N- Fourier Series Equations Involving Jacobi $H_m$ Polynomials

Dr. A. P. Dwivedi¹, Dr. Virendra Upadhyay², Indu Shukla³

¹Former, Head of Mathematics Department, H. B. Tech. Univ. Kanpur, India
²Department of Physical science, M.G.C.G.V., Chitrakoot, Satna, M.P., India
³Research Scholar, M. G. C. G. V., Chitrakoot, Satna, M.P., India

Abstract: In this paper, we have considered the N-Fourier series equations involving Jacobi $H_m$ polynomials of the first and second kind and solved the two sets of series equations.

Keywords: Integral equation, Series equation, Fourier series, integral theorems, Jacobi polynomials

1. Introduction

If we review the literature then we observe that the existing solutions on series equations are derived only from dual to six Fourier series equations. No further generalizations are available till date. This tempted us to find the solution of N-Fourier series equations involving some special functions and in this paper we have obtained certain results. By considering the special values of $n = 2, 3, 4, 5, 6$ we shall be able to derive solutions of dual, triple, quadruple, 5-tuple and 6-tuple Fourier series equations involving respective special functions.

2. N-Series Equations of the first kind

(i) N-series equations of the first kind

$$\sum_{m=0}^{\infty} A_{m,j}(\alpha, \gamma; \sigma) = f_j(\sigma), a_{i-1} < \sigma < a_i$$ (1)

where, $i = 1, 3, 5, \ldots, n$ and $a_0 = 0$.

$$\sum_{m=0}^{\infty} B_{m,j}(\alpha, \gamma; \sigma) = f_j(\sigma), a_{i-1} < \sigma < a_i$$ (2)

where, $j = 2, 4, 6, \ldots, n$ and $a_n = 1$.

Here $n$ is taken as an even number. If $n$ is odd then the equations will be

$$\sum_{m=0}^{\infty} B_{m,j}(\alpha, \gamma; \sigma) = f_j(\sigma), a_{i-1} < \sigma < a_i$$ (3)

where, $i = 1, 3, 5, \ldots, n$ and $a_0 = 0$.

(ii) N-series equations of the second kind

$$\sum_{m=0}^{\infty} D_{m,j}(\alpha, \gamma; \sigma) = g_j(\sigma), a_{i-1} < \sigma < a_i$$ (4)

where, $j = 2, 4, 6, \ldots, n$ and $a_n = 1$.

Here also $n$ is taken as an even number. If $n$ is odd then the equations will be

$$\sum_{m=0}^{\infty} E_{m,j}(\alpha, \gamma; \sigma) = g_j(\sigma), a_{i-1} < \sigma < a_i$$ (5)

where, $i = 1, 3, 5, \ldots, n$ and $a_0 = 0$.

$$\sum_{m=0}^{\infty} F_{m,j}(\alpha, \gamma; \sigma) = g_j(\sigma), a_{i-1} < \sigma < a_i$$ (6)

where, $j = 2, 4, 6, \ldots, n$ and $a_1 = 1$.

3. Preliminary Results

In the analysis, we shall use the following results:

(i) The orthogonality relation for the Jacobi polynomial is,

$$\int_{0}^{1} t^{\lambda-1}(1-t)^{\mu-1}j_m(\alpha, \gamma; t)j_n(\alpha, \gamma; t)dt = \frac{\delta_{mn}}{\Delta^2}$$ (7)

where $\Delta^2 = (\alpha+2\lambda)(\alpha+2\mu)/(\alpha+2\nu)$.

(ii) When $\alpha+1>\gamma>0$, $\delta_{mn}$ is the kronecker delta.

3.1 Derivation of series equation

Here we solve only equations (1), (2) of first kind and equations (5), (6) of the second kind and the solution of equation (3), (4) of first kind and equations (7), (8) of the second kind can be obtained easily by following similar procedure.

Volume 5 Issue 12, December 2016

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4. The Solution

(i) Equations of the first kind:

Let us assume
\[ \sum_{n=0}^{\infty} A_{jm}(\alpha, y; \sigma) = \phi_i(\sigma), \quad a_{i-1} < x < a_i \tag{18} \]
where, \( i = 2, 4, 6, \ldots, n \) and \( \phi_i(x) \) are unknown functions. Using orthogonality relation it follows from equations (1) and (18).

\[ A_m = \Delta^2 \sum_{i=0}^{n-2} \int_{a_{2i+1}}^{a_{2i+2}} \int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma) + \phi_{2i+1}(\sigma) \sigma^{-1} \int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma) + \phi_{2i+1}(\sigma) \sigma^{-1} \int_{a_{2i+1}}^{a_{2i+2}} \]
\[ = \frac{\int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma)}{\int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+1}(\sigma)} \int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma) + \phi_{2i+1}(\sigma) \sigma^{-1} \int_{a_{2i+1}}^{a_{2i+2}} \]
\[ = M_k(\sigma), a_{k-1} < \sigma < a_k \tag{20} \]

Substituting the above expression for \( A_m \) in equation (2) we obtain
\[ \sum_{i=0}^{n-2} \int_{a_{2i+1}}^{a_{2i+2}} \int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma) \left( K(\sigma, t) + S(\sigma, t) \right) dt \]
\[ + \int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+1}(\sigma) \left( K(\sigma, t) + S(\sigma, t) \right) dt \]
\[ = \frac{\int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma) \left( K(\sigma, t) + S(\sigma, t) \right) dt}{\int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+1}(\sigma) \left( K(\sigma, t) + S(\sigma, t) \right) dt} \cdot \frac{\int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma)}{\int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+1}(\sigma)} \]
\[ = M_k(\sigma), a_{k-1} < \sigma < a_k \tag{25} \]

Where,
\[ M_k(\sigma) = \frac{\int_{a_{2i+1}}^{a_{2i+2}} f_{2i+1}(\sigma)}{\int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+1}(\sigma) \left( K(\sigma, t) + S(\sigma, t) \right) dt} \]

Using equation (13) in equation (25) , we get
\[ \int_{a_{2i-1}}^{a_{2i}} \phi_{2i}(t) \int_{a_{2i-1}}^{a_{2i}} m(x)(\sigma - x)^{\alpha-1}(t - x)^{\gamma-\lambda+p+1} dx dt + \int_{a_{2i}}^{a_{2i+1}} \phi_{2i}(t) \int_{a_{2i}}^{a_{2i+1}} m(x)(\sigma - x)^{\alpha-1}(t - x)^{\gamma-\lambda+p+1} dx dt \]
\[ = M_k(\sigma) \sum_{i=0}^{k-4} \int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+2}(\sigma)\int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+2}(\sigma) \]

Using the results,

**Volume 5 Issue 12, December 2016**

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Paper ID: ART20163375

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Now equation (31) will be
\[
m(x)\vec{b}_k(x) = A_k(x) - \frac{\sin(1 - \rho)\pi}{\pi} \int_0^{(k-1)/2} \left( a_{k-1} - \xi \right)^{\rho} m(\xi) dx + \frac{\sin(1 - \rho)\pi}{\pi} \int_0^{(k+1)/2} \left( a_{k+1} - \xi \right)^{\rho} m(\xi) dx
\]

For all \( k = 2,4,6,\ldots,n \).

Applying the equation (36) in equation (34) and also applying the Leibnitz theorem, we get
\[
m(x)\vec{b}_k(x) = A_k(x) - \frac{\sin(1 - \rho)\pi}{\pi} \int_0^{(k-1)/2} \left( a_{k-1} - \xi \right)^{\rho} m(\xi) dx + \frac{\sin(1 - \rho)\pi}{\pi} \int_0^{(k+1)/2} \left( a_{k+1} - \xi \right)^{\rho} m(\xi) dx
\]

(ii) Equations of the second kind:
Let us assume

\[
\sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \Psi_i(x) \psi_i(x) dx = \Psi_i(\sigma), \quad a_{i-1} < x < a_i, \quad (i = 1,3,5,\ldots,n-1) \text{ and } \Psi_i(x) \text{ are unknown functions in the given interval.}
\]

Using orthogonality relation it follows from equations (6) and (42).

\[
D_m = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \Psi_i(\sigma) \psi_i(x) dx = \psi_i(x) \psi_i(x) = \delta_{ij}
\]

we get,
\[
D_m = A_m \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \psi_i(x) \psi_i(x) dx = \psi_i(x) \psi_i(x) = \delta_{ij}
\]
Substituting this expression for $D_n$ in equation (5) we get on interchanging the order of integration and summation,

\[
\sum_{i=0}^{n-2} \int_{a_{2i+1}}^{a_{2i+2}} \int_{t_i}^{t_{i+1}} \psi_{2i+1}(t) \{K_s(t) + S(t, t)\} dt + \int_{t_i}^{t_{i+1}} g_{2i+1}(t) \{K_s(t) + S(t, t)\} dt + \int_{t_i}^{t_{i+1}} \psi_{2i+1}(t) S(t, t) dt + \int_{t_i}^{t_{i+1}} \psi_{2i+1}(t) K_s(t) dt = M_k(\sigma)
\]

where,

\[
N_k(\sigma) = \frac{\Gamma(\rho \sigma)^{\rho \sigma}}{\Gamma(\sigma + \rho \sigma)} g_0(\sigma) - \sum_{i=0}^{n-2} \int_{a_{2i+1}}^{a_{2i+2}} \int_{t_i}^{t_{i+1}} g_{2i+1}(t) \{K_s(t) + S(t, t)\} dt
\]

Using equation (13) in equation (48), we get

\[
\sum_{i=0}^{n-k-3} \int_{a_{2i+1}}^{a_{2i+2}} \int_{t_i}^{t_{i+1}} m(x)(\sigma - x)^{\rho - 1}(t - x)^{\gamma - 1} dx dt + \int_{t_i}^{t_{i+1}} \int_{0}^{m(x)(\sigma - x)^{\rho - 1}(t - x)^{\gamma - 1}} dx dt = N_k(\sigma) - \sum_{i=0}^{n-2} \int_{a_{2i+1}}^{a_{2i+2}} \int_{t_i}^{t_{i+1}} m(x)(\sigma - x)^{\rho - 1}(t - x)^{\gamma - 1} dx dt
\]

Inverting the order of integration, we obtain

\[
\sum_{i=0}^{n-2} \int_{a_{2i+1}}^{a_{2i+2}} \int_{t_i}^{t_{i+1}} m(x)(\sigma - x)^{\rho - 1}(t - x)^{\gamma - 1} dx dt
\]
Applying the equation (57) in equation (55) and also applying the Leibnitz theorem, we get

Therefore, for all $k = 1, 3, 5$

Using the equations (32) and (33) we get,

Equation (51) is also an Abel type integral equation and its solution is given by

For all $k = 1, 3, 5, \ldots, n-1$. Therefore,

Applying the equation (57) in equation (55) and also applying the Leibnitz theorem, we get

\[
\int_{a_{k-1}}^{a_k} \frac{\Psi_k'(t)dt}{(t-\xi)^{1-r+\lambda-\rho}} = -\frac{\sin(1-\rho+\lambda-\rho)\pi}{\pi^2(x-a_{k-1})^\rho} \int_{a_{k-1}}^{a_k} \Psi_k(x)dx
\]

For all $k = 1, 3, 5, \ldots, n-1$. (57)
With the help of the result of these N-series equation it is easy to find the solution of corresponding dual, triple, quadruple etc. series equations.

For all \( k = 1, 3, 5, \ldots, n-1 \), we have

\[
P_k(x, y) = \frac{\sin(1-\rho)\pi \sin(1-\gamma+\lambda-\rho)\pi}{\pi^2(x-a_{k-1})^\rho} \int_0^{a_{k-1}} \frac{(x-a_{k})^\rho}{(x-a_{k-1})^\rho} \sin(1-\rho)\pi \sin(1-\gamma+\lambda-\rho)\pi dt
\]

\[
Q_{2k+1}(x, y) = \frac{1}{\pi^2(x-a_{k-1})^\rho} \int_0^{a_{k-1}} \frac{(x-a_{k})^\rho}{(x-a_{k-1})^\rho} \sin(1-\gamma+\lambda-\rho)\pi dt
\]

\[
R_{2k+1}(x, y) = \frac{1}{\pi^2(x-a_{k-1})^\rho} \int_0^{a_{k-1}} \frac{(x-a_{k})^\rho}{(x-a_{k-1})^\rho} \sin(1-\gamma+\lambda-\rho)\pi dt
\]

\[
S_{2k+1}(x, y) = \frac{1}{\pi^2(x-a_{k-1})^\rho} \int_0^{a_{k-1}} \frac{(x-a_{k})^\rho}{(x-a_{k-1})^\rho} \sin(1-\gamma+\lambda-\rho)\pi dt
\]

Substituting \( k = 1, 3, 5, \ldots, n-1 \) in equation (58) we will get \( n/2 \) simultaneous Fredholm Integral equations of the second kind. With the help of these \( n/2 \) simultaneous equations we can calculate \( \Psi_2(x), \Psi_4(x), \ldots, \Psi_n(x) \) and then the values of \( \Psi_2(t), \Psi_4(t), \ldots, \Psi_n(t) \) can be determined. After all these calculations we can compute the coefficient \( D_m \) with the help of equation (46).

5. Particular Cases

With the help of the result of these N-series equation it is easy to find the solution of corresponding dual, triple, quadruple etc. series equations.

6. Acknowledgement

Author is thankful to Dr. A.P. Dwivedi for his co-operation & support provided to me during the preparation of this paper. Author is also thankful to Dr. Brajesh Mishra for his support.

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