Generalized Study of Commuting Self-Maps and Fixed Points

Dolhare U.P.[1], Khanpate V.B.[2]

[1, 2]Department of Mathematics, D. S. M. College, Jintur. [M.S.] India

Abstract: ***In this paper, some common fixed point theorems are proved related to complete & compact metric space, in which the fixed point theorems in G.Jungck[17], B.E.Rhoades [5] , G.Das & J.P. Dabata [16] and U.P.Dolhare.[9,10,11, 34] as a special case. By using weakly commuting pairs, commuting maps & find out fixed point for self-maps in complete metric space. We proved some fixed point results for self-mappings satisfying weakly contractive conditions with fixed point results involving altering distance in complete metric space.. In this paper we extend and generalize the results obtained by K. Goebel and W.A. Kirk and prove fixed point theorem for self maps.***

Keywords: Fixed points, self Maps

1. Introduction

Fixed point is center of vigorous research activity & most of these results deal with commuting mappings. Concept of commuting mapping has useful for generalizing in the context of metric space for proving fixed point. Generalized theorem is a major research activity in fixed point theory and its applications. The concept of commuting maps developed in [16 ,17]. In 1969, A Meri & E. Keeler [1] obtained a remarkable generalization of the Banach contraction principle, G.Jungek [15] introduced fixed points in commuting mappings in 1976. Shin-sen chang [30] generalized some theorem for commuting mapping in complete metric space. In 1986, R.P.Pant [29] prove that, common fixed point theorem of two pairs of commuting mappings satisfies Meir & Keeler type condition. In 2012 M.A. Alighamdi, S.Rademovic & N.Shahzad generalized some commuting mappings. In this paper we used common fixed point theorems in commuting mappings for proving common fixed point in complete metric space & prove some interesting results on commuting mappings.

Study of fixed points of self mappings satisfying contractive conditions which is one of the research activity. Large number of fixed point results for self-mappings satisfying various types of contractive inequalities. Which is in [9,10,12] . Fixed point results involving altering distances have been introduced in [11] . In 1977 Rhoades [5] proved fixed points for extended forms of contraction pairs. In 1986 Jungck [15] introduced the notion of compatible mappings, in 1998 Rhoades and Jungck [18] introduced the concept of weakly compatible maps. K. Goebel and W. A. Kirk [25] in 1990, V. I. Istratescu [33] in 1981 and M. S. Khan [27] in 1984 proved fixed point results for altering distances. In 1992 J. Meszaros [22] , In 1998 K. P. R. Sastry and G. V. R. Babu [24] proved fixed point theorems in Metric spaces by altering distances between the points. Naidu [28] in 2003, Singh and Dimri [20] in 2011 proved fixed point theorems for altering distance function.. In 2011 Popescu proved fixed point theorems which involving weak contractive type inequalities and weak contraction. In 2013 Gairola and Ram krishan [32] , Lj. B. Ciric [26] , Naidu S.V.R. [28] proved some fixed point results for self-maps satisfying a generalized weak contraction conditions. Number of interesting results obtained various authors for the study of common fixed point of mappings satisfying some contractive type condition.

In this paper we introduce the generalized altering distance function and prove fixed point theorems for to obtain unique fixed point.

2. Preliminaries and Definitions

Definition :2.1 :[**Khan,** 1998][27] : A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an **altering distance** function, if the following properties are satisfied.

(i) ψ is monotonically increasing and continuous

(ii) $\psi(t) = 0$ if and only if $t = 0$

Fixed point results involving altering distance have been introduced in [12]. An altering distance is a mapping, $F : [0, \infty) \rightarrow [0, \infty)$ which satisfies,

(a) F is increasing and continuous and

(b) $F(t) = 0$ if and only if $t = 0$

Fixed point result for altering distances have also studied in [8, 9]

Definition : 2.2 : [**Rhoades**, 2001] [5] : A mapping $T : X \rightarrow X$, where (X, d) is a Metric space, is said to be weakly contractive for $x, y \in X$,

$d(Tx, Ty) \leq d(x,y) - \emptyset(d(x,y))$

where $\emptyset : [0, \infty) \rightarrow [0, \infty)$ is continuous non-decreasing function such that

$\emptyset(t) = 0$ if and only if $t = 0$. If $\emptyset(t) = (1-k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

Definition 2.3 : (**Jungck** [15]) Let (X, ϱ) metric space, we say (f, g) is a weakly commuting pair (w.c.p.) or f, g commute weakly or f, g are weakly commutative if,

$$(fgx, gfx) \leq \varrho(fx, gx), \forall x \in X.$$

Definition 2.4] [4] Let (X, ϱ) be a metric space, self maps f,g are called commuting iff $fgx = gfx$, for all $x \in X$.

Definition :_2.5 : **Jungck** [19] : Let S and T be mappings from a Metric space

(X, d) into itself. The mapping S and T are said to be compatible if,

$\lim_{n\to\infty} d(ST_{x_n}, TS_{x_n}) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} S_{x_n} = t = \lim_{n\to\infty} T_{x_n} \; for\ some\ t \in X.$$

Definition : 2.6 : Jungck and Rhoodes [18] : Let S and T be mappings from a Metric space (X , d) into itself. The mappings S and T are said to be weakly compatible if, they commute at their coincidence points, that is, if $Tu = Su$ for some u ∈ X, than $TS_u = ST_u$. in this connection if we write v $= Tu = Su$. Then we say that v is a point of coincidence of (S , T)

Definition 2.7 (**R. P. Pant** [29]) : Let (X , ϱ) be a metric space a pair (f, g) of self-maps is said to be K-weakly commuting if there exists some $K > 0$ such that,

$$\varrho\,(fgx, gfx) \le K\varrho\,(fx, gx)\,, \forall\, x \in X.$$

3. Unique fixed point for self-maps

In this paper we obtain a new fixed point result for self-mappings defined on complete metric space which satisfying a contractive conditions which involves a function of two variables and acts on distances of two pair of points in a Metric spaces

Theorem 3.1 :(Cicic [26]) Let (X , ϱ) be a complete metric space. Let f be a self-map on X such that, for some constant $\beta \in (0,1)$ &

$$\varrho\,(fx, fy) \le \beta \max\{\,\varrho\,(x,y), \varrho\,(x, fx)\,, \varrho\,(y, fy), \varrho\,(x, fy), \varrho\,(y, fx)\,\} \qquad (3.1)$$

$\forall\, x, y \in X$. Then f possess a unique fixed point

Theorem 3.2: G.Jungck [19] Let (X , ϱ) be a complete metric space. Let f & g commuting continuous self-maps on X such that,

$$f(X) \supset g(X) \qquad (3.2)$$

Further, let there exist a constant $\beta \in (0,1)$ such that

$$\varrho\,(gx, gy) \le \beta\,\varrho(fx, fy)\,, \forall\, x, y \in X. \qquad (3.3)$$

Then f & g have a unique common fixed point

Theorem 3.3 : **Dolhare U.P**.[10] *Let* (X, d) *be a complete metric space and* $T: X \to X$ *be satisfying*

$$d(T_x, T_y) \le \alpha \cdot d(x, y), \forall\, x, y \in X \;\ldots\ldots\ldots (3.3)$$

Where $0 \le \alpha < 1$. Then T has a unique fixed point in X.

Theorem 3.4 : **Dolhare U.P.and Bele C.D.**[11] *Let f be a self maps of f-orbitally complete D- metric space X satisfying*

$(x, y, z)\rho(C\,, f_y\,, f_z\,) = \lambda\,\rho$ (*where* $0 \le \lambda \le 1$ *then f has a unique fixed point.*

Theorem 3.5: **Dolhare U.P**.[9] :*Let* (X, ρ) *be a complete metric space and* f *be a self map on X such that* f^2 *is continuous if* $g: f(x) \to X$ *such that* $g\,f(x) \subset f^2(x)$ *and* $g\,(f(x)) = f\,(g\,(x))$ *both sides are defined for all* $x, y \in f(x)$. *Then f and g have unique common fixed point.*

In Khan (1976), the following fixed point theorem is generalized as follows.

Theorem 3.6 : **Khan** [27] Let T be a self mapping of a complete metric space (X, d) and satisfying

$$d(T_x, T_y) \le \alpha \left(d(x, T_y)d(y, T_x)\right)^{1/2} \;\ldots\ldots\ldots 3.4$$

for all $x, y \in X$ and $0 \le \alpha < 1$, then T has a unique fixed point.

Theorem 3.7 [**Patil S. and Dolhare U.P**]: [29]- Let (X, d) be a metric space and T be a map of x into itself such that,

i) $d(Tx, Ty) \le g(d(x, y), d(x, Tx), d(y, Ty)) \forall x, y \in X$,

Where g Є G.

ii) T is continuous at a point u Є X.

iii) There exist a point x Є X such that the sequence of iterates $\{T^n(x)\}$ has a Subsequence $\{T^{ni}(x)\}$ on veering to u. then u is the unique fixed point of T.

Theorem 3.8 [Patil S. and Dolhare U.P] [29]:- Let (X, d) be a metric space and let S and T be two weakly compatible self mappings such that

1) S and T satisfying the (E.A) property

$$d(Tl, Tm) < \max\{d(Sl, Sm), [d(Tl, Sl) + d(Tm, Sm)]/2, [d(Tm, Sl) + d(Tl, Sm)]/2\} \forall l \ne m \in X$$

2)

3) Tx ⊂ Sx

If Sx or Tx is a complete subspace of X then T and S have a unique common fixed point.

Theorem 3.9 [**Patil S. and Dolhare U.P**][29]:- In a metric space (X, d) let S,T,U and V be self mappings such that

1) (S, V) and (T, U) are weakly compatibles.
2) $d(Sx,Ty) \le F\,[\max\{d(Vx, Uy), d(Vx, Ty), d(Ux, Ty)\}]$ for all $(x, y) \in X^2$.
3) (S, V) or (T, U) satisfies the property (E.A).
4) Sx is the subset Ux and Tx is subset of Vx.

If the range of one of the mappings S, T, U and V is complete subspace of X then S, T, U and V have a unique common fixed point.

Theorem 3.10 : **Dolhare** [9] Let $f: X \to X$ be a contraction of the complete metric space (X, ρ) so that $d(f(x), f(y)) \le \lambda\, d(x, y)$ for some $0 \le \lambda \le 1$, and let x_0 be any point X. Then the sequence $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ converges to unique fixed point x. Further more for any n^{th} value we have

$$d(x_n, x) \le \frac{\lambda^n}{1-\lambda}\, d(x_0, f(x_0)) \quad 3.5$$

Then f has unique fixed point.

Theorem 3.11 : [**Jaggi D. S**.] [21] : Let f be a continuous (t) = 0 if and only if t = 0.self-map defined on a complete metric space (X , d). Further, Let f satisfy the following condition.

$$d(f(x), f(y)) \le \frac{\propto d(x, f(x))\, d(y, f(y))}{d(x,y)} + \beta\, d(x, y) \quad 3.6$$

for all $x, y \in X$, $x \ne y$ and for some $\propto, \beta \in [0, 1)$ with $\propto + \beta < 1$, then f has a unique fixed point in X.

D.S. Jaggi generalized theorem 3.3 for some integer m as follows.

Theorem:3.12 [**Jaggi D. S.**] [21] : Let ap define on a complete metric space (X , d) such that for some positive integer m, f satisfy the condition

$$d(f^m(x), f^m(y)) \leq \frac{\propto d\left(x, f^m(x)\right).d(y, f^m(y))}{d(x,y)} + \beta d(x,y) \quad 3.7$$

for all $x, y \in X$, $x \neq y$ and for some $\propto, \beta \in [0, 1)$ with $\propto + \beta < 1$. if f^m is continuous then f has a unique fixed point.
Jungck proved the following theorem for f–contractive point- to- point mapping for fixed point .

Theorem 3.13: [**K. Goebel and W. A. Kirk**] [25] Let $F : X \rightarrow X$ be a self – mapping from a complete Metric space to itself and satisfy

$[(\delta(f_x, f_y))^p + r(\delta(x, f_x))^q]^k + [(\delta(y, f_y))^p + r(\delta(y, f^2_x))^q]^k \leq \lambda [(\delta(x,y))^p + r(\delta(x, f_x))^q]^k + \lambda'[(\delta(y,f_y))^P + r(\delta(y, f_y))^q]^k$ here $x, y \in X$; $p, k > 0$; $r, q \geq 0$ and $0 < \lambda < 1$; $0 < \lambda' \leq 1$, then f has a unique fixpoint..

Theorem 3.14: Let (X, ϱ) be a complete metric space. Let f be a continuous self-maps on X & g be self-maps on X that commute with f. Also f & g satisfy (3.2) & (2.1). Then f & g have a unique common fixed point.

4. Main Result for Fixed point

In the main result we prove the result for unique fixed point by generalizing the theorem 3.1 & 3.3 by using condition in definition 2.2 & 2.3 then unique fixed point result are as follows.

Theorem 4.1 : Let f & g be commuting self-maps of a compact metric space (X, ϱ) such that, gf is continuous if,
$x \neq gy \Rightarrow \varrho(fx, gy) < diam\{h(z)/ z \epsilon \{x, y\} \& h \epsilon C_{gf}\}$
(4.1)
Then there is a unique point $a \epsilon X$ such that $a = fa = ga$

Theorem 4.2 : Let (X, ϱ) be a complete metric space. Consider f be a self-map on X such that f^2 in continuous. Let $g: f(X) \rightarrow X$ be such that
$gf(X) \subset f^2(X)$ (4.1)
$g(f(x)) = f(g(x))$ whenever both sides are defined .
Further, let there exist a number $\beta \epsilon (0,1)$ such that (2.1) holds $\forall x, y \epsilon f(X)$. Then f & g have a unique common fixed point .

Proof : Firstly we will show that the theorem is true for $n = 2$. Since f is a contraction consider $\lambda < 1$, then
$d(f(x), f(y)) \leq \lambda d(x, y)$
We can apply f to $f(x)$ and $f(y)$ such that
$d(f^2(x), f^2(y)) \leq \lambda d(f(x), f(y))$
Since $d(f(x), f(y)) \leq \lambda d(x, y)$
$d(f^2(x), f^2(y)) \leq \lambda d(f(x), f(y)) \leq \lambda^2 d(x, y)$
Thus, $d(f^2(x), f^2(y)) \leq \lambda^2 d(x, y)$
Since $\lambda < 1, \lambda^2 < 1$ then f^n is a contraction,
Since f^n is a contraction then it is true that f^{n+1} is also contraction

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \lambda^{n+1} d(f(x), f(y)) \leq \lambda^{n+1} d(x, y)$$

Thus, $d(f^{n+1}(x), f^{n+1}(y)) \leq \lambda^{n+1} d(x, y)$
Thus by induction the theorem is true for all n. If $f(x) = x$, then

$$f^2(x) = f(f(x)) = f(x) = x$$

Then $f^n(x) = x$
Hence by induction f^n is contraction and f^n has the unique fixed point.

Theorem 4.3 : *Let (X, ρ) be a complete metric space suppose f is contraction mapping and λ is a constant for f and λ^n is the constant for f^n. Then f^n is also a contraction and f^n has a fixed point.*

Proof : Firstly we will show that the theorem is true for $n = 2$. Since f is a contraction consider $\lambda < 1$, then
$d(f(x), f(y)) \leq \lambda d(x, y)$

We can apply f to $f(x)$ and $f(y)$ such that
$d(f^2(x), f^2(y)) \leq \lambda d(f(x), f(y))$
Since $d(f(x), f(y)) \leq \lambda d(x, y)$
$d(f^2(x), f^2(y)) \leq \lambda d(f(x), f(y)) \leq \lambda^2 d(x, y)$
Thus, $d(f^2(x), f^2(y)) \leq \lambda^2 d(x, y)$
Since $\lambda < 1, \lambda^2 < 1$ then f^n is a contraction,
Since f^n is a contraction then it is true that f^{n+1} is also contraction

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \lambda^{n+1} d(f(x), f(y)) \leq \lambda^{n+1} d(x, y)$$

Thus, $d(f^{n+1}(x), f^{n+1}(y)) \leq \lambda^{n+1} d(x, y)$
Thus by induction the theorem is true for all n. If $f(x) = x$, then

$$f^2(x) = f(f(x)) = f(x) = x$$

Then $f^n(x) = x$
Hence by induction f^n is contraction and f^n has the unique fixed point

5. Conclusion

In the present paper we used commuting maps, weakly commuting pair. K-weakly commuting pairs for to find out common fixed point of self-maps in complete and compact metric space.

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