Binet’s Formula for the Tetranacci Sequence

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1. Introduction
The terms of a recursive sequence are usually defined by a recurrence procedure; this means that any term is the sum of preceding terms. Such a definition might not be entirely satisfactory because to compute any term, we require computing all of its preceding terms. An alternate definition gives any term of a recursive sequence as a function of index of the term. For the simplest non trivial recursive sequence, the Fibonacci sequence, Binet’s formula [1]

\[ u_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}) \]

defines any Fibonacci number as a function of its index and the constants
\[ \alpha = \frac{1}{2} (1 + \sqrt{5}) \text{ and } \beta = \frac{1}{2} (1 - \sqrt{5}) \]

W.R. Spickerman [2] derived an analog of Binet’s formula for the Tribonacci Sequence 1, 1, 2, 4, 7, … with initial terms
\[ [u_0, u_1, u_2] = [1, 1, 2] \]
with recurrence relation \( u_n = u_{n-1} + u_{n-2} + u_{n-3}, \ n \geq 3. \)
In this paper we derive an analog of Binet’s formula for the Tetranacci sequence [3] denoted by \( \{t_n\}_{n=0}^\infty \) with initial terms
\[ [t_0, t_1, t_2, t_3] = [0, 0, 0, 1] \]
and with recurrence relation,
\[ t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}, \ n \geq 4 \]
In this note, we also provide a property of the generalized Tetranacci sequence \( \{T_n\}_{n=0}^\infty \) with arbitrary initial values \( T_0, T_1, T_2, T_3 \) (not all simultaneously zero). We prove that the ratio of two terms \( T_{n+1} \) and \( T_n \) of generalized Tetranacci sequence approaches the value \( \alpha^4 \) as \( n \) tends to infinity. Where, \( \alpha \) is the Tetranacci constant.

2. Binet’s Formula for the Tetranacci Sequence \( \{t_n\}_{n=0}^\infty \)
The Binet’s formula is derived by determining the generating function for the difference equation,
\[ t_0 = t_1 = t_2 = 0 \text{ and } t_3 = 1 \]
\[ t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}, \ n \geq 4 \]
Let,
\[ T(x) = t_0 + t_1 x + t_2 x^2 + \ldots + t_n x^n + \ldots = \sum_{i=0}^{\infty} t_i x^i \]
be the generating function then,
\[ (1 - x - x^2 - x^3 - x^4)T(x) = x^3 \]
\[ \Rightarrow T(x) = \frac{x^3}{1 - x - x^2 - x^3 - x^4} \]
\[ = \frac{x^3}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} \]
The roots of \( p(x) = 0 \) are \( 1/\alpha, 1/\beta, 1/\gamma \) and \( 1/\delta \).
where \( \alpha, \beta, \gamma \) and \( \delta \) are the roots of
\[ \frac{1}{x} = x^4 - x^3 - x^2 - x - 1 = 0. \]
Applying Cardano’s formula to \( \frac{1}{x} = 0 \), yields
\[ \alpha = \frac{1}{4} + \frac{1}{2} R + \frac{1}{2} \sqrt{\frac{11}{4} - R^2 + \frac{13}{4} R^{-1}} \]
\[ \beta = \frac{1}{4} + \frac{1}{2} R - \frac{1}{2} \sqrt{\frac{11}{4} - R^2 + \frac{13}{4} R^{-1}} \]
\[
\gamma = \frac{1}{4} - \frac{1}{2} R + \frac{1}{2} \sqrt{\frac{11}{4} - R^2 - \frac{13}{4} R^{-1}}
\]
\[
\delta = \frac{1}{4} - \frac{1}{2} R - \frac{1}{2} \sqrt{\frac{11}{4} - R^2 - \frac{13}{4} R^{-1}}
\]

Where,
\[
R = \sqrt[1/3]{\frac{11}{12} + \left(\frac{-65}{54} + \frac{563\sqrt{108}}{108}\right) + \left(\frac{-65}{54} - \frac{563\sqrt{108}}{108}\right)}
\]

Approximate numerical values for \(\alpha, \beta, \gamma\) and \(\delta = \overline{\gamma}\) are:
\(\alpha = 1.92756, \quad \beta = -0.774804, \quad \gamma = -0.0764 + 0.8147i, \quad \delta = -0.0764 - 0.8147i\)

Since, the roots of \(p(x) = 0\) are distinct, by partial fractions,
\[
T(x) = \sum_{i=0}^{\infty} \frac{x^i}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}
\]
\[
T(x) = \frac{A}{(1 - \alpha x)} + \frac{B}{(1 - \beta x)} + \frac{C}{(1 - \gamma x)} + \frac{D}{(1 - \delta x)} \quad (2.1)
\]

Here,
\[
A = \frac{1}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}
\]
\[
B = \frac{1}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}
\]
\[
C = \frac{1}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}
\]
\[
D = \frac{1}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]

From (2.1), we have
\[
T(x) = \sum_{i=0}^{\infty} \frac{a^i}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} x^i + \frac{1}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} x^i + \frac{1}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} x^i + \frac{1}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} x^i
\]
\[
\therefore T(x) = \sum_{i=0}^{\infty} \left(\frac{a^i}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^i}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^i}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^i}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}\right) x^i
\]

Thus, the Binet’s formula for the Tetranacci sequence is:
\[
t_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
\[
= \frac{(\alpha - \delta)(\alpha - \gamma)(\alpha - \overline{\gamma})}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{(\beta - \delta)(\beta - \gamma)(\beta - \overline{\gamma})}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{(\gamma - \delta)(\gamma - \beta)(\gamma - \overline{\gamma})}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{(\delta - \delta)(\delta - \beta)(\delta - \overline{\gamma})}{(\delta - \alpha)(\delta - \beta)(\delta - \overline{\gamma})}
\]
\[
= \frac{(\alpha - \delta)(\alpha - \gamma)(\alpha - \overline{\gamma})}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{(\beta - \delta)(\beta - \gamma)(\beta - \overline{\gamma})}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{(\gamma - \delta)(\gamma - \beta)(\gamma - \overline{\gamma})}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{(\delta - \delta)(\delta - \beta)(\delta - \overline{\gamma})}{(\delta - \alpha)(\delta - \beta)(\delta - \overline{\gamma})}
\]
\[
\therefore T_n = a_n + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
\[
\therefore T_n = a_n + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]

Now, multiply the numerators and denominators of the last two terms by \((\alpha - \gamma)(\alpha - \overline{\gamma})\) and \((\beta - \gamma)(\beta - \overline{\gamma})\) respectively, we get,
\[
t_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \overline{\gamma})} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \overline{\gamma})} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \overline{\gamma})} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \overline{\gamma})}
\]
\[
\therefore T_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
\[
\therefore T_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
\[
\therefore T_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
\[
\therefore T_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
\[
\therefore T_n = \frac{\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}
\]
Approximate values for the constants are:

\[ r = 0.81827, \quad \theta = 1.6643, \quad p = 0.07908 \]
\[ Q = -0.32136, \quad R = 0.24228, \quad S = -0.47 \]

3. An Application

The value of

\[ R \cos n\theta + S \sin n\theta = 0.24228 \cos n\theta - 0.47 \sin n\theta \]

is

\[ M = \sqrt{R^2 + S^2} = 0.52877 \]

We have,

\[ M \cos \omega = R \quad \text{and} \quad M \sin \omega = S \]

\[ \therefore \omega = -\tan^{-1}\left(\frac{S}{R}\right) = -1.09483 \]

\[ \therefore R \cos n\theta + S \sin n\theta = 0.52877 \cos(n\theta + 1.09483) \]

The maximum value of \( |R \cos n\theta + S \sin n\theta| = 0.52877 \) is at

\[ n\theta = -1.09483 \pm k\pi, \quad k = 0, 1, 2, 3, \ldots \]

So, the value of

\[ |r^n(R \cos n\theta + S \sin n\theta)| < \frac{1}{2}, \quad \text{for} \quad n \geq 0 \]

Therefore, a short form of the formula that is appropriate calculating the terms of the Tetranacci sequence is:

\[ t_n = [Pa^n + QB^n + \beta^n + \gamma^n], \quad \text{for} \quad n \geq 0. \]

(\text{where,} \lfloor \cdot \rfloor \text{ is the greatest integer function}).

4. Ratio of Generalized Tetranacci Sequence through Limits

The roots \( \alpha, \beta, \gamma \) and \( \bar{\gamma} \) have the following properties:

\[ \lim_{n\to\infty} \frac{\beta^n}{\alpha^n} = 0, \quad \lim_{n\to\infty} \frac{\gamma^n}{\alpha^n} = 0, \quad \lim_{n\to\infty} \frac{\bar{\gamma}^n}{\alpha^n} = 0 \]

and

\[ \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \alpha \]

where, the root \( \alpha \) is called Tetranacci constant. The generalized Tetranacci sequence denoted by \( \{T_n\}_{n=0}^{\infty} \) satisfy the recurrence relation

\[ T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}, \quad n \geq 4 \]

Where the initial terms \( T_0, T_1, T_2, T_3 \) are arbitrary but not all simultaneously zero. Mansi N. Zaveri and Dr. J.K.Patel [4] provided a formula for finding \( n \)th term of a generalized Tetranacci sequence defined by the formula,

\[ T_n = T_0 t_{n-1} + T_1 (t_{n-1} + t_{n-2}) + T_2 (t_{n-1} + t_{n-2} + t_{n-3}) + T_3 t_{n-3} \]

Our goal in this paper is to study the generalized Tetranacci sequence through limits [5]. Particularly, where we will be dealing on the limit given by \( \lim_{n\to\infty} \frac{t_{n+i}}{t_n} \), where \( i \) is the positive integer.
\[
\lim_{n \to \infty} \frac{\alpha^{n+i}/\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \varphi)} + \frac{\beta^{n+i}/\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \varphi)} + \frac{\gamma^{n+i}/\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \varphi)} + \frac{\varphi^{n+i}/\varphi^n}{(\varphi - \alpha)(\varphi - \beta)(\varphi - \gamma)}
\]

\[
= \lim_{n \to \infty} [\alpha^{n+i}/\alpha^n]
\]

(\because \text{by the properties of roots of } \alpha, \beta, \gamma \text{ & } \varphi)

= \alpha^i

Continuing and using proven claim, we obtain,

\[
\lim_{n \to \infty} \frac{T_{n+i}}{T_n}
\]

\[
= \frac{T_0\alpha^{i-1} + T_1(\alpha^{i-1} + \alpha^{i-2}) + T_2(\alpha^{i-1} + \alpha^{i-2} + \alpha^{i-3}) + T_3\alpha^i}{T_0\alpha^{-1} + T_1(\alpha^{-1} + \alpha^{-2}) + T_2(\alpha^{-1} + \alpha^{-2} + \alpha^{-3}) + T_3}
\]

\[
= \alpha^i
\]

\[
= \lim_{n \to \infty} \frac{T_{n+i}}{T_n} = \alpha^i
\]

References:


