Binet's Formula for the Tetranacci Sequence

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Abstract: *In this paper, we derive an analog of Binet's formula for the Tetranacci sequence with initial terms* $t_0 = t_1 = t_2 = 0$ & $t_3 = 1$ *and with recurrence relation* $t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}$, $n \geq 4$. *This formula gives* t_n *explicitly as a function of index* n *and the roots of the associated characteristic equation* $x^4 - x^3 - x^2 - x - 1 = 0$. *In this study we also prove that the ratio of two terms* T_{n+i} *and* T_n *of the generalized Tetranacci sequence approaches the value* α^i *as* n *tends to infinity. where,* α *is the Tetranacci constant.*

Keywords: Tetranacci sequence, Tetranacci numbers, Binet's formula**.** Generalized Tetranacci Sequence, Tetranacci Constant

1. Introduction

The terms of a recursive sequence are usually defined by a recurrence procedure; this means that any term is the sum of preceding terms. Such a definition might not be entirely satisfactory because to compute any term, we require computing all of its preceding terms. An alternate definition gives any term of a recursive sequence as a function of index of the term. For the simplest non trivial recursive sequence, the Fibonacci sequence, Binet's formula [1]

$$u_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

defines any Fibonacci number as a function of its index and the constants

$$\alpha = \tfrac{1}{2}\left(1 + \sqrt{5}\right) \quad \text{and} \quad \beta = \tfrac{1}{2}\left(1 - \sqrt{5}\right)$$

W.R. Spickerman [2] derived an analog of Binet's formula for the Tribonacci Sequence 1, 1, 2, 4, 7, …..with initial terms

$$[u_0,\ u_1,\ u_2] = [1,\ 1,\ 2]$$

with recurrence relation $u_n = u_{n-1} + u_{n-2} + u_{n-3}$, $n \geq 3$. In this paper we derive an analog of Binet's formula for the Tetranacci sequence [3] denoted by $\{t_n\}_{n=0}^{\infty}$ with initial terms

$$[t_0,\ t_1,\ t_2,\ t_3] = [0,\ 0,\ 0,\ 1]$$

and with recurrence relation,

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}, \qquad n \geq 4$$

In this note, we also provide a property of the generalized Tetranacci sequence $\{T_n\}_{n=0}^{\infty}$ with arbitrary initial values $T_0,\ T_1,\ T_2,\ T_3$ (not all simultaneously zero). We prove that the ratio of two terms T_{n+i} and T_n of generalized Tetranacci sequence approaches the value α^i as n tends to infinity. Where, α is the Tetranacci constant.

2. Binet's Formula for the Tetranacci Sequence $\{t_n\}_{n=0}^{\infty}$

The Binet's formula is derived by determining the generating function for the difference equation,

$$t_0 = t_1 = t_2 = 0 \text{ and } t_3 = 1$$

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}, \qquad n \geq 4$$

Let,

$$T(x) = t_0 + t_1 x + t_2 x^2 + \ldots + t_n x^n + \ldots. = \sum_{i=0}^{\infty} t_i x^i$$

be the generating function then,

$$(1 - x - x^2 - x^3 - x^4)T(x) = x^3$$

$$\therefore\ T(x) = \frac{x^3}{1 - x - x^2 - x^3 - x^4}$$

$$= \frac{x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}$$

$$= \frac{x^3}{p(x)}$$

The roots of $p(x) = 0$ are $1/_{\alpha}, 1/_{\beta}, 1/_{\gamma}$ and $1/_{\delta}$.

where $\alpha,\ \beta,\ \gamma$ and δ are the roots of

$$p\left(\frac{1}{x}\right) = x^4 - x^3 - x^2 - x - 1 = 0.$$

Applying Cardano's formula to $p\left(\frac{1}{x}\right) = 0$, yields

$$\alpha = \frac{1}{4} + \frac{1}{2}R + \frac{1}{2}\sqrt{\frac{11}{4} - R^2 + \frac{13}{4}R^{-1}}$$

$$\beta = \frac{1}{4} + \frac{1}{2}R - \frac{1}{2}\sqrt{\frac{11}{4} - R^2 + \frac{13}{4}R^{-1}}$$

$$\gamma = \frac{1}{4} - \frac{1}{2}R + \frac{1}{2}\sqrt{\frac{11}{4} - R^2 - \frac{13}{4}R^{-1}}$$

$$\delta = \frac{1}{4} - \frac{1}{2}R - \frac{1}{2}\sqrt{\frac{11}{4} - R^2 - \frac{13}{4}R^{-1}}$$

Where,

$$R = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}$$

Approximate numerical values for $\alpha,\ \beta,\ \gamma$ and $\delta = \overline{\gamma}$ are:

$$\alpha = 1.92756,\ \ \beta = -0.774804,\ \ \gamma = -0.0764 + 0.8147i\,,$$
$$\overline{\gamma} = \delta = -0.0764 - 0.8147i$$

Since, the roots of $p(x) = 0$ are distinct, by partial fractions,

$$T(x) = \frac{x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}$$
$$= \frac{A}{(1-\alpha x)} + \frac{B}{(1-\beta x)} + \frac{C}{(1-\gamma x)} + \frac{D}{(1-\overline{\gamma} x)} \quad \mathbf{(2.1)}$$

Here,

$$A = \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})},$$
$$B = \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})},$$
$$C = \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})},$$

and $$D = \frac{1}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)}$$

From (2.1), we have

$$T(x) = \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})}\sum_{i=0}^{\infty}\alpha^i x^i + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})}\sum_{i=0}^{\infty}\beta^i x^i + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})}\sum_{i=0}^{\infty}\gamma^i x^i + \frac{1}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)}\sum_{i=0}^{\infty}\overline{\gamma}^i x^i$$

$$\therefore T(x) = \sum_{i=0}^{\infty}\left(\frac{\alpha^i}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})} + \frac{\beta^i}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})} + \frac{\gamma^i}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})} + \frac{\overline{\gamma}^i}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)}\right)x^i$$

Thus, the Binet's formula for the Tetranacci sequence is:

$$t_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})} + \frac{\overline{\gamma}^n}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)} \quad \mathbf{(2.2)}$$

Now, multiply the numerators and denominators of the last two terms by $(\alpha-\gamma)(\alpha-\overline{\gamma})$ and $(\beta-\gamma)(\beta-\overline{\gamma})$ respectively, we get,

$$t_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})} + \frac{(\alpha-\gamma)(\alpha-\overline{\gamma})(\beta-\gamma)(\beta-\overline{\gamma})\,\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})(\alpha-\gamma)(\alpha-\overline{\gamma})(\beta-\gamma)(\beta-\overline{\gamma})} + \frac{(\alpha-\gamma)(\alpha-\overline{\gamma})(\beta-\gamma)(\beta-\overline{\gamma})\,\overline{\gamma}^n}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)(\alpha-\gamma)(\alpha-\overline{\gamma})(\beta-\gamma)(\beta-\overline{\gamma})}$$

$$\therefore t_n = \frac{\alpha^n}{(\alpha-\beta)|\alpha-\gamma|^2} + \frac{\beta^n}{(\beta-\alpha)|\beta-\gamma|^2} + \frac{(\alpha-\overline{\gamma})(\beta-\overline{\gamma})\,\gamma^n}{(\gamma-\overline{\gamma})\,|\alpha-\gamma|^2|\beta-\gamma|^2} + \frac{(\alpha-\gamma)(\beta-\gamma)\,\overline{\gamma}^n}{(\overline{\gamma}-\gamma)\,|\alpha-\gamma|^2|\beta-\gamma|^2} \quad \mathbf{(2.3)}$$

Using the relation,

$$\gamma = r(\cos\theta + i\sin\theta),$$
$$\gamma^n = r^n(\cos n\theta + i\sin n\theta), \qquad \theta = \tan^{-1}\{Im(\gamma)/Re(\gamma)\}$$
$$\&\ \ \gamma - \overline{\gamma} = 2i\,Im(\gamma)$$

$\therefore$ From (2.3), we have,

$$t_n = \frac{\alpha^n}{(\alpha-\beta)|\alpha-\gamma|^2} + \frac{\beta^n}{(\beta-\alpha)|\beta-\gamma|^2} + \frac{(\alpha-\overline{\gamma})(\beta-\overline{\gamma})\,\gamma^n - (\alpha-\gamma)(\beta-\gamma)\,\overline{\gamma}^n}{2ir\sin\theta|\alpha-\gamma|^2|\beta-\gamma|^2}$$

This can be simplified as:

$$t_n = \frac{\alpha^n}{(\alpha-\beta)|\alpha-\gamma|^2} + \frac{\beta^n}{(\beta-\alpha)|\beta-\gamma|^2} + \frac{(\alpha+\beta-2r\cos\theta)}{|\alpha-\gamma|^2\,|\beta-\gamma|^2}\,r^n\cos n\theta + \frac{(\alpha\beta-(\alpha+\beta)\,r\cos\theta + r^2\cos 2\theta)}{r\sin\theta\,|\alpha-\gamma|^2\,|\beta-\gamma|^2}\,r^n\sin n\theta$$

Denoting the coefficients of $\alpha^n,\ \beta^n,\ r^n\cos n\theta\ \&\ r^n\sin n\theta$ by $P,\ Q,\ R\ \&\ S$ respectively we get,

$$t_n = P\alpha^n + Q\beta^n + r^n(R\cos n\theta + S\sin n\theta) \qquad \mathbf{(2.4)}$$

Approximate values for the constants are:

$$r = 0.81827, \quad \theta = 1.6643, \quad p = 0.07908$$

$$Q = -0.32136, \quad R = 0.24228, \quad S = -0.47$$

3. An Application

The value of

$$R\cos n\theta + S\sin n\theta = 0.24228\cos n\theta - 0.47\sin n\theta$$
$$= M\cos(n\theta - \omega)$$

Where, $M = \sqrt{R^2 + S^2} = 0.52877$

We have,

$$M\cos\omega = R \quad \& \quad M\sin\omega = S$$

$$\therefore \omega = -\tan^{-1}\left|\frac{S}{R}\right| = -1.09483$$

$$\therefore R\cos n\theta + S\sin n\theta = 0.52877\cos(n\theta + 1.09483)$$

The maximum value of $|R\cos n\theta + S\sin n\theta| = 0.52877$ is at

$$n\theta = -1.09483 \pm k\Pi, \quad k = 0, 1, 2, 3, \ldots\ldots$$

So, the value of

$$|\, r^n(R\cos n\theta + S\sin n\theta)\,| < \frac{1}{2}, \quad \text{for } n \ge 0$$

Since, $|r| < 1$, the n[th] Tetranacci number is the integer nearest $P\alpha^n + Q\beta^n$ when

$$|\, r^n(R\cos n\theta + S\sin n\theta)\,| < \frac{1}{2}$$

Therefore, a short form of the formula that is appropriate calculating the terms of the Tetranacci sequence is:

$$t_n = [P\alpha^n + Q\beta^n + .5\,], \quad for\ n \ge 0.$$

(where, [ ] is the greatest integer function).

4. Ratio of Generalized Tetranacci Sequence through Limits

The roots $\alpha,\ \beta,\ \gamma$ and $\overline{\gamma}$ have the following properties:

$$\lim_{n\to\infty}\frac{\beta^n}{\alpha^n} = 0, \quad \lim_{n\to\infty}\frac{\gamma^n}{\alpha^n} = 0, \quad \lim_{n\to\infty}\frac{\overline{\gamma}^n}{\alpha^n} = 0$$

and

$$\lim_{n\to\infty}\frac{t_{n+1}}{t_n} = \alpha$$

where, the root α is called Tetranacci constant.

The generalized Tetranacci sequence denoted by $\{T_n\}_{n=0}^{\infty}$ satisfy the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}, \quad n \ge 4$$

Where the initial terms $T_0,\ T_1,\ T_2,\ T_3$ are arbitrary but not all simultaneously zero. Mansi N. Zaveri and Dr. J.K.Patel [4] provided a formula for finding n[th] term of a generalized Tetranacci sequence defined by the formula,

$$T_n = T_0 t_{n-1} + T_1(t_{n-1} + t_{n-2}) + T_2(t_{n-1} + t_{n-2} + t_{n-3}) + T_3 t_n \qquad \mathbf{(2.5)}$$

Our goal in this paper is to study the generalized Tetranacci sequence through limits [5]. Particularly, where we will be dealing on the limit given by $\lim_{n\to\infty}\frac{T_{n+i}}{T_n}$, where i is the positive integer.

4.1. Theorem: Prove that

$$\lim_{n\to\infty}\frac{T_{n+i}}{T_n} = \alpha^i$$

Proof: From (2.5), we can write,

$$\lim_{n\to\infty}\frac{T_{n+i}}{T_n}$$
$$= \lim_{n\to\infty}\frac{T_0 t_{n+i-1} + T_1(t_{n+i-1} + t_{n+i-2}) + T_2(t_{n+i-1} + t_{n+i-2} + t_{n+i-3}) + T_3 t_{n+i}}{T_0 t_{n-1} + T_1(t_{n-1} + t_{n-2}) + T_2(t_{n-1} + t_{n-2} + t_{n-3}) + T_3 t_n}$$

Divide by t_n, both in numerator and denominator, we get,

$$= \lim_{n\to\infty}\left[\frac{T_0\left(\frac{t_{n+i-1}}{t_n}\right) + T_1\left(\frac{t_{n+i-1} + t_{n+i-2}}{t_n}\right) + T_2\left(\frac{t_{n+i-1} + t_{n+i-2} + t_{n+i-3}}{t_n}\right) + T_3\left(\frac{t_{n+i}}{t_n}\right)}{T_0\left(\frac{t_{n-1}}{t_n}\right) + T_1\left(\frac{t_{n-1} + t_{n-2}}{t_n}\right) + T_2\left(\frac{t_{n-1} + t_{n-2} + t_{n-3}}{t_n}\right) + T_3}\right]$$

Claim : $\lim_{n\to\infty}\frac{t_{n+i}}{t_n} = \alpha^i$

From result (2.2), we can write,

$$\lim_{n\to\infty}\frac{t_{n+i}}{t_n}$$
$$= \lim_{n\to\infty}\left[\frac{\frac{\alpha^{n+i}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})} + \frac{\beta^{n+i}}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})} + \frac{\gamma^{n+i}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})} + \frac{\overline{\gamma}^{n+i}}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)}}{\frac{\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\overline{\gamma})} + \frac{\beta^{n}}{(\beta-\alpha)(\beta-\gamma)(\beta-\overline{\gamma})} + \frac{\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\overline{\gamma})} + \frac{\overline{\gamma}^{n}}{(\overline{\gamma}-\alpha)(\overline{\gamma}-\beta)(\overline{\gamma}-\gamma)}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\alpha^{n+i}/\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\bar{\gamma})}+\frac{\beta^{n+i}/\alpha^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\bar{\gamma})}+\frac{\gamma^{n+i}/\alpha^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\bar{\gamma})}+\frac{\bar{\gamma}^{n+i}/\alpha^n}{(\bar{\gamma}-\alpha)(\bar{\gamma}-\beta)(\bar{\gamma}-\gamma)}}{\frac{\alpha^n/\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\bar{\gamma})}+\frac{\beta^n/\alpha^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\bar{\gamma})}+\frac{\gamma^n/\alpha^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\bar{\gamma})}+\frac{\bar{\gamma}^n/\alpha^n}{(\bar{\gamma}-\alpha)(\bar{\gamma}-\beta)(\bar{\gamma}-\gamma)}}\right]$$

$$= \lim_{n\to\infty}\left[\alpha^{n+i}/\alpha^n\right]$$

($\because$ by the properties of roots of α, β, γ & $\bar{\gamma}$)

$$= \alpha^i$$

Continuing and using proven claim, we obtain,

$$\lim_{n\to\infty}\frac{T_{n+i}}{T_n}$$

$$= \frac{T_0\alpha^{i-1}+T_1(\alpha^{i-1}+\alpha^{i-2})+T_2(\alpha^{i-1}+\alpha^{i-2}+\alpha^{i-3})+T_3\alpha^i}{T_0\alpha^{-1}+T_1(\alpha^{-1}+\alpha^{-2})+T_2(\alpha^{-1}+\alpha^{-2}+\alpha^{-3})+T_3}$$

$$= \frac{\alpha^i[T_0\alpha^{-1}+T_1(\alpha^{-1}+\alpha^{-2})+T_2(\alpha^{-1}+\alpha^{-2}+\alpha^{-3})+T_3]}{T_0\alpha^{-1}+T_1(\alpha^{-1}+\alpha^{-2})+T_2(\alpha^{-1}+\alpha^{-2}+\alpha^{-3})+T_3}$$

$$= \alpha^i$$

$$\therefore \lim_{n\to\infty}\frac{T_{n+i}}{T_n} = \alpha^i$$

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