q(0) \) is the popular class of Schwarz Carathéodory functions. The popular class of Schwarz's functions \& Carathéodory functions plays very much useful role in geometrical function theory of Regular function. This is investigated and studied by various researchers. It’s very simple to check \( \forall q \in q(\alpha) \) if \& only if the following condition is satisfied
\[
\frac{p(z) - \infty}{1 - \infty} \in P \quad (1.3)
\]
Using (1.3) and applying the properties of the functions which are in \( q \), the lemmas given below for the above defined functions in the class \( q(\alpha) \) can be obtained.

2. Preliminary Lemmas:

Lemma 1.1 Let us assume \( p \in \mathfrak{m} \) i.e. here after it is to be taken as \( p \in P(\alpha) \) \( \iff \exists w \in \Omega \) satisfying
\[
q(z) = \frac{1 - \alpha q(z)}{1 - w(z)} \quad (1.4)
\]

Lemma 1.2 (Herglotz formula): A function \( p \in \mathfrak{m} \) contained in the class \( P(\alpha) \) \( \exists a \) a probability measure \( \mu(y)\) on \( \partial D \) satisfying the following Equation given by
\[
q(z) = \int_{1}^{+\infty} \frac{1 - \alpha q(z)}{1 - y(z)} \, dq(y), \quad (\forall z \in D). \quad (1.5)
\]
We know that if \( g < f \) i.e. here after it is to be taken as \( g \) at \( z = 0 \) is equal to 0 and \( f \) at \( z = 0 \) and \( \text{gcd}(f) \) is subset of \( f(D) \). Particularly, if \( f \) is univalent in \( D \) we will get the equivalence relation \( g(f) = f(f) \) if \& only if \( g(z) < f(z) \) \& \( g(D) \subset f(D) \). Let us assume that \( \sum_{\alpha} \) represents the classes of all meromorphic functions \( g \) given by
\[
g(z) = \frac{1}{z^{r}} + \sum_{n=1-s}^{\infty} b_{n} z^{n} \quad (\forall b_{n} \geq 0, q \in N). \quad (1.6)
\]
The above defined functions are multivalent Analytic in the punctured unit disk \( D^{*} = D \setminus \{ 0 \} \), given by \( \sum_{\alpha} \) represents subclass of \( \sum_{\alpha} \) which contains the functions given by
\[
g(z) = \frac{1}{z^{r}} + \sum_{n=1-s}^{\infty} b_{n} z^{n} \geq 0 \quad (\forall z \in D^{*}) \quad (1.7)
\]
Function \( g \in \sum_{\alpha} \) is said to be meromorphic univalent starlike of order \( 0 < \infty \). Where,
\[
\text{Re} \left( \frac{1}{\sqrt{q(\alpha)}} \right) \geq (z \in U).
\]
The class \( \sum_{\alpha} \) represents such functions that are as given above. If \( g \in \sum_{\alpha} \) is given by (1.6) \& \( f \in \sum_{\alpha} \) given below
\[
f(z) = \frac{1}{z^{r}} + \sum_{n=1-s}^{\infty} a_{n} z^{n}, \quad (1.8)
\]
I. e. here after it is to be taken as the Convolution that is the \( g * f \) as
\[
(g * f) = \frac{1}{z^{r}} + \sum_{n=1-s}^{\infty} b_{n} a_{n} z^{n} \geq 0 \quad (\forall q \in N \text{ & } \forall z \in D^{*}).
\]
For all functions \( g \in \sum_{\alpha} \), we are defining the differential operator \( D_{\alpha}^{\delta} \) as given as follows
\[
D_{\alpha}^{\delta} q(g(z)) = g(z) \quad (1.9)
\]
Where it is obviously for all,
\[
0 < q \leq \frac{1}{2}, 0 \leq \xi < 1, \tau \geq 1, 0 \leq \xi < 1, 0 \leq \eta < 1 \}
\]
If the function $f \in \Sigma_q$ as given in Equation (1.6) i.e. here after it is to be taken as, from (1.8) and (1.9), we have been obtained that

$$D^k_{\rho \zeta \sigma} g(z) = \frac{1}{z^k} + \sum_{n=1}^{\infty} \Phi_n(q, \tau, \zeta, k, q)b_nz^n$$  \hspace{1cm} (1.10)\hspace{1cm} (k in N, q in N, z in D^*)$$

Here over it is,

$$\Phi_n(q, \tau, c, \xi, k, q) = \left(1 + \left[\frac{(n+q)}{e^\tau} + (n+q+1)(\tau - \xi)(\tau - \xi)\right]\right)^k.$$  \hspace{1cm} (1.11)$$

from (1.10) $\Rightarrow D^k_{\rho \zeta \sigma} g(z)$ in terms of convolution can be given as follows

$$D^k_{\rho \zeta \sigma} g(z) = (g \ast h)(z)$$  \hspace{1cm} (1.12)$$

Where it is,

$$h(z) = \frac{1}{z^k} + \sum_{n=1}^{\infty} \Phi_n(q, \tau, c, \xi, k, q)z^n.$$  \hspace{1cm} (1.13)$$

Note that, the case $\xi = \frac{1}{2}$ and $\tau = c$ of $D^k_{\rho \zeta \sigma} g(z)$ which denotes Differential operator was investigated and studied by researchers Srivastava and Patel [60]. The differential operator $D^p_{\rho \zeta \sigma} f(z)$ for $p = 1$ was considered in [50]. For the differential operator $D^k_{\rho \zeta \sigma} g(z)$, we are going to define a new subclass of functions given by $\Sigma_q^k$. 

**Definition 1.1** A function given by $g \in \Sigma_q$ contained in the class $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$, if it will satisfy the inequality given as follows.

$$\left|\frac{z[D^k_{\rho \zeta \sigma} g(z)]'}{D^k_{\rho \zeta \sigma} g(z)}\right| < \beta$$

And \hspace{1cm} (0 < q \leq \frac{1}{2}, 0 < \xi < 1, \tau \geq 1, 0 \leq \xi \leq 1, 0 < \eta < 1, 0 < \kappa_1 < 1 \text{ and } k = 0$$ \text{ is the class of Meromorphic univalent starlike functions of order } \alpha \text{ and type } \beta.$$

Recently Mogura et al.[62] introduced and studied given by

$$\Sigma_{q\rho \zeta \theta} (\alpha, \beta, \eta) \ni q = 1 \text{ and } k = 0$$ is the class of $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$.

Let us assume another subclass of $\Sigma_q$ as follows

$$\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta) \ni \Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta) \cap \Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta).$$

And \hspace{1cm} (0 < q \leq \frac{1}{2}, 0 < \xi < 1, \tau \geq 1, 0 \leq \xi \leq 1, 0 < \eta < 1, 0 < \kappa_1 < 1 \text{ and } k = 0$$ \text{ is the class of Meromorphic univalent starlike functions of order } \alpha \text{ and type } \beta.$$

The main purpose of this chapter is to study and represent systematically investigation of the classes $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$ and $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$. 

4.2 Properties of the Class $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$

We are going to start this part of the chapter for function which contained in given by $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$ with respect required necessary & sufficient conditions in terms of subordinations.

**Theorem 2.1** A function $g \in \Sigma_q$ contained in $\Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$ if & only if that will satisfy the condition as given as follows.

$$\left|\frac{z[D^k_{\rho \zeta \sigma} g(z)]'}{D^k_{\rho \zeta \sigma} g(z)}\right| < \frac{q(1-\eta)(1-\beta)z-q}{1-\beta z} \hspace{1cm} (\forall z \in D).$$

**Proof** Let $g \in \Sigma_{q\rho \zeta \sigma} (\alpha, \beta, \eta)$. I. e. here after it is to be taken as, from (1.6), we have
(1.14) becomes
\[ \left| \frac{z^p g(z)}{g'(z)} \right|^{1+2p(\eta-1)} \leq \beta \]  
\[ \frac{z^p g(z)}{g'(z)} \leq 2q(1-\eta) \frac{1-(1+(4\eta-1)x)yz}{1-\beta z} \]
\[ \frac{z^p g(z)}{g'(z)} \leq \frac{2q(1-\eta)+2q(1-\eta)(1+(4\eta-1)x)yz}{1-\beta z} \]

Hence the theorem proved.

Remark 2.1 Since
\[ \Re \left( \frac{1}{z} g(z) \right) > 0 \]
It follows that
\[ z^p \Re \left( \frac{z^p g(z)}{g'(z)} \right) > 0, \]  
\[ z^p g(z) \in \mathbb{S}_p(\alpha). \]

Structural formula for the class \( \mathbb{S}_{\zeta, \tau, \kappa}(\alpha, 1, \eta) \):

Theorem 2.2 Function \( g \in \mathbb{S}_q \) said to be in the class \( \mathbb{S}_{\zeta, \tau, \kappa}(\alpha, 1, \eta) \) if \& only if \( \exists \) a probability measure given by \( \mu(y) \) on \( \mathbb{D} \) such as
\[ g(z) = \frac{1}{1-z} + \sum_{n=1}^{\infty} \frac{n}{q(n \cdot r, s, c, k, q)} z^n \]
\[ \cdot \left\{ z^{-q} \exp \int_y^{2q(1-\eta) \left[ 1+4(4\eta-1) \alpha+1 \right]} \ln(1-1-xyz) d\mu(y) \right\} \]

Hence \( g(z) \) is in \( \mathbb{D}^* \). The mapping between \( \mathbb{S}_{\zeta, \tau, \kappa}(\alpha, 1, \eta) \) and the probability measure \( \mu(y) \) is injective mapping.

Proof In agreement with the subordination condition (2.5), we can say that
\[ g \in \mathbb{S}_{\zeta, \tau, \kappa}(\alpha, 1, \eta) \] if \& only if
\[ z^p \Re \left( \frac{z^p g(z)}{g'(z)} \right) \in q(\alpha). \]

From Lemma 1.2, we obtained
\[ z^p \Re \left( \frac{z^p g(z)}{g'(z)} \right) = 2q(1-\eta) \int_y^{1-(4\eta-1)x+1} \frac{1-\beta z}{1-z} \mu(y) \]

Which is equivalent to
\[ \frac{z^p g(z)}{g'(z)} = \int_x^{1-(4\eta-1)x+1} \frac{1-\beta z}{1-z} \mu(y) \]

Integrating both the sides of above inequality, we have obtained
\[ z^p g(z) = \exp \int_y^{2q(1-\eta) \left[ 1+4(4\eta-1) \alpha+1 \right]} \ln(1-1-xyz) d\mu(y) \]

Or
\[ g(z) = z^{-q} \cdot \exp \int_y^{2q(1-\eta) \left[ 1+4(4\eta-1) \alpha+1 \right]} \ln(1-1-xyz) d\mu(y) \]

Using (2.10)
\[ g(z) = \left( 1 + \sum_{n=1}^{\infty} \frac{n}{q(n \cdot r, s, c, k, q)} \right) \left( \ln(1-1-xyz) \right) d\mu(y) \]

Where it is obviously for all \( \forall z \in U' \).
Proof Let $g \in \Sigma_{\alpha, \beta, \eta}$. i.e. here after it is to be taken as, from (2.1) gives the following
\[
\begin{align*}
-2q(1-\eta)z^{-q} + (q+1)z^{1-q} &\neq 0, \\
(1-\beta \epsilon^{i\theta})z^{D_{q}^{e}g(z)} &\neq 0, \\
\end{align*}
\]
\[
\forall z \in D, \theta \in (0, 2\pi).
\]

Coeficient Estimates

Theorem 2.6 Let function $g$ be of the form (1.6) contained in the class $\Sigma_{\alpha, \beta, \eta}$. i.e. here after it is to be taken as, for $m \geq 3 - 2q(1-\eta)$
\[
|\Lambda_{m+2q(1-\eta)}(\alpha, \beta, \eta, k, q)| \leq \beta, \quad m \geq 1.
\]

Theorem 2.5 If function $g \in \Sigma_{\alpha, \beta, \eta}$ is said to be in the class $\Sigma_{\alpha, \beta, \eta}$, i.e. here after it is to be taken as
\[
\begin{align*}
D_{q}^{e}g(z) &\neq 0, \\
(1-\beta \epsilon^{i\theta})z^{D_{q}^{e}g(z)} &\neq 0, \\
\forall z \in D, \theta \in (0, 2\pi).
\end{align*}
\]

Using the equations (1.10) & (2.17), here we are going to obtain
\[
\sum_{n=1}^{\infty}(n+q)\Phi_{n}(q, r, \epsilon, k, q)a_{n}\zeta^{n+q} = \left\{ -q[1+(4(q-1)\beta+1)] \right\}
\]
\[
\sum_{n=1}^{\infty}(n+q)\Phi_{n}(q, r, \epsilon, k, q)a_{n}\zeta^{n+q} \leq 1/\pi + q\beta[1+(4(q-1)\beta+1)]\sum_{n=1}^{\infty}(n+q)\Phi_{n}(q, r, \epsilon, k, q).
\]

Theorem 2.6 Corollary 2.1 If $g \in \Sigma_{\alpha, \beta, \eta}$ is said to be in the class $\Sigma_{\alpha, \beta, \eta}$, i.e. here after it is to be taken as, for $m \geq 1$
\[
|g(z)| \leq \frac{1}{\pi} + q\beta[1+(4(q-1)\beta+1)]\sum_{n=1}^{\infty}(n+q)\Phi_{n}(q, r, \epsilon, k, q).
\]
\[ x^{r_1 - q} \sum_{n=1-q}^{\infty} (n+2q(1-n)) \phi_n(\rho, \tau, \xi, \kappa, \eta, \alpha) \]

And
\[ |g'(z)| \leq \frac{2q(1-\eta)}{\alpha^2} + 2q(1-\eta) \beta [1 + (4(\eta - 1) + 1)] \]

\[ x^{r_2 - q} \sum_{n=1-q}^{\infty} (n+2q(1-n)) \phi_n(\rho, \tau, \xi, \kappa, \eta, \alpha) \]

\[ |g'(z)| \leq \frac{2q(1-\eta)}{\alpha^2} + 4q(1-\eta)^2 \beta [1 + (4(\eta - 1) + 1)] \]

It will contained in the class given as
\[ \sum_{\epsilon_\rho}(\alpha, \beta, \eta) \], which is sufficient condition

**Theorem 2.7** Let us assume that \( g \in \sum_{\rho} \) represented by (1.6).

If \( \forall \alpha (0 \leq \alpha < 1) \) and \( \forall \beta (0 < \beta \leq 1) \),
\[ \sum_{n=1-q}^{\infty} \left\{ (2q(1-\eta)[1 + \beta(4(\eta - 1) + 1)] \right\} \times \phi_\rho, \tau, \xi, \kappa, \eta, \alpha \leq q(1-\beta)[1 + (4(\eta - 1) + 1)] \] (2.22)

Where it is, \( g \in \sum_{\epsilon_\rho}(\alpha, \beta, \eta) \).

**Proof.** We assume \( g(z) = z^{-q} + \sum_{n=1-q}^{\infty} a_n z^n \).

We get
\[ M = \left| \frac{1}{\epsilon_\rho}(\rho, \tau, \xi, \kappa, \eta, \alpha) \right| \]

\[ \sum_{n=1-q}^{\infty} \left\{ (n+2q(1-n)) \phi_n(\rho, \tau, \xi, \kappa, \eta, \alpha) \right\} \leq q(1-\beta)[1 + (4(\eta - 1) + 1)] \] (2.22)

For which is taken as we say that for
\[ M = \sum_{n=1-q}^{\infty} (n+2q(1-n)) \phi_n(\rho, \tau, \xi, \kappa, \eta, \alpha) \]

If we clear the denominator in the Equation (2.9.1) and consider as \( z \rightarrow 1 \) all positive values i. e. here after it is to be taken as we get
\[ \sum_{n=1-q}^{\infty} (n+2q(1-n)) \phi_n(\rho, \tau, \xi, \kappa, \eta, \alpha) \leq q(1-\beta)[1 + (4(\eta - 1) + 1)] \] (3.1)

take the values of \( z \) on x-axis of z-plane s. t.
\[ |g'(z)| \leq \frac{2q(1-\eta)}{\alpha^2} + 2q(1-\eta) \beta [1 + (4(\eta - 1) + 1)] \times \phi_\rho, \tau, \xi, \kappa, \eta, \alpha \]

**Corollary 3.1** If a function \( g \in \sum_{\rho}^{\rho} \) defined as in equation (1.7) contained the class \( \sum_{\epsilon_\rho}(\alpha, \beta, \eta) \) i.e. here after it is to be taken as we say that for
\[ b_n \leq \frac{2q(1-\eta)[1 + (4(\eta - 1) + 1)]}{\rho \rho_n(\rho, \tau, \xi, \kappa, \eta, \alpha)} \] (3.2)

for the functions given below the equality
\[ g_n(z) = \frac{1}{z^q} - \frac{2q(1-\eta)[1 + (4(\eta - 1) + 1)]}{\rho \rho_n(\rho, \tau, \xi, \kappa, \eta, \alpha)} \] (3.3)

**Theorem 3.2** If a function \( g \in \sum_{\epsilon_{\rho,\kappa}}(\alpha, \beta, \eta) \) and using the inequality given as
\[ \sum_{n=1-q}^{\infty} b_n \leq \frac{2q(1-\eta)[1 + (4(\eta - 1) + 1)]}{\rho \rho_n(\rho, \tau, \xi, \kappa, \eta, \alpha)} \]

**Proof.** Suppose \( g \in \sum_{\epsilon_{\rho,\kappa}}(\alpha, \beta, \eta) \) and using the inequality given as
\[ \sum_{n=1-q}^{\infty} b_n \leq \frac{2q(1-\eta)[1 + (4(\eta - 1) + 1)]}{\rho \rho_n(\rho, \tau, \xi, \kappa, \eta, \alpha)} \].

3. Properties of the class \( \sum_{\epsilon_{\rho,\kappa}}(\alpha, \beta, \eta) \):

In this part of the chapter we are proving that \( \forall \) functions which are in the class \( \sum_{\epsilon_{\rho,\kappa}}(\alpha, \beta, \eta) \) the Equation (2.22) is required necessary and sufficient condition.

**Theorem 3.1** Let us assume a function \( g \in \sum_{\rho}^{\rho} \) i. e. here after it is to be taken as \( \forall \) in the class \( \sum_{\epsilon_{\rho,\kappa}}(\alpha, \beta, \eta) \). If & only if
Theorem 3.3 Let us assume that function $h(z)$ which is Analytic in $U^*$ and given $h(z) = z^{-\eta} + \sum_{n=1}^{\infty} c_n z^n$ implies that $\eta > 1$. If function $g$ given in (1.7) is in the class $\Sigma_{\text{crmp}}(\alpha, \beta, \eta)$ i.e. here it is to be taken as the convolution of functions $g \ast h$ ($g \ast h$) contained in $\Sigma_{\text{crmp}}(\alpha, \beta, \eta)$.

Proof Since $f \in \Sigma_{+}^{\text{crmp}}(\alpha, \beta, \eta)$, i.e. here after it is to be taken as by Theorem 3.1, $\sum_{n=1}^{\infty} \left\{n(\beta + 1) + 2q(1-\eta) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]$. 

Theorem 3.4 If the function $g$ defined as $g \in \Sigma_{+}^{\text{crmp}}(\alpha, \beta, \eta)$, and the integral operator as given below

\[ G_c(z) = \frac{c}{c + 2z(1 - \eta)} \int_{c}^{z} e^{t+2p(1-\eta)} dt, c > 0. \]

\[ \sum_{k=1}^{\infty} \left(2q(1-\eta) + 1 + 2q-1\alpha + 1\right) \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

4. Neighborhoods and Partial Sum

The well-known $(n, \delta)$ — neighborhoods of Regular functions & partial sums was earlier investigated and studied by many authors Aouf [71], Goodman [19], [20], [21], Orah [46], Ruscheweyh [56], [59], Liu [54], Altina [3], [46], [58], Deniz and Orah [14] and more recently by Liu and Srivastava [38]. We have defined the $(n, \delta)$ — neighborhoods of Analytic functions $g \in \Sigma_{+}$ of the form (2.76).

Definition 4.1 \( \forall \), $\delta = \frac{2(\varepsilon - t)^2 + \frac{1}{2} \varepsilon (\varepsilon - t)}{1 + 2(\varepsilon - t)^2 + \frac{1}{2} \varepsilon (\varepsilon - t)} > 0$ & non negative sequence $s = \{s_n\}_{n=1}^{\infty}$.

Where it is obviously for

\[ S_n = \frac{\beta [1-2q(1-\eta)] + 2q(1-\eta) + 3q(1-\eta) + 1 + 1\} \Phi_1(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

The (n, \delta) — neighborhoods of Analytic functions $g \in \Sigma_{+}$ given by (1.6) defined as

\[ N_\delta(g) = \left\{g \in \Sigma_{+}; g(z) = \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} s_n |b_n - a_n| \leq \delta \right\}. \]

\[ \forall s_n = n, \text{ definition 1.4 corresponds to the } (n, \delta) — \text{ neighborhoods of analytic functions considered by Ruscheweyh [56]. Using definition 4.1, we are proving results for } (n, \delta) — \text{ neighborhoods of Analytic functions of the class } \Sigma_{\text{crmp}}(\alpha, \beta, \eta). \]

Theorem 4.1 Let $g \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta)$ denoted by Equation (1.6). If a function $g$ satisfies the following relation

\[ (g(z) = \sum_{n=1}^{\infty} b_n z^n) \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta) \]

\[ (\varepsilon \in C, \ |\varepsilon| < \delta, \delta \geq 0) \]

\[ i. e. \text{ after it is to be taken as } N_\delta(f) \subset \Sigma_{\text{crmp}}(\alpha, \beta, \eta). \]

Proof $g \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta)$ if \& only if

\[ z^{\delta} \in \sum_{n=1}^{\infty} b_n z^n \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta). \]

\[ \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

\[ (g(z) = \sum_{n=1}^{\infty} b_n z^n) \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta) \]

\[ \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

\[ \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

\[ (g(z) = \sum_{n=1}^{\infty} b_n z^n) \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta) \]

\[ \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

Further in accordance with (4.3), and by using (4.6) we have obtained the following result

\[ (g(z) = \sum_{n=1}^{\infty} b_n z^n) \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta) \]

\[ \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]

Further in accordance with (4.3), and by using (4.6) we have obtained the following result

\[ (g(z) = \sum_{n=1}^{\infty} b_n z^n) \in \Sigma_{\text{crmp}}(\alpha, \beta, \eta) \]

\[ \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} \Phi_k(\alpha, \tau, \zeta, \xi, \kappa, q) \sum_{n=1}^{\infty} n(\beta + 1) + 1 + 2q-1\alpha + 1\right\} 
\leq 2q(1 - \beta)[(4n - 1) + 1]. \]
t. \( g \in \sum^+ \) as given in (1.7) which is given by
\[
N_q(g) = \left\{ f \in \sum^+ : f(z) = z^{-q} + \sum_{n=1-q} \alpha_n z^n \right\}
\]
and \( \sum_{n=1-q} s_n |a_n - b_n| \leq \delta \) \hspace{1cm} (4.10)
We have obtained the following result on the \((n, \delta)\) – neighborhoods of analytic functions of the class \( \sum_{\alpha \epsilon C_{mp}} (\alpha, \beta, \eta) \).

**Theorem 4.2** If the function \( g \) as given in Equation (1.7) belonging to the class defined \( \sum_{\alpha \epsilon C_{mp}} (\alpha, \beta, \eta) \),
I. e. here after it is to be taken as 
\( N_q(g) \subset \sum^+_{\epsilon C_{mp}} (\alpha, \beta, \eta) \),
where it is obviously for all,
\[
\delta = \frac{2(\tau - (r - c))\left(\sum_{n=1-q} |a_n|^q \right)^{\frac{1}{q}}}{1+2(r - c)\left(\sum_{n=1-q} |a_n|^q \right)^{\frac{1}{q}} + (\sum_{n=1-q} |a_n|^q)^{\frac{1}{q}}}
\]
From above it is clear that means \( \delta \) can’t increase.

**Proof** \( \forall g \in \sum^+_{\epsilon C_{mp}} (\alpha, \beta, \eta) \) of the form (1.7), theorem 3.1
immediately yields
\[
\sum_{n=1-q} |a_n|^q \leq \Phi_{\epsilon q}(r, \tau, \epsilon, \alpha, \beta, \eta)
\]
Using (4.12) & (4.13), we have obtained
\[
\sum_{n=1-q} s_n |a_n - b_n| \leq \delta
\]
Thus in accordance with theorem 3.1, we get \( f \in \sum^+_{\epsilon C_{mp}} (\alpha, \beta, \eta) \). To prove more accuracy of functions 
\( g \in \sum^+_{\epsilon C_{mp}} (\alpha, \beta, \eta) \) and \( g \in \sum^+ \) as follows
\[
g(z) = z^{-q} + \sum_{n=1-q} a_n z^n \leq \delta,
\]
Thus the function \( f \in N_q(g) \) but in agreement with 
Theorem 3.1, \( g \in \sum^+_{\epsilon C_{mp}} (\alpha, \beta, \eta) \). Consequently, our theorem is proved. In the next part, we have introduced as given in Equation (1.6)
\[
k(z) = \left\{ \begin{array}{ll}
    z^{-q}, & k = 1,2,\ldots, q \\
    z^{-q} + \sum_{n=1-q} a_n z^n, & k = 1 - q, 2 - q,\ldots
\end{array} \right.
\]
we have determined lower bounds which are sharp for Rep 
\( k(z) \) and \( \Re \left\{ k(z) \right\} \).

**Theorem 4.3** Let \( g \in \sum_q \) be given by (1.6) and let \( k(z) \) be given by (4.16). Suppose that 
\( \sum_{n=1-q} \theta_n |a_n| \leq 1 \)
(4.17)
Here it is obviously for all,


