Geometric Decomposition of Spider Tree

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Abstract: Let \( G = (V, E) \) be a simple connected graph with \( p \) vertices and \( q \) edges. If \( G_1, G_2, G_3, \ldots, G_n \) are connected edge disjoint subgraphs of \( G \) with \( E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \ldots \cup E(G_n) \), then \( (G_1, G_2, G_3, \ldots, G_n) \) is said to be a decomposition of \( G \). A decomposition \( (G_1, G_2, G_3, \ldots, G_n) \) of \( G \) is said to be an Arithmetic Decomposition if each \( G_i \) is connected and \( |E(G_i)| = a + (i - 1)d \), for every \( i = 1, 2, 3, \ldots, n \) and \( a, d \in \mathbb{N} \). In this paper, we introduced a new concept Geometric Decomposition. A decomposition \( (G_1, G_2, G_3, \ldots, G_n) \) of \( G \) is said to be a Geometric Decomposition (GD) if each \( G_{\omega^i} \) is connected and \( |E(G_{\omega^i})| = a \omega^{i-1} \), for every \( i = 1, 2, 3, \ldots, n \) and \( a, \omega \in \mathbb{N} \). Clearly \( q = \frac{a(\omega^{n-1})}{\omega - 1} \) if \( a = 1 \) and \( r = 2 \), then \( q = 2^n - 1 \). In this paper we study the Geometric Decomposition of spider tree.

Keywords: Decomposition, Arithmetic Decomposition (AD), Geometric Decomposition (GD), Geometric Path Decomposition (GPD), Geometric Star Decomposition (GSD).

1. Introduction

In this paper, we consider simple undirected graph without loops or multiple edges. For all other standard terminology and notations we follow Harary [1].

N. Gnanadhas and J. Paulraj Joseph introduced the concept of Continuous Monotonic Decomposition (CMD) of graphs [2]. E. Ebin Raja Merly and N. Gnanadhas introduced the concept of Arithmetic Odd Decomposition (AOD) of spider tree [3].

Definition: 1.1

Let \( G = (V, E) \) be a simple connected graph with \( p \) vertices and \( q \) edges. If \( G_1, G_2, G_3, \ldots, G_n \) are connected edge disjoint subgraphs of \( G \) with
\[
E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \ldots \cup E(G_n),
\]
then \( (G_1, G_2, G_3, \ldots, G_n) \) is said to be a decomposition of \( G \).

Definition: 1.2

A decomposition \( (G_1, G_2, G_3, \ldots, G_n) \) of \( G \) is said to be an Arithmetic Decomposition (AD) if each \( G_i \) is connected and
\[
|E(G_i)| = a + (i - 1)d,
\]
for every \( i = 1, 2, 3, \ldots, n \) and \( a, d \in \mathbb{N} \).

Definition: 1.3

Let \( G \) be a connected graph. The diameter of \( G \) is defined as \( \max \{d(u, v) : u, v \in V(G)\} \) and is denoted by \( \text{diam}(G) \).

2. Geometric Decomposition of Graphs

Definition: 2.1

A decomposition \( (G_1, G_2, G_3, \ldots, G_n) \) of \( G \) is said to be a Geometric Decomposition (GD) if each \( G_{\omega^i} \) is connected and
\[
|E(G_{\omega^i})| = a \omega^{i-1},
\]
for every \( i = 1, 2, 3, \ldots, n \) and \( a, \omega \in \mathbb{N} \). Clearly \( q = \frac{a(\omega^{n-1})}{\omega - 1} \). If \( a = 1 \) and \( r = 2 \), then \( q = 2^n - 1 \).

We know that \( 2^n - 1 \) is the sum of \( 2^0, 2^1, 2^2, 2^3, \ldots, 2^n - 1 \). That is, \( 2^n - 1 \) is the sum of \( 1, 2, 4, 8, \ldots, 2^n - 1 \). Thus we denote the GD as \( (G_1, G_2, G_3, \ldots, G_{2^{n-1}}) \).

Example: 2.2

Figure 1: A Petersen graph admits GD \( (G_1, G_2, G_3, G_8) \) of \( G \).

Theorem 2.3: A graph \( G \) admits GD \( (G_1, G_2, G_3, \ldots, G_{2^{n-1}}) \) if and only if \( q = 2^n - 1 \) for each \( n \in \mathbb{N} \).

Proof: Let \( G \) be a connected graph with \( q = 2^n - 1 \). Let \( u, v \) be two vertices of \( G \) such that \( d(u,v) \) is maximum. Let \( N(u) = \{v \in V(G) : d(u,v) = r\} \). If \( d(u) = 2^n - 1 \), choose \( 2^n - 1 \) edges incident with \( u \). Let \( G_{\omega^i} \) be a subgraph induced by these \( 2^n - 1 \) edges. If \( d(u) < 2^n - 1 \), then choose \( 2^n - 1 \) edges incident with...
successively such that the subgraph $G_{2^{n-2}}$ induced by these edges is connected. In both cases $G - G_{2^{n-2}}$ has a connected component $H_1$ with $2^n - 2^{n-1} - 1$ edges.

Now, consider $H_1$ and proceed as above to get $G_{2^{n-2}}$ such that $H_1 - G_{2^{n-2}}$ has a connected component $H_2$ of size $2^n - 2^{n-1} - 2^{n-2} - 1$ edges. Proceeding like this we get a connected subgraph $G_i$ such that $G_{2^{n-2}}$ is a graph with one edge taken as $G_i$. Thus $(G_1, G_2, G_3, \ldots, G_{2^{n-1}})$ is a GD of $G$.

Conversely, Suppose $G$ admits GD $(G_1, G_2, G_3, \ldots, G_{2^n})$. Then obviously, $q(G) = 1 + 2 + 4 + \ldots + 2^n - 1 = 2^n - 1$ for each $n \in N$.

**Definition 2.4:**
A GD in which each $G_{2^{i-1}}$ is a path of size $2^{i-1}$ is said to be a Geometric Path Decomposition (GPD).

**Example 2.5:**

![Figure 2: A triangular snake graph $T_5$ admits GPD.](image)

**Definition 2.6:**
A GD in which each $G_{2^{i-1}}$ is a star of size $2^{i-1}$ is said to be a Geometric star Decomposition (GSD).

**Example 2.7:**

![Figure 3: Fish graph admits GSD.](image)

3. **Geometric Decomposition of Spider Graphs**

**Definition 3.1:** A tree $T$ with exactly one vertex of degree $\geq 3$ is called a Spider tree.

**Notation 3.2:** Let $W$ denote the set of pendent vertices of $T$ and $u$ be the vertex of degree $\geq 3$ in $T$.

**Theorem 3.3:** If $T$ is a spider tree with $\text{diam}(T) = t$, $2 \leq t \leq 5$ with $d(u) = (2^n - 1) - (t - 2)$, then $T$ admits GSD.

**Proof:**

Case (i): $t = 2$. Since $\text{diam}(T) = 2$, $T$ is a star. Also, since $d(u) = 2^n - 1$, $T = K_1, 2^{n-1}$. Therefore, $q(T) = 2^n - 1$. Hence $T$ admits GSD.

Case (ii): $t = 3$. Since $\text{diam}(T) = 3$ and $d(u) = (2^n - 1) - 1$, there are $(2^n - 1) - 2$ pendent edges incident with $u$. Let $S_1 = \emptyset$. Then $T - e$ is a star $K_{1, 2^{n-1}}$, and $q(T - e) = (2^n - 1) - 1$. Then we can easily decompose $T - e$ into $S_2, S_3, S_4, \ldots, S_{2^{n-1}}$. Hence $T$ admits GSD.

Case (iii): $t = 4$.

Subcase (i): $u$ is the origin of $P_3$.

Let $u_1$ be a non pendent vertex adjacent to $u$ and $u_2$ be a terminus of $u - u_1$ path of length 3. Let $S_1 = u_1u$ and $S_2 = u_2 - u_1$. Then the remaining edges of $u$ is a star which can be decomposed into $S_3, S_4, S_5, \ldots, S_{2^{n-1}}$.

Subcase (ii): $u$ is not the origin of $P_3$.

Let $u_1$ and $u_2$ be the two non pendent vertices adjacent to $u$ and let $v_1$ and $v_2$ be the pendent vertices adjacent to $u_1$ and $u_2$ respectively. Then $S_1 = u_1v_1$ and $S_2 = u_2 - v_2$. Then we can easily decompose into $S_3, S_4, S_5, \ldots, S_{2^{n-1}}$.

Case (iv): $t = 5$.

Subcase (i): $u$ is the origin of $P_4$.

Let $u_1$ be a non pendent vertex adjacent to $u$ and $u_2$ be a terminus of $u - u_1$ path of length 4. Then $u_2 - u_1$ path can be decomposed into $S_5, S_6, S_7, \ldots, S_{2^{n-1}}$.

Subcase (ii): $u$ is not the origin of $P_4$.

Let $u_1$ and $u_2$ be the two non pendent vertices adjacent to $u$ and $v_1$ be a pendent vertex adjacent to $u_1$. Let $v_2$ be the pendent vertex of $T$ such that there is a $u_2 - v_2$ path of length 2 is adjacent to $u_2$. Then $S_1 = u_1v_1$ and $S_2 = u_2 - v_2$. Then we can easily decompose into $S_3, S_4, S_5, \ldots, S_{2^{n-1}}$.

**Theorem 3.4:** If $T$ is a spider tree with $\text{diam}(T) = t$, $3 \leq t \leq 5$ and $d(u) = (2^n - 1) - (t - 2)$ admits GSD if and only if $T - W = P_t$ where $x \leq 3$.

**Proof:**

Assume $T - W = P_t$, where $x \leq 3$. Then by previous theorem $T$ admits GSD. Conversely, the result is obvious.
Result 3.5: If \( T \) is a spider tree with \( \text{diam}(T) = 2 \) and \( d(u) = 3 \), then \( T \) admits GSD and GPD.

**Proof:**
Since \( \text{diam}(T) = 2 \) and \( d(u) = 3 \). Clearly \( T \) is a spider tree with 3 edges. Then we can easily decompose \( T \) into paths \( P_1 \) and \( P_2 \). Therefore, by theorem(3.3) \( T \) admits GSD and GPD.

Result 3.6: If \( T \) is a spider tree with \( \text{diam}(T) = 4 \) and \( d(u) = 5 \), then \( T \) admits GSD and GPD.

**Proof:**
Since \( \text{diam}(T) = 4 \) and \( d(u) = 5 \), then there is a path of length 4. Therefore, the spider tree can be decomposed into \( P_1, P_2 \) and \( P_3 \). Also by theorem (3.3) \( T \) admits GSD and GPD.

Result 3.7: If \( T \) is a spider tree with \( \text{diam}(T) = 5 \) and \( d(u) = 4 \), then \( T \) admits GSD and GPD.

**Proof:**
Since \( \text{diam}(T) = 5 \), then there is path of length 5. Then \( P_1 \) can be decomposed into \( P_1 \) and \( P_4 \). Also by theorem (3.3) \( T \) admits GPD and GSD.

Results 3.8:
(i) If \( T \) is a spider tree with \( 2^n - 1 \leq \text{diam}(T) \leq (2^n - 1) - 1 \), then \( T \) admits GPD but not GSD.
(ii) If \( T \) is a spider tree with \( 6 \leq \text{diam}(T) \leq (2^n - 1) - 6 \), then \( T \) admits neither GPD nor GSD.

Example 3.9: Consider a spider tree \( T \) with \( q = 15 \).

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References