Development of Infinite Series in India

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Abstract: ***India has a long tradition of study of Mathematics. In India, all Mathematicians and mathematics were deeply respected. The development of mathematical ideas in India should be studied from the history of Indian Mathematicians and their texts. Indian Mathematicians made significant contribution in the field of mathematics and produced a large contribution in which some concepts of calculus are present. The idea of calculus in India starts with the development of decimal number system, concept of zero and infinity, irrational numbers, trigonometry, arithmetic and geometric progression, finding the value of π and calculating the first order and second order differences of sine values. Now our aim is to discuss the background of Indian Mathematicians in the field of calculus. The most important core concept of calculus is infinite series. Indian mathematicians developed the concept of infinite series without using the concept of set theory and functions. We discuss how the concept of infinite series is development in India.***

Keywords: Arithmetic Progression, Geometric Progression, Infinite geometric series, Infinite series for π, Sine series and Cosine series

1. Introduction

Indian mathematicians studied arithmetic and geometric series at a very early date. Arithmetic and geometric progressions are found in the Vedic literature of Indians: *Taittirīya-Saṃhitā*, *Vājasaneyī Saṃhitā* and *Kalpa-sūtra* of Bhadrabāhu etc (2000 B.C.). Āryabhata – I (499 A.D.), Brahmagupta (628) and most of Indian mathematicians have stated various formulae for sum of n terms of series, arithmetic mean of a finite sequence, number of terms in series and common difference in Arithmetic progression. For the first time Mahāvīra (815 to 878 A.D.) gives the formula for the geometric progression. He also gave a rule of beautiful series known as aritmetico- geometric series. Indian mathematicians knew that the infinite geometric series converges if its common ratio is less than one.

The first and most important result stated by Nārāyaṇa Paṇḍita (1340 to 1400 A.D.) is the general formula for the sum of any triangular number (The triangular number T_n is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one). He stated it as *sum of* sums called as **Vārasaṅkalita**. He generalised the formula of Vārasaṅkalita. The first Vārasaṅkalita will give sum of first order triangular numbers; the second Vārasaṅkalita will give sum of second order triangular numbers and finally stated the result for the sum of k^{th} order triangular numbers or k^{th} Vārasaṅkalita of an arithmetic progression.

Various geometrical proofs for the convergence of finite or infinite geometric series are given by the followers of Kerala School of mathematics. Mādhava of Saṅgamagrāma stated an infinite series for π. He stated the value of π as 3.141592653592...which is correct up to 11 decimal places. He used the method of infinite series with end correction for finding the value of π. He is also the founder of series for sine and cosine function. Here we discuss the contribution of Indian mathematicians in infinite series with special reference to Yukti-bhāṣa of Jyeṣṭhadeva.

2. Arithmetic Progression

Āryabhata – I (499 A.D.) stated following results on arithmetic progression in his text *Āryabhatīya*. For an arithmetic series, a + (a+d) + (a+2d) + (a+3d) + ...+ (a+(n-1) d)

The arithmetic mean of the n terms of series is M = a + (n-1) d/2

The **sum of series** is $S = \frac{n}{2}[a + (a + (n-1)d)]$
= sum of first and last term multiplied by half number of terms

Number of terms in arithmetic progression
If S is sum, d is common difference and a is first term of arithmetic progression then Āryabhata – I gives the formula for number of terms n as

$$n = \frac{1}{2d}\left(\sqrt{8dS + (2a-d)^2} - 2a + d\right)$$

Sum of the series 1 + (1+2) + (1+2+3) +........... upto n terms

$1 + (1+2) + (1+2+3) + \ldots\ldots = \frac{n(n+1)(n+2)}{6} = \frac{(n+1)^3 - (n+1)}{6}$

$1^2 + 2^2 + 3^2 + \ldots + \text{upto n terms} = \frac{n(n+1)(2n+1)}{6}$ and

$1^3 + 2^3 + 3^3 + \ldots + \text{upto n terms} = (1+2+3+\ldots+n)^2 = (n(n+1)/2)^2$

Other mathematicians Mahāvīra (815 to 878 A.D.), Bhāskara (1114 to 1193 A.D.), Nārāyaṇa (1340 to 1400 A.D.) stated the results on arithmetic progression in their texts Gaṇitasārasaṅgrhaḥ, Pātīgaṇitam, Gaṇitakaumudī respectively. In these texts there are many examples on arithmetic progression.

A commentary on *Āryabhaṭīya*, the *bhāṣya* of Nīlakaṇṭha is the most elaborate and interesting. The *gaṇitapāda* of this *bhāṣya* contains rationale and logical explanation to various mathematical concepts discussed in *Āryabhaṭīya*. Similarly the commentary on *Līlāvatī* of Bhāskara II, *Kriyākramakari* authored by Saṅkara Vāriyar and Nārāyaṇa (completed the text), contains proof and logical explanations to the mathematics in *Līlāvatī*.

The **geometrical approach** to arithmetic progression is a special feature of the *bhāṣya* of Nīlakaṇṭha and *Kriyākramakari* of Saṅkara Vāriyar. In both the works 8 arithmetical progression are used to find number of terms in arithmetic progression. The geometrical diagrams itself derive the formula. This shows the development of mathematics in India.

3. Nārāyaṇa: Vārasaṅkalita of Natural Numbers

In India the significant development of series occurred after Bhāskara's time. These developments of series occur because of treatment of two ideas Vārasaṅkalita in *Gaṇita Kaumudī* and Sama-ghāta-saṅkalita in *Yukti-bhāṣa*. These work leads to evaluate integration as the summation of series. Now we elaborate the concept of Vārasaṅkalita discussed by Nārāyaṇa in *Gaṇita Kaumudī*.

3.1 Definition Vārasaṅkalita of Natural Numbers (^kV_n)

Vārasaṅkalita is defined as follows

$^1V_n = \sum n = 1 + 2 + 3 +$ --------- + up to n terms

$^2V_n = \sum\sum n = 1 + 3 + 6 + 10 +$ --------- + up to n terms

$^3V_n = \sum\sum\sum n = 1 + 4 + 10 + 20 +$ --------- + up to n terms

Formula for ^1V_n

The first order Vārasaṅkalita is the sum of first n natural numbers. This formula is coated by most of all Indian Mathematicians. The formula coated by Nārāyaṇa Paṇḍita is

सैकपदघ्नपदार्धम् सङ्कलितम् ।

or

saikapadaghnapadārdham saṅkalitam

$^1V_n = \frac{\quad}{2} = \underline{\quad\quad}$

Formula for ^2V_n

Bhāskara II coated the formula for second order Vārasaṅkalita

सैकपदघ्नपदार्धमथैकाद्यङ्कयुतिः किल सङ्कलिताख्या ।
सा द्वियुतेन पदेन विनिघ्नी स्यात् त्रिहृता खलु सङ्कलितै

saikapadaghnapadārdhamathaikādyankayutih kila saṅkalitākhyā |
sā dviyutena padena vinighnī syāt trihṛtā khalu saṅkalitākhyā ||

Meaning:- The sum of n numbers starting from 1 up to n (by common difference 1) is (n+1) multiplied by (n/2) this sum is called saṅkalita.

That is (sum of natural numbers) is multiplied by (n+2) and then divided by 3 is the sum of sum of natural numbers (second order Vārasaṅkalita).

$$^2V_n = \frac{n \quad n+1)}{2} \quad \frac{(n+2)}{3} = \frac{n \quad n \qquad n+2)}{} = \frac{n \quad n \qquad n+2)}{3!}$$

This result can be proved as follows

$$^2V_n = \sum\sum n = \sum \frac{\quad}{2}$$

$$= \frac{1}{2}\sum(n^2 + n)$$

$$= \frac{n \quad (n+1) \quad n+2)}{}$$

Formula for ^3V_n

The formula for 3rd order triangular number or third order Vārasaṅkalita is

$$^3V_n = \frac{\quad (n+1) \quad (n+2) \quad}{4!}$$

Formula for k^{th} order Vārasaṅkalita in *Gaṇita Kaumudī* of Nārāyaṇa Paṇḍita ^kV_n

The formula stated by Nārāyaṇa Paṇḍita is

एकाधिकवारमिताः पदादिरूपोत्तराःपृथक् तेंघ्शाः
एकाद्येकचयहरास्तद्घातो वारसङ्कलितम् ।

Meaning:- The numbers beginning with the number of terms in the series increasing by one and equal in number to one more than the number representing the order of summation separately from the numerators. The corresponding denominators are the natural numbers beginning with one. Product of these (fractions) is the Vārasaṅkalita.

$$^kV_n = \frac{\quad (n+1) \quad (n+2) \quad \ldots \quad}{}$$

3.2 Approach to modern methods

According to modern text Vārasaṅkalita is the integration of integration. For the large value of n, put n = (n+1) the result is

$$^1V_n = \frac{\quad}{} \approx \frac{n^2}{2} \text{ or } {}^1V_n = \int x\,dx = \frac{x^2}{2}$$

$$^2V_n = \frac{n \quad (n+1) \quad n+2)}{} \approx \frac{n^3}{6} \text{ or } {}^2V_n = \iint x\,dx = \frac{x^3}{6}$$

Similarly

$$^kV_n = \frac{(n+1) \quad (n+2) \quad \ldots}{} \approx \frac{n^{k+1}}{(k+1)!} \text{ or}$$

$$^kV_n = \iint.(\text{k times}).\int x\,dx = \frac{x^{k+1}}{(k+1)!}$$

Vārasaṅkalita of terms in Arithmetic progression was also discussed by Nārāyaṇa Paṇḍita and gave the formula for k^{th} order Vārasaṅkalita of terms in Arithmetic progression. He constructed example which elaborate this idea clearly.

Example:- A cow gives birth to one calf every year. The calves become young and themselves are being giving birth to calves when they are three years old. O learned man, tell me the number of progeny produced during twenty years by one cow.

4. Sum of $1^k + 2^k + \ldots\ldots + n^k$ or *sama-ghāta-saṅkalita* for large n.

The method of calculating circumference without finding squares roots is the greatest contribution of Mādhava. It is based on concept of infinite series and covers the idea of Integration. This result gives us values of π (for ratio of circumference to diameter) in terms of infinite series. **We use the results of *sama-ghāta-saṅkalita* in *Yukti-bhāṣa* stated by Jyeṣṭhadeva for finding infinite series of π.**

Calculation of *sama-ghāta-saṅkalita* is given in *Yukti-bhāṣa*

$S_n = 1^{\square} + 2^{\square} + \cdots \ldots \ldots \ldots . + n^{\square}$ where n is very large.

The method is as follows:

4.1 Sum of first n natural numbers *Mūla-saṅkalita*

$S_n^{(1)} = 1 + 2 + 3 + \cdots \ldots \ldots \ldots .. + n$
$= n + (n-1) + (n-2) + \cdots \ldots \ldots \ldots .. + 2 + 1$
$= n + (n-1) + (n-2) + \cdots \ldots \ldots \ldots .. + (n-(n-2)) + (n-(n-1))$
$- (1 + 2 + 3 + \cdots \ldots \ldots + (n-1))$
$- S_{n-1}^{(1)}$

For large n, $S_n^{(1)} \approx S_{n-1}^{(1)}$

$S_n^{(1)} \approx n^2 - S_{n-1}^{(1)}$

So $S_n^{(1)} \approx \frac{n^2}{2}$ ....(1)

4.2 Sum of square of first n natural numbers (*Varga-saṅkalita*)

Sum of squares of first n natural numbers is known as *Varga-saṅkalita* given by

$S_n^{(2)} = 1^2 + 2^2 + 3^2 + \ldots \ldots \ldots . + n^2$

$S_n^{(2)} = n^2 + (n-1)^2 + (n-2)^2 + \ldots \ldots \ldots . + 2^2 + 1^2$ ....(2)

We use $_n^{(1)}$ and

$(nS_n^{(1)})$ $(n-1) + (n-2) + \cdots \ldots . + 2 + 1)$ ....(3)

Now (3) – (2)

$(nS_n^{(1)}) - S_n^{(2)} = \frac{1}{2} S_{n-1}^{(2)}$ ....(4)

But when n is very large $S_{n-1}^{(2)} \approx S_n^{(2)}$ and $S_n^{(1)} \approx \frac{n^2}{2}$

So equation (4) becomes

$(nS_n^{(1)}) - S_n^{(2)} \approx \frac{S_n^{(2)}}{2}$ or $n\frac{n^2}{2} - S_n^{(2)} \approx \frac{S_n^{(2)}}{2}$ or $S_n^{(2)} \approx \frac{n^3}{3}$

4.3 Sum of $1^k + 2^k + \ldots\ldots + n^k$ or *sama-ghāta-saṅkalita* for large n.

Sum of k^{th} power of first n natural numbers is called as *sama-ghāta-saṅkalita* given by

$S_n = n^{\square} + (n-1)^{\square} + (n-2)^{\square} + \ldots \ldots \ldots . + 2^{\square} + 1^{\square}$

So that

$(nS_n^{(\square-1)}) - S_n^{(\square)} = S_{n-1}^{-1)} + S_{n-2}^{-1)} + S_{n-3}^{-1)} + \ldots\ldots\ldots\ldots$

and $S_n^{(\square-1)} \approx \frac{n^{\square}}{\square}$

results $S_n^{(\square)} \approx \frac{n}{\quad}$

(In modern text we use $\int x^{\square} dx = \frac{x}{\quad}$, in Indian mathematics, repeated summations are also calculated)

5. Infinite Series

The formula for the sum of an infinite geometric series, with common ratio less than unity was known to Jain mathematician Mahāvīra. Application of this formula was made to find the volume of the frustum of a cone.

The mathematicians of Kerala school of Mathematics have made a significant contribution in the theory of infinite series. In the fifteenth century they discovered infinite series for π, Sine and cosine of an arc. Use of this series is made in astronomy.

Mādhava obtained the basic π series also discuss the method of end correction. Complete chapter VI of *Gaṇita-Yukti-bhāṣā* popularly known as *Yukti-bhāṣā* written by Jyeṣṭhadeva is devoted to find the ratio of circumference and diameter and development of series for π. The beauty of this chapter is that all the lemma and propositions needed in developing the series for π taken up and proved in logical sequence. This series for π is slowly converging. The method of end correction after finite number of terms is discussed. They also discovered the arctangent series.

In the process of finding the infinite series for π, infinitesimal calculus is developed. This is one of the most significant events in the history of mathematics.

Mr. Charles M. Whish, a civil servant in Madras establishment of east India Company was the first Westerner to take note of the work of the Kerala School. In his paper in 1832, he drew attention of the world for first time on Kerala School of Mathematics and Kerala Mathematicians.

6. Mādhava's Series for π

The infinite series is credited to Mādhava but quoted by Śaṅkara Vāriyar in his commentary *Yukti-dīpikā* and *Kriyā-kramakarī*. The method of calculating circumference without finding square roots is the greatest contribution of Mādhava based on concept of infinite series. Also it covers the idea of Integration. This result gives us values of π. The quoted verse is as follows

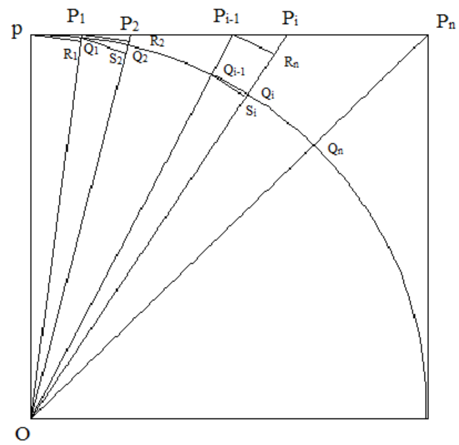
व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादिविषमसंख्याभक्तं ऋणं स्वं पृथक्क्रमात् कुर्यात।।

Meaning:- The diameter multiplied by four and divided by unity, decreases and increases should be made in turn of diameter multiplied by four and divided one by one by the odd numbers beginning with 3 and 5.

This series is written as

Circumference $= 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots$

Or $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$



The circle is inscribed in a square of side equal to the diameter of the circle. This circle touches the middle points of the sides of square. A quarter of the circle with the circumscribing square is shown in the figure. The half side of square PP_n is divided into small number of equal parts PP_1, P_1P_2, P_2P_3, ... $P_{n-1}P_n$ each of length Δr. PP_1, PP_2, ...

are called *bhujās*. The points P, P_1, P_2,.........$P_n$are joined to centre O. The line OP_1 cuts the circumference at Q_1, similarly line OP_2 cuts the circumference at Q_2,......and OP_n cuts the circumference at Q_n.

Lines OP_1, OP_2, ... OP_n are called as *karṇas*. From P, P_1, P_2,....perpendiculars PR_1, P_1R_2, ... are drawn on next *karṇas*. Means PR_1 is perpendicular from P on OP_1, P_1R_2 is perpendicular from P_1 on OP_2 similarly $P_{i-1}R_n$ is perpendicular from P_{i-1} on OP_i. Now from Q_1, Q_2,.. which are the points of intersection of *karṇas* and circumference of circle perpendiculars are drawn on next *karṇas*. Q_1S_2 is perpendicular on OP_2, Q_2S_3 is perpendicular on OP_3 and $Q_{i-1}S_i$ is perpendicular on OP_i.

Here the arc circle is divided in large number of small arcs. Length of every small arc is measured in terms of side of square which is also divided into small arcs. This is the important idea of "infinitesimal calculus" is "infinitely large" and "Infinitely small". The detail proof of this series for π is very interesting. The proof of series for π is depending on the properties of similar triangles and several techniques including the ideas of integration and differentiation. The detailed proof is avoided.

This series does not converge rapidly. It is so slow that even for obtaining the value of π correct to two decimal places we have to find hundreds of terms and for getting π correct to 4 or 5 decimal places we have to find millions of terms.

To find the accurate value circumference we have to derive rapidly converging series. *Yukti-bhāṣa* deals with the rapidly converging series and contains many series derived from original series by grouping the elements. The new series obtained converges rapidly.

$$C = 4d\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$$

$$= 4d\left(\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots\right)$$

$$= 8d\left(\left(\frac{1}{2^2-1}\right)+\left(\frac{1}{6^2-1}\right)+\left(\frac{1}{10^2-1}\right)+\cdots\ldots.\right)$$

Similarly

$$C = 4d - 4d\left(\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots\ldots\right)$$

$$= 4d - 8d\left(\left(\frac{1}{4^2-1}\right)+\left(\frac{1}{8^2-1}\right)+\cdots\right)$$

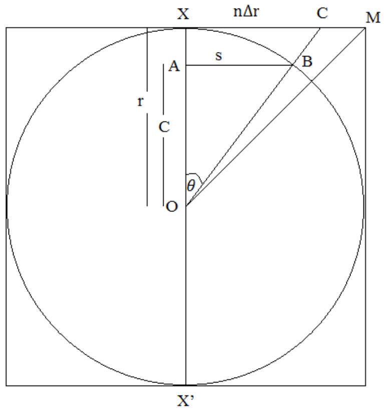
7. Mādhava's Series for θ in terms of tan θ. (Mādhava Gregory Series)

Mādhava beautifully expressed circumference in terms of diameter and the detail proof is also given in the last article. The other series represented is following verses is a Mādhava-Gregory Series quoted in *Kriyā-kramakarī* and assigned to Mādhava

इष्टज्याित्रिज्ययोर्घातात् कोट्याप्तं प्रथम् फलं ।
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम् ।।
प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहः ।
एकत्र्याद्योजसंख्याभिर्भक्तेष्वेतेष्वनुक्रमात ।
ओजनां संयुतेस्त्यक्त्वा युग्मयोगं धनुर्भवेत् ।
दोः कोट्योरल्पमेवेष्टं कल्पनीयमिह स्मृतम् ।।

Kriyā-kramakarī 692-693.

Meaning:- The product of the given sine- chord and radius, divided by the cosine chord, is the first result. Then a series of results are to be obtained from this first result and the succeeding ones by making the square of the sine-chord the multiplier and the square of the cosine-chord the divisor. When these are divided in order by the odd numbers 1, 3, 5...etc, the sum of the terms in the even places is to be subtracted from the sum of the terms in the odd places to get the arc. The smaller of the sine and cosine-cord is to be used for calculation.



The detailed explanation is given in *Kriyā-kramakarī*. Idea of calculus of dividing XC in to n parts each of equal length Δr is used.

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \frac{\tan^7\theta}{7} + \cdots$$

This series was derived by Mādhava in 14th century before Gregory. Now it is called as Mādhava- Gregory series.

8. Geometric progression and Infinite geometric series.

Mahāvīra Jain mathematician of ninth century gives the formula for sum first n terms of geometric progression with common ratio less than unity

$$a + ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$$

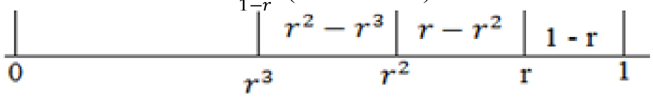
The geometrical representation of convergence of summation of geometric series is beautifully explained in Nīlakaṇṭha's Aryabhatiyabhasya and Jyeṣṭhadeva's *Yukti-bhāṣa*.

Visual Demonstration of convergence of geometric series for r < 1

Let r be a real number such that -1< r < 1 show that

$1 + r + r^2 + r^3 + \ldots\ldots\ldots = \frac{1}{1-r}$ (Infinite Series) and

$1 + r + r^2 + r^3 + \ldots\ldots\ldots = \frac{1-r^n}{1-r}$ (finite Series)



In above figure

$(1 - r) + (r - r^2) + (r^2 - r^3) + \ldots\ldots\ldots\ldots = 1$

$(1 - r)(1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots\ldots\ldots) = 1$

Therefore

$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{1-r}$

In Finite case

$(1 - r) + (r - r^2) + (r^2 - r^3) + \ldots\ldots + (r^k - r^{k+1}) = 1 - r^{k+1}$

Taking the factor (1 - r) common from each term on left hand side and rearrange we get

$$1+ r +r^2 + r^3 +................+ r^k = \frac{1-r^{k+1}}{1-r}$$

9. Derivation of binomial series for $\frac{1}{(1+x)}$

In *Yukti-bhāṣa* Nīlakaṇṭha presented a beautiful derivation of binomial series for $\frac{1}{(1+x)}$ with the use of iterative substitution.

Using above explanation in sun of infinite geometric series obtained series for $a\left(\frac{c}{b}\right)$. For this series we have sequences as follows.

$$a\left(\frac{c}{b}\right) = a - a\frac{(b-c)}{b}$$

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c} \quad \frac{(b-c)}{b}\right)$$

$$\left(a\frac{(b-c)}{c} \quad \frac{(b-c)}{b}\right) = \left(a\frac{(b-c)}{c} \quad \frac{(b-c)}{c}\right) - \left(a\frac{(b-c)}{c} \quad \frac{(b-c)}{c}\right.$$

$$(b-c)b$$

Continuing the process of replacing divisor b present in the last term of the bracket by c, we have to make a subtractive correction every time. We obtain the series in which all even terms are negative, and series is as follows,

$$a\left(\frac{c}{b}\right) =$$

$$a - a\frac{(b-c)}{c} + \left(a\frac{(b-c)}{c} \quad \right) - \cdots + (-1)^{n-1}a\left(\frac{(b-c)}{c}\right)^{n-1}$$

$$+(-1)^n a\left(\frac{(b-c)}{c}\right)^{n-1}\frac{(b-c)}{b} + \cdots ...$$

It is mentioned in *Yukti-bhāṣa* and *Kriyā-kramakarī* that there is no logical end to this process. However the process may be terminated after obtaining desired accuracy by neglecting subsequent *phalas* (terms) as their value become smaller and smaller.

Explanation: - put $\frac{(b-c)}{c} = x$, then $\frac{c}{b} = \frac{1}{1+x}$ hence above series is nothing but a known binomial series

$$\frac{a}{(1+x)} = a - ax + ax^2 + \cdots ....+(-1)^n ax^n + \cdots ......$$

or $$\frac{1}{(1+x)} = 1 - x + x^2 + \cdots ....+(-1)^n x^n + \cdots ......$$

which is convergent for $-1 < x < 1$.

10. Series for sine and cosine

The derivation of sine and cosine series is obtained in *Yukti-bhāṣā* by making use of the saṅkalita (method of integration). The derivation is obtained in two steps. The first step is to find (calculate) change in **two** consecutive sine chords (*bhujajyā*) and obtain the expression for this change, same for cosine chords. This change in sine chord is called as *bhujākhaṇḍa* and change in cosine chord is called as *Koṭikhaṇḍa*.

The second step is to obtain the difference between two *bhujākhaṇḍas* which is sine difference of second order. Third step is to find the difference between arc and sine chord. Finally the result is

$$\sin\theta = \frac{a}{r} - \frac{a^3}{3!r^3} + \frac{a^5}{5!r^5} - \frac{a^7}{7!r^7} + ... \text{ but } a = r\theta$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \;......$$

Similarly

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \;......$$

11. Conclusion

1) Āryabhata Discussed arithmetic progression in details, also obtained $\sum n^2$ and $\sum n^3$. Mahāvīra, Bhāskara, Nārāyaṇa stated the results on arithmetic progression in their texts Gaṇita Sārasaṅgrhaḥ, Pātīgaṇitam, gaṇitakaumudī. Mahāvīra gave the formula of cube of number in terms of arithmetic progression of n terms.
2) Bhāskara II preliminary concept of infinitesimal calculus, along with contributions towards integral calculus and conceived the differential calculus after discovering the derivative and differential coefficient.
3) The important idea of "infinitesimal calculus" is "infinitely large" and "Infinitely small". So when n increases Sama-ghāta-saṅkalita is the integration and Vārasaṅkalita is integration of integration.
4) Mādhava (Founder of Kerala School Mathematics (1340 – 1425)) expressed the approximate value of π using by the method of calculating circumference without finding square roots is one of the contributions of Mādhava based on concept of infinite series. Also he expressed the approximate value of π in terms of infinite series. This series converges slowly.
5) The infinite series is credited to Mādhava but quoted by Śaṅkara Vāriyar in his commentary *Yukti-dīpikā* and *Kriyā-kramakarī*. The method of calculating circumference without finding square roots is the greatest contribution of Mādhava based on concept of infinite series. Also it covers the idea of Integration.
6) Mādhava stated the foundation of calculus.
7) Mādhava found the approximate value of π using by the method of calculating circumference without finding square roots is one of the contributions of Mādhava based on concept of infinite series. Also found the value of π in terms of infinite series. This series converges slowly.
8) Mādhava found the series for arctan ($\tan^{-1}$), sine, versed sine and cosine.
9) The geometrical representation of convergence of summation of geometric series is beautifully explained in Nīlakaṇṭha's Aryabhatiyabhasya and Jyeṣṭhadeva's *Yukti-bhāṣa*.
10) In *Yukti-bhāṣa* Nīlakaṇṭha presented a beautiful derivation of binomial series for $\frac{1}{(1+x)}$ with the use of iterative substitution. It is mentioned in *Yukti-bhāṣa* and *Kriyā-kramakarī* that there is no logical end to this process. However the process may be terminated after obtaining desired accuracy by neglecting subsequent *phalas* (terms) as their value become smaller and smaller.
11) The proofs of sine and cosine series are obtained by sine and cosine difference of second order.

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