Generalized Fibonacci-Type Sequence and its Properties

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Abstract: ***The Fibonacci sequence is a source of many nice and interesting identities. Fibonacci sequence is famous for possessing wonderful and amazing properties. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions and others by preserving the recurrence relation. In this paper, generalized Fibonacci-Type sequence is introduced and defined by*** $Y_{n+2} = Y_{n+1} + aY_n,\ n \ge 0$ ***,*** $Y_0 = 2$ ***and*** $Y_1 = 2 + b$***, where*** a ***and*** b ***are integer. Further some standard identities and determinant identities of generalized Fibonacci sequence are presented.***

Keywords: Generalized Fibonacci sequence, Generalized Fibonacci-Like sequence, Generating function, Binet's Formula

1. Introduction

It is well-known that the Fibonacci and Lucas sequences are most prominent example of second order recursive sequences. The Fibonacci sequence is a sequence of numbers starting with integer 0 and 1, where each next term of the sequence calculated as the sum of the previous two. The Fibonacci sequence [3] is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2},\ n \ge 2 \text{ With } F_0 = 0,\ F_1 = 1 \qquad (1.1)$$

The similar interpretation also exists for Lucas sequence. Lucas sequence [3] is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}\ \ n \ge 2 \text{ With } L_0 = 0,\ L_1 = 1 \qquad (1.2)$$

The second order recurrence sequence has been generalized in two ways mainly first by preserving the initial conditions and second by preserving the recurrence relation.

Horadam [1] defined generalized Fibonacci sequence $\{H_n\}$ by $H_{n+2} = H_{n+1} + H_n$, $H_0 = q$ and $H_1 = p$, $n \ge 0$ where p, q are arbitrary integers. (1.3)

In [2], Horadam introduced and studied properties of another generalized Fibonacci sequence $\{w_n\}$ and defined generalized Fibonacci sequence $\{w_n\}$ by the recurrence relation:

$w_n = pw_{n-1} - qw_{n-2},\ n \ge 2$ with $w_0 = a$, $w_1 = b$ where a, b, p and q are arbitrary integers. (1.4)

Waddill and Sacks [7] extended the Fibonacci numbers recurrence relation and defined the sequence $\{P_n\}$ by the recurrence relation:

$$P_n = P_{n-1} + P_{n-2} + P_{n-3},\ n \ge 3 \qquad (1.5)$$

Where P_0, P_1 and P_2 are not all zero given arbitrary algebraic integers.

Jaiswal [5] defined generalized Fibonacci sequence $\{T_n\}$ by

$$T_{n+1} = T_n + T_{n-1},\ n \ge 1 \text{ With } T_1 = a \text{ and } T_2 = b \qquad (1.6)$$

Falcon and Plaza [8] introduced k^{th} Fibonacci sequence $\{F_{k,n}\}_{n \in N}$ and defined it by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1},\ n \ge 1 \text{ With } F_{k,0} = 0, F_{k,1} = 1 \qquad (1.7)$$

Edson and Yayenie [6] studied a new generalization $\{q_n\}$, with initial condition $q_0 = 0$ and $q_1 = 1$ which is generated by the recurrence relation (when n is even) or $q_n = bq_{n-1} + q_{n-2}$ (when n is odd), where are non zero real numbers.

B.singh, S.Bhatnagar and O.Sikhwal [4] defined Fibonacci-Like sequence $\{S_n\}$ by the recurrence relation: $S_n = S_{n+1} + S_n,\ n \ge 0$ With $S_0 = 2, S_1 = 2$

The associated initial conditions S_0 and S_1 are the sum of the Fibonacci and Lucas sequences respectively, i.e. $S_0 = F_0 + L_0$ and $S_1 = F_1 + L_1$ (1.8)

In this paper, we introduce generalized Fibonacci-Type sequence. Also we establish some of the interesting properties of generalized Fibonacci-Type sequence like Catalan's identity, Cassini's identity' d'ocagnes's identity, Binets formula, Generating function and some determinant identities.

2. Generalized Fibonacci-Type Sequence

Generalized Fibonacci-Type sequence $\{Y_n\}_{n=0}^{\infty}$ is introduced and defined by the recurrence relation

$$Y_{n+2} = Y_{n+1} + aY_n,\ n \ge 0 \text{ with } Y_0 = 2 \text{ and } Y_1 = 2 + b, \qquad (2.1)$$

Where a and b are integers.

The first few terms of are as follows:

$Y_1 = 2 + b,$

$Y_2 = 2 + 2a + b,$

$Y_3 = 2 + 4a + b + ab,$

$Y_4 = 2 + 6a + b + 2ab + 2a^2,$

$Y_5 = 2 + 8a + b + 3ab + 6a^2 + a^2b, \ldots$

The characteristic equation of recurrence relation (2.1) is $t^2 - t - a = 0$. Which has two real roots:

$$\alpha = \frac{1 + \sqrt{1 + 4a}}{2} \text{ and } \beta = \frac{1 - \sqrt{1 + 4a}}{2}.$$

Also, $\alpha\beta=-a,\ \alpha+\beta=1,\ \alpha-\beta=\sqrt{1+4a},\ \alpha^2+\beta^2=1+2a.$ (2.2)

Generating function of generalized Fibonacci-Type sequence is

$$\sum_{n=0}^{\infty} Y_n t^n = \frac{2+bt}{1-t-at^2}. \qquad (2.3)$$

Hypergeometric representation of generating function of generalized Fibonacci-Type sequence

$$\sum_{n=0}^{\infty} \frac{Y_n}{n!} t^n = (2+bt)e^t {}_2F_1\left(n+1,\ 1;\ 1;\ at^2\right). \qquad (2.4)$$

By generating function (2.3), we have

$$\sum_{n=0}^{\infty} Y_n t^n = \frac{2+bt}{1-t-at^2},$$

$$=(2+bt)\left[1-(1+at)t\right]^{-1},$$

$$=(2+bt)\sum_{n=0}^{\infty}(1+at)^n t^n,$$

$$=(2+bt)\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k}(1)^{n-k}(at)^k,$$

$$=(2+bt)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!\,n-k!}(1)^{n-k} t^{n+k} a^k,$$

$$=(2+bt)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!}(1)^{n} t^{n+2k} a^k,$$

$$=(2+bt)\sum_{n=0}^{\infty}\frac{(t)^n}{n!}\sum_{k=0}^{\infty}\frac{n+k!}{k!} t^{2k} a^k,$$

$$=(2+bt)e^t\sum_{k=0}^{\infty}\frac{n+k!}{k!}\left(at^2\right)^k,$$

$$\sum_{n=0}^{\infty}\frac{Y_n}{n!}t^n=(2+bt)e^t\sum_{k=0}^{\infty}\frac{n+k!}{n!}\frac{\left(at^2\right)^k}{k!},$$

$$\sum_{n=0}^{\infty}\frac{Y_n}{n!}t^n=(2+bt)e^t\sum_{k=0}^{\infty}(n+1)_k\frac{(1)_k}{(1)_k}\frac{\left(at^2\right)^k}{k!},$$

Hence, $\sum_{n=0}^{\infty}\frac{Y_n}{n!}t^n=(2+bt)e^t{}_2F_1\left(n+1,\ 1;\ 1;\ at^2\right).$

Binet's formula of generalized Fibonacci-Type sequence is defined by

$$Y_n = A\alpha^n + B\beta^n = A\left(\frac{1+\sqrt{1+4a}}{2}\right)^n + B\left(\frac{1-\sqrt{1+4q}}{2}\right)^n \qquad (2.5)$$

Here, $A=\frac{(2+b)-2\beta}{\sqrt{1+4a}}$ and $B=\frac{2\alpha-(2+b)}{\sqrt{1+4a}}.$

Also, $AB=\frac{4(a+1)+2b-(2+b)^2}{(\alpha-\beta)^2},\ A\beta+B\alpha=-b$

And $A\beta^2+B\alpha^2=2a-b.$

Generalized Fibonacci-Type sequence generalizes many sequences for different values of a and b. Examples of such sequences are Lucas sequence, Fibonacci-Like sequences, Jacobsthal-Lucas sequence, companion Fibonacci-Like sequence, etc.

- For $a=1$ and $b=-1$, we obtain Lucas sequence.
- For $a=1$ and $b=0$, we obtain Fibonacci-Like sequence.
- For $a=2$ and $b=-1$, we obtain Jacobsthal-Lucas sequence.
- For $a=2$ and $b=0$, we obtain Companion Fibonacci-Like sequence.
- For $a=2$ and $b=-2$, we obtain Companion Fibonacci-Like sequence.

3. Identities of Generalized Fibonacci-Type Sequence

Now some identities of generalized Fibonacci-Type sequence are present using generating function and Binet's formula.

Theorem 1. (Explicit Sum Formula) Let Y_n be the n^{th} term of generalized Fibonacci-Type sequence. Then

$$Y_n = 2\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}a^k + b\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}a^k. \qquad (3.1)$$

Proof. By generating function (2.3), we have

$$\sum_{n=0}^{\infty} Y_n t^n = \frac{2+bt}{1-t-at^2},$$

$$=(2+bt)\left[1-(1+at)t\right]^{-1},$$

$$=(2+bt)\sum_{n=0}^{\infty}(1+at)^n t^n,$$

$$=(2+bt)\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k}(1)^{n-k}(at)^k,$$

$$=(2+bt)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!\,n-k!}(1)^{n-k} t^{n+k} a^k,$$

$$=(2+bt)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!}(1)^{n} t^{n+2k} a^k,$$

$$=(2+bt)\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}(1)^{n-2k} t^{n} a^k,$$

$$=\sum_{n=0}^{\infty}\left[2\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}(1)^{n-2k}a^k\right]t^n+\sum_{n=0}^{\infty}\left[b\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!\,n-2k!}(1)^{n-2k+1}a^k\right]t^{n+1}.$$

Equating the coefficient of 't^n' on both sides, we obtain

$$Y_n = 2\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}a^k + b\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}a^k.$$

Theorem 2. (Sum of First n terms) sum of first n terms of generalized Fibonacci-Type sequence is given by

$$\sum_{k=0}^{n-1} Y_k = \frac{Y_{n+1}-(2+b)}{a}. \qquad (3.2)$$

Proof. Using the Binet's formula (2.5) we have

$$\sum_{k=0}^{n-1} Y_k = \sum_{k=0}^{n-1}\left[A\alpha^k + B\beta^k\right] = A\left[\frac{1-\alpha^n}{1-\alpha}\right]+B\left[\frac{1-\beta^n}{1-\beta}\right],$$

$$=\frac{(A+B)-(A\beta+B\alpha)-(A\alpha^n+B\beta^n)+\alpha\beta(A\alpha^{n-1}+B\beta^{n-1})}{1-(\alpha+\beta)+\alpha\beta},$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} Y_k = \frac{Y_{n+1}-(2+b)}{a}.$$

Theorem 3. (Sum of First n terms with odd indices) Sum of first n terms (with odd indices) of generalized Fibonacci-Type sequence is given by

$$\sum_{k=0}^{n-1} Y_{2k+1} = \frac{Y_{2n+1}-a^2 Y_{2n-1}-(2+b)+ab}{2a-a^2}. \qquad (3.3)$$

Proof. Using the Binet's formula (2.5), we have

$$\sum_{k=0}^{n-1} Y_{2k+1} = \sum_{k=0}^{n-1}\left[A\alpha^{2k+1} + B\beta^{2k+1}\right] = A\left[\frac{\alpha(1-\alpha^{2n})}{1-\alpha^2}\right] + B\left[\frac{\beta(1-\beta^{2n})}{1-\beta^2}\right],$$

$$= \frac{(A\alpha+B\beta) - \alpha\beta(A\beta+B\alpha) - (A\alpha^{2n+1}+B\beta^{2n+1}) + (\alpha\beta)^2(A\alpha^{2n-1}+B\beta^{2n-1})}{1-(\alpha^2+\beta^2)+(\alpha\beta)^2},$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} Y_{2k+1} = \frac{Y_{2n+1} - a^2 Y_{2n-1} - (2+b) + ab}{2a-a^2}.$$

Theorem 4. (Sum of First n terms with even indices) Sum of first n terms (with even indices) of generalized Fibonacci-Type sequence is given by

$$\sum_{k=o}^{n-1} Y_{2k} = \frac{Y_{2n} - a^2 Y_{2n-2} + (2a-b) - 2}{2a-a^2} \qquad (3.4)$$

Proof. Using the Binet's formula (2.5), we have

$$\sum_{k=0}^{n-1} Y_{2k} = \sum_{k=0}^{n-1}\left[A\alpha^{2k} + B\beta^{2k}\right] = A\left[\frac{(1-\alpha^{2n})}{1-\alpha^2}\right] + B\left[\frac{(1-\beta^{2n})}{1-\beta^2}\right],$$

$$= \frac{(A+B) - (A\beta^2+B\alpha^2) - (A\alpha^{2n}+B\beta^{2n}) + (\alpha\beta)^2(A\alpha^{2n-2}+B\beta^{2n-2})}{1-(\alpha^2+\beta^2)+(\alpha\beta)^2},$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=o}^{n-1} Y_{2k} = \frac{Y_{2n} - a^2 Y_{2n-2} + (2a-b) - 2}{2a-a^2}.$$

Theorem 5. (Catalan's Identity) Let Y_n be the n^{th} term of generalized Fibonacci-Type sequence. Then

$$Y_n^2 - Y_{n+r}Y_{n-r} = \frac{(-a)^{n-r}}{(2+b)^2 - 2(2+b) - 4a}\left[(2+b)Y_r - 2Y_{r+1}\right]^2, n > r \geq 1. \qquad (3.5)$$

Proof. Using the Binet's formula (2.5), we have

$$\text{(LHS)} = \left(A\alpha^n + B\beta^n\right)^2 - \left(A\alpha^{n+r} + B\beta^{n+r}\right)\left(A\alpha^{n-r} + B\beta^{n-r}\right),$$

$$= AB(\alpha\beta)^{n-r}\left(2\alpha^r\beta^r - \alpha^{2r} - \beta^{2r}\right),$$

$$= -AB(-a)^{n-r}\left(\alpha^r - \beta^r\right)^2,$$

$$= \frac{\left[(2+b)^2 - 4(a+1) - 2b\right]}{(\alpha-\beta)^2}(-a)^{n-r}\left(\alpha^r - \beta^r\right)^2,$$

$$= \left[(2+b)^2 - 4(a+1) - 2b\right](-a)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha-\beta}\right)^2,$$

From $\dfrac{\alpha^r - \beta^r}{\alpha-\beta} = \dfrac{(2+b)Y_r - 2Y_{r+1}}{(2+b)^2 - 2(2+b) - 4a}$, we obtain

$$Y_n^2 - Y_{n+r}Y_{n-r} = \frac{(-a)^{n-r}}{(2+b)^2 - 2(2+b) - 4a}\left[(2+b)Y_r - 2Y_{r+1}\right]^2.$$

Corollar 6. (Cassini's Identity) Let Y_n be the n^{th} term of generalized Fibonacci-Type sequence. Then

$$Y_n^2 - Y_{n+1}Y_{n-1} = (-a)^{n-1}\left[(2+b)^2 - 2(2+b) - 4a\right], n \geq 1. \qquad (3.6)$$

Taking $r = 1$ in the Catalan's identity (3.5), the required identity is obtained.

Theorem 7. (d'Ocagne's Identity) Let Y_n be the n^{th} term of generalized Fibonacci-Type sequence. Then

$$Y_m Y_{n+1} - Y_{m+1}Y_n = (-a)^n\left[(2+b)Y_{m-n} - 2Y_{m-n+1}\right], m > n \geq 0. \quad (3.7)$$

Proof. Using the Binet's formula (2.5), we have

$$\text{(LHS)} = \left(A\alpha^m + B\beta^m\right)\left(A\alpha^{n+1} + B\beta^{n+1}\right) - \left(A\alpha^{m+1} + B\beta^{m+1}\right)\left(A\alpha^n + B\beta^n\right),$$

$$= AB\left(\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^n\beta^{m+1} - \alpha^{m+1}\beta^n\right),$$

$$= AB(\alpha\beta)^n\left[\beta\left(\alpha^{m-n} - \beta^{m-n}\right) - \alpha\left(\alpha^{m-n} - \beta^{m-n}\right)\right],$$

$$= -AB(-a)^n(\alpha-\beta)\left(\alpha^{m-n} - \beta^{m-n}\right),$$

$$= \frac{\left[(2+b)^2 - 4(a+1) - 2b\right]}{(\alpha-\beta)^2}(-a)^n(\alpha-\beta)\left(\alpha^{m-n} - \beta^{m-n}\right).$$

From $\dfrac{\alpha^{m-n} - \beta^{m-n}}{\alpha-\beta} = \dfrac{(2+b)Y_{m-n} - 2Y_{m-n+1}}{(2+b)^2 - 2(2+b) - 4a}$, we obtain

$$Y_m Y_{n+1} - Y_{m+1}Y_n = (-a)^n\left[(2+b)Y_{m-n} - 2Y_{m-n+1}\right].$$

Theorem 8. (Generalized Identity) Let Y_n be the n^{th} term of generalized Fibonacci-Type sequence. Then

$$Y_m Y_n - Y_{m-r}Y_{n+r} = (-a)^{m-r}\frac{\left[(2+b)Y_r - 2Y_{r+1}\right]\left[(2+b)Y_{n-m+r} - 2Y_{n-m+r+1}\right]}{\left[(2+b)^2 - 2(2+b) - 4a\right]}, n > m \geq r \geq 1. \quad (3.8)$$

Proof. Using the Binet's formula (2.5), we have

$$\text{(LHS)} = \left(A\alpha^m + B\beta^m\right)\left(A\alpha^n + B\beta^n\right) - \left(A\alpha^{m-r} + B\beta^{m-r}\right)\left(A\alpha^{n+r} + B\beta^{n+r}\right),$$

$$= AB\left(\alpha^r - \beta^r\right)\left[\frac{\alpha^m\beta^n}{\alpha^r} - \frac{\alpha^n\beta^m}{\beta^r}\right],$$

$$= AB\frac{\left(\alpha^r - \beta^r\right)}{(\alpha\beta)^r}\left(\alpha^m\beta^{n+r} - \alpha^{n+r}\beta^m\right),$$

$$= -AB(-a)^{m-r}\left(\alpha^r - \beta^r\right)\left(\alpha^{n-m+r} - \beta^{n-m+r}\right),$$

$$= \frac{\left[(2+b)^2 - 4(a+1) - 2b\right]}{(\alpha-\beta)^2}(-a)^{m-r}\left(\alpha^r - \beta^r\right)\left(\alpha^{n-m+r} - \beta^{n-m+r}\right),$$

From $\dfrac{\alpha^r - \beta^r}{\alpha-\beta} = \dfrac{\left[(2+b)Y_r - 2Y_{r+1}\right]}{\left[(2+b)^2 - 2(2+b) - 4a\right]}$

And $\dfrac{\alpha^{n-m+r} - \beta^{n-m+r}}{\alpha-\beta} = \dfrac{\left[(2+b)Y_{n-m+r} - 2Y_{n-m+r+1}\right]}{\left[(2+b)^2 - 2(2+b) - 4a\right]}$, we obtain

$$Y_m Y_n - Y_{m-r} Y_{n+r} = (-a)^{m-r} \frac{\left[(2+b)Y_r - 2Y_{r+1}\right]\left[(2+b)Y_{n-m+r} - 2Y_{n-m+r+1}\right]}{\left[(2+b)^2 - 2(2+b) - 4a\right]}.$$

The identity (3.8) provides Catalan's, Cassini's and d'Ocagne's identities as below:

- If $m = n,$, the Catalan's identity (3.5) is obtained.
- If $m = n,$ and $r = 1$, the Cassini's identity (3.6) is obtained.
- If $n = m, m = n+1$ and $r = 1$, d'Ocagne's identity (3.7) is obtained.

4. Determinant Identities

There is a long tradition of using matrices and determinants to study Fibonacci numbers. T. Koshy [9] explained two chapters on the use of matrices and determinants in Fibonacci numbers. In this section, some determinant identities are presented.

Theorem 1: For any integers $n \ge 0$ prove that

$$\begin{vmatrix} Y_{n+1} & Y_{n+2} & Y_{n+3} \\ Y_{n+4} & Y_{n+5} & Y_{n+6} \\ Y_{n+7} & Y_{n+8} & Y_{n+9} \end{vmatrix} = 0$$

Proof. Let $\Delta = \begin{vmatrix} Y_{n+1} & Y_{n+2} & Y_{n+3} \\ Y_{n+4} & Y_{n+5} & Y_{n+6} \\ Y_{n+7} & Y_{n+8} & Y_{n+9} \end{vmatrix},$

Applying $C_1 \to C_1(a)$, we get

$$\Delta = \begin{vmatrix} aY_{n+1} & Y_{n+2} & Y_{n+3} \\ aY_{n+4} & Y_{n+5} & Y_{n+6} \\ aY_{n+7} & Y_{n+8} & Y_{n+9} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ and using the definition of generalized Fibonacci-Type recurrence relation, we get

$$\Delta = \begin{vmatrix} Y_{n+3} & Y_{n+2} & Y_{n+3} \\ Y_{n+6} & Y_{n+5} & Y_{n+6} \\ Y_{n+9} & Y_{n+8} & Y_{n+9} \end{vmatrix}.$$

Since two columns are identical, thus we obtained required results.

Theorem 2. For any integers $n \ge 0$, prove that

$$\begin{vmatrix} Y_n - Y_{n+1} & Y_{n+1} - Y_{n+2} & Y_{n+2} - Y_n \\ Y_{n+1} - Y_{n+2} & Y_{n+2} - Y_n & Y_n - Y_{n+1} \\ Y_{n+2} - Y_n & Y_n - Y_{n+1} & Y_{n+1} - Y_{n+2} \end{vmatrix} = 0.$$

Proof. Let $Let\,\Delta = \begin{vmatrix} Y_n - Y_{n+1} & Y_{n+1} - Y_{n+2} & Y_{n+2} - Y_n \\ Y_{n+1} - Y_{n+2} & Y_{n+2} - Y_n & Y_n - Y_{n+1} \\ Y_{n+2} - Y_n & Y_n - Y_{n+1} & Y_{n+1} - Y_{n+2} \end{vmatrix}$

By applying $C_1 \to C_1 + C_2 + C_3$ and expanding along first row, we obtained required result.

Theorem 3. For any integers $n \ge 0$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ aY_n & Y_{n+1} & Y_{n+2} \\ Y_{n+1} + Y_{n+2} & aY_n + Y_{n+2} & aY_n + Y_{n+1} \end{vmatrix} = 0.$$

Proof. $Let\,\Delta = \begin{vmatrix} 1 & 1 & 1 \\ aY_n & Y_{n+1} & Y_{n+2} \\ Y_{n+1} + Y_{n+2} & aY_n + Y_{n+2} & aY_n + Y_{n+1} \end{vmatrix}$

Applying $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ aY_n & Y_{n+1} & Y_{n+2} \\ 2Y_{n+2} & 2Y_{n+2} & 2Y_{n+2} \end{vmatrix}$$

Taking common out $2Y_{n+2}$ from third row, we get

$$\Delta = 2Y_{n+2} \begin{vmatrix} 1 & 1 & 1 \\ aY_n & Y_{n+1} & Y_{n+2} \\ 1 & 1 & 1 \end{vmatrix}.$$

Since two rows are identical, thus we obtained required results.

Theorem 4. For any integers $n \ge 0$, prove that

$$\begin{vmatrix} Y_n & Y_n + Y_{n+1} & Y_n + Y_{n+1} + Y_{n+2} \\ 2Y_n & 2Y_n + 3Y_{n+1} & 2Y_n + 3Y_{n+1} + 4Y_{n+2} \\ 3Y_n & 3Y_n + 6Y_{n+1} & 3Y_n + 6Y_{n+1} + 12Y_{n+2} \end{vmatrix} = 3Y_n Y_{n+1} Y_{n+2}.$$

Proof. Let $\Delta = \begin{vmatrix} Y_n & Y_n + Y_{n+1} & Y_n + Y_{n+1} + Y_{n+2} \\ 2Y_n & 2Y_n + 3Y_{n+1} & 2Y_n + 3Y_{n+1} + 4Y_{n+2} \\ 3Y_n & 3Y_n + 6Y_{n+1} & 3Y_n + 6Y_{n+1} + 12Y_{n+2} \end{vmatrix},$

Applying $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} Y_n & Y_n + Y_{n+1} & Y_n + Y_{n+1} + Y_{n+2} \\ 0 & Y_{n+1} & Y_{n+1} + 2Y_{n+2} \\ 0 & 3Y_{n+1} & 3Y_{n+1} + 9Y_{n+2} \end{vmatrix},$$

Applying $R_3 \to R_3 - 3R_2$ and expanding along first row,we obtained required result.

Theorem 5. For any integers $n \ge 0$, prove that

$$\begin{vmatrix} 0 & Y_n Y_{n+1}^2 & Y_n Y_{n+2}^2 \\ Y_n^2 Y_{n+1} & 0 & Y_{n+1} Y_{n+2}^2 \\ Y_n^2 Y_{n+2} & Y_{n+2} Y_{n+1}^2 & 0 \end{vmatrix} = 2Y_n^3 Y_{n+1}^3 Y_{n+2}^3.$$

Proof. Let $\Delta = \begin{vmatrix} 0 & Y_n Y_{n+1}^2 & Y_n Y_{n+2}^2 \\ Y_n^2 Y_{n+1} & 0 & Y_{n+1} Y_{n+2}^2 \\ Y_n^2 Y_{n+2} & Y_{n+2} Y_{n+1}^2 & 0 \end{vmatrix},$

Taking common $Y_n^2, Y_{n+1}^2, Y_{n+2}^2$ from C_1, C_2, C_3 respectively, we get

$$\Delta = Y_n^2 Y_{n+1}^2 Y_{n+2}^2 \begin{vmatrix} 0 & Y_n & Y_n \\ Y_{n+1} & 0 & Y_{n+1} \\ Y_{n+2} & Y_{n+2} & 0 \end{vmatrix},$$

Now taking common Y_n, Y_{n+1}, Y_{n+2} from R_1, R_2, R_3 respectively and expanding along first row, we get required result.

5. Conclusions

In this paper, generalized Fibonacci-Type sequence is introduced. Some standard identities of generalized Fibonacci-Type sequence have been obtained and derived using generating function and Binet's formula. Also some determinant identities have been established and derived.

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