Some Results on $I$-cordial Graph

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Abstract: An $I$-cordial labeling of a graph $G (V, E)$ is an injective map $f$ from $V$ to $[-\frac{p}{2}, \frac{p}{2}]$ or $[-\frac{p}{2}, \frac{p}{2}]$ as $p$ is even or odd, respectively be an injective mapping such that $f(u) + f(v) ≠ 0$ and induces an edge labeling $f^* : E → {0, 1}$ where, $f^*(uv) = 1$ if $f(u) + f(v) > 0$ and $f^*(uv) = 0$ otherwise, such that the number of edges labeled with 1 and the number of edges labeled with 0 differ at most by 1. If a graph has $I$-cordial labeling, then it is called $I$-cordial graph. In this paper, we introduce the concept of $I$-cordial labeling and prove that some standard graphs that are $I$-cordial and some graph that are not $I$-cordial.

Notation: Here $[-x..x] = \{t \mid t ≤ x\}$ and $[-x..x]^* = [-x..x] - \{0\}$.

Keywords: Cordial labeling; $I$-cordial labeling

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1. Introduction

By a graph we mean a finite undirected graph without loops and multiple edges. For terms not defined here we refer to Harary [9].

An $I$-cordial labeling of a graph $G (V, E)$ is an injective map $f$ from $V$ to $[-\frac{p}{2}, \frac{p}{2}]$ or $[-\frac{p}{2}, \frac{p}{2}]$ as $p$ is even or odd, respectively, be an injective mapping such that $f(u) + f(v) ≠ 0$ and induces an edge labeling $f^* : E → \{0, 1\}$ where, $f^*(uv) = 1$ if $f(u) + f(v) > 0$ and $f^*(uv) = 0$ otherwise such that the number of edges labeled with 1 and the number of edges labeled with 0 differ at most by 1. If a graph has $I$-cordial labeling, then it is called $I$-cordial graph. The concept of cordial graph originated from I.Cahit [1, 2] in 1987 as a weaker version of graceful and harmonious graphs and was based on $\{0, 1\}$ binary labeling of vertices.

Let $f : V → \{0, 1\}$ be a mapping that induces an edge labeling $f^* : E → \{0, 1\}$ defined by $f^*(uv) = |f(u) - f(v)|$. Cahit called such a labeling cordial if the following condition is satisfied: $|v_i(0) - v_i(1)| ≤ 1$ and $e_i(0) - e_i(1) ≤ 1$, where $v_i(0)$ and $e_i(0)$, $i = 0, 1$ are the number of vertices and edges of $G$ respectively with label $i$ (under $f$ and $f^*$ respectively). A graph $G$ is called cordial if it admits cordial labeling.

In [1], Cahit showed that (i) every tree is cordial (ii) $K_n$ is cordial if and only if $n ≤ 3$ (iii) $K_{r,s}$ is cordial for all $r$ and $s$ (iv) the wheel $W_n$ is cordial if and only if $n = 3$ (mod 4) (v) $C_n$ is cordial if and only if $n ≠ 2$ (mod 4) (vi) an Eulerian graph is not cordial if its size is congruent to 2 modulo 4.

Du [4] investigated cordial complete k-partite graphs. Kuo et al. [13] determined all $m$ and $n$ for which $mK_n$ is cordial. Lee et al. [14] exhibited some cordial graphs. Generalised Petersen graphs that are cordial are characterised in [7]. Ho et.al [6] investigated the construction of cordial graphs using Cartesian products and composition of graphs. Shee and Ho [7] determined the cordialities of $C_m^{(n)}$; the one–point union of $n$ copies of $C_m$. Several constructions of cordial graphs were proposed in [10-12, 15-18]. Other results and open problems concerning cordial graph are seen in [2, 5]. Other types of cordial graphs were considered in [3, 4, 8, 20]. Vaidya et.al [21] has also discussed the cordiality of various graphs.

Definition 1.1 [23]
Let $f$ be a binary edge labeling of graph $G = (V, E)$ and the induced vertex labeling is given by $f(v) = \sum_{e \in E} f(e) (\mod 2)$ where $v \in V$ and $(u,v) \in E$. $f$ is called an $E$-cordial labeling of $G$ if $|e(0) - e(1)| ≤ 1$ and $|v(0) - v(1)| ≤ 1$, where $e(0)$ and $e(1)$ denote the number of edges, and $v(0)$ and $v(1)$ denote the number of vertices with 0's and 1's respectively. The graph $G$ is called $E$-cordial if it admits $E$-cordial labeling.

In 1997 Yılmaz and Cahit [23] have introduced $E$-cordial labeling as a weaker version of edge–graceful labeling. They proved that the trees with $n$ vertices, $K_n$, $C_n$ are $E$-cordial if and only if $n$ is even or odd, respectively. $C_m$ is $E$-cordial if and only if $n ≠ 2$ (mod 4) while $K_m,n$ admits $E$-cordial labeling if and only if $m + n ≠ 2$ (mod 4).

Definition 1.2 [20]
A prime cordial labeling of a graph $G$ with vertex set $V$ is a bijective function $f$ from $V$ to $\{1, 2, 3, \ldots, |V|\}$ where each edge $uv$ is assigned the label 1 if gcd $(f(u), f(v)) = 1$ and 0 if gcd $(f(u), f(v)) > 1$, such that the number of edges having label 0 and edges having label 1 differ by at most 1.

Sundaram et.al. [19] introduced the notion of prime cordial labeling. They proved the following results are prime cordial: $C_n$ if and only if $n ≥ 6$; $P_n$ if and only if $n ≠ 3$ or 5; $K_{1,n}(n, odd)$; the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n ≥ 3$; bi-stars; dragons; crowns; triangular snakes if and only if the graph has at least three triangles; ladders. J. Babujee and L.Shobana [22] proved the existence of prime cordial labeling for sun graph, kite graph and coconut tree and $Y$-tree, $< K_{1,n}; K_{2} >$ ( $n ≥ 1$); Hoffman tree, and $K_{2} \cup C_n (C_n)$.
In this paper we introduce the concept of integer I-cordial labeling and we prove that some standard graphs such as cycle $C_n$, Path $P_n$, Friendship $F_n$, Helm graph $H_n$, Closed graph $CH_n$, Double Fan $DF_n$, $n \geq 2$, are I-cordial; Wheel $W_n$ and Fan graph $fn$ are I-cordial if and only if $n$ is even.

### Main Results

#### Notation.1.3

Here $[-x..x] = \{t \div t$ is an integer and $|t| \leq x\}$ and $[-x..x]^* = [-x..x] - \{0\}$.

#### 2. Main Results

#### Definition.2.1

Let $G = (V,E)$ be a simple connected graph with $p$ vertices. Let $f: V \rightarrow [-p/2..p/2]$ or $[-p/2..p/2]^*$ as $p$ is even or odd respectively be an injective mapping such that $f(u) + f(v) \neq 0$ and induces an edge labeling $f^*: E \rightarrow \{0, 1\}$ where $f(v_{uv}) = 1$, if $f(u) + f(v) > 0$ and $f(v_{uv}) = 0$ otherwise. Let $e_f(i) =$ number of edges labeled with $i$, where $i = 0$ or $1$. $f$ is said to be I-cordial if $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called I-cordial if it admits a I-cordial labeling.

#### Figure 1: I-cordial Graph

**Theorem 2.2** The cycle $C_p$ is I-cordial.

**Proof.** Let $v_1, v_2, \ldots, v_p$ be the $p$ vertices of the cycle $C_p$. Here $q = p$.

**CASE 1.** $p$ is even.

Let $p = 2n$. We define $f: V \rightarrow [-n..n]^*$ as follows: $f(v_i) = -i$; $1 \leq i \leq n$; $f(v_{i+n}) = i$; $1 \leq i \leq n$.

When $f(v_n) = -n$ and $f(v_{n+1}) = 1$, the edge labeling $f^*(v_nv_{n+1}) < 0$.

This implies that $f^*(v_nv_{n+1}) < 0$.

Similarly, $f(v_{p-1}) = -1$ and $f(v_p) = n$ yield $f^*(v_{p-1}v_p) = n - 1$, which is positive when $n \geq 1$.

Therefore, we assign $f^*(v_{p-1}v_p) = 1$. Obviously, the sum of consecutive negative (positive) integers is negative (positive). As there are $\frac{q}{2}$ such negative labels and $\frac{q}{2}$ positive labels, $e_f(0) = e_f(1) = \frac{q}{2}$.

**CASE 2.** $p$ is odd and $p > 3$.

Let $p = 2n + 1$, when $n > 1$. We define $f: V \rightarrow [-n..n]$ as follows: $f(v_i) = -i$, $1 \leq i \leq n - 1$; $f(v_{n+1}) = i + 1$, $1 \leq i \leq n - 1$; and $f(v_n) = 0$.

Let us consider the edge $v_nv_{n+1} \in E$ then $f(v_n) = -(n - 1)$ and $f(v_{n+1}) = 1$.

That is, $f(v_n) + f(v_{n+1}) < 0$, which implies $f^*(v_nv_{n+1}) = 0$.

Similarly, for the edge, $v_nv_{n+1}$, $f^*(v_nv_{n+1}) = -1 + 0 = -1$, so that $f^*(v_nv_{n+1})$ receives label 0. From the observation $n + 1$ edges receive label 0 and $n$ edges receive label 1.

Therefore, $e_f(0) = n + 1$ and $e_f(1) = n$. Thus $|e_f(0) - e_f(1)| = 1$.

The case when $p = 3$ does not yield any I-cordial labeling for $C_3$ by Theorem 2.3.

Thus $C_p$ is I-cordial graph.

#### Figure 2: $C_{12}$ and $C_{11}$ are I-cordial graph.

**Theorem 2.5** Path $P_n\ n > 2$ is I-cordial.

**Proof.** When $n = 3$, we label $\{-1, 0, 1\}$ corresponding to the vertices $\{v_1, v_2, v_3\}$ which implies $P_3$ is I-cordial.

For the case $n > 3$, we labeling is similar to Theorem 2.4.

**Theorem 2.6** Complete graph $K_p$ is not I-cordial.

**Proof holds from Theorem 2.3 and 2.4.

**Theorem 2.7** The Wheel graph $W_n$; $n > 3$ is I-cordial if and only if $n$ is even.

**Proof.** Let $u$ be the apex vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices. Here $|V| = n + 1$ and $|E| = 2n$. Let us consider two cases.

**Case 1.** $|V|$ is odd. ($n$ is even)

Let $n = 2m$. We define $f: V \rightarrow [-m..m]$ as follows: $f(v_i) = -i$, $1 \leq i \leq m$ and $f(v_{m+i}) = i$, $1 \leq i \leq m$.

Consider the vertex label $f(v_{m}) = -n$ and $f(v_{m+1}) = 1$ then, $f^*(v_{m}v_{m+1}) < 0$. Also $f^*(v_{m+i}v_{i}) < 0$ for all $i = 1, 2, \ldots, m$ and $f^*(v_{m+i}v_{m+i}) > 0$ for all $i = m+1, 2m-1$. Also, $f(v_{2m}) = n$ and $f(v_{m}) = -1$.

Therefore, $f^*(v_{2m}v_{i}) > 0$. Now $f^*(uv_i) > 0$ for all $i = 1, 2, \ldots, m$ and $f^*(uv_i) > 0$ for all $i = m+1, 2m$. Hence, the q edges equally share label 0 and 1. That is, $e_f(0) = e_f(1) = \frac{q}{2}$ which imply $|e_f(0) - e_f(1)| = 1$.

Thus from both the cases $|e_f(0) - e_f(1)| \leq 1$.

Hence $W_n$; $n > 3$ is I-cordial.
Case 2. |V| is even
That is, when n is odd, by Theorem 2.4, \( W_n \); \( n \geq 3 \) is not \( I \)-cordial.

\[ \text{Figure 3: } W_n \text{ is } I\text{-cordial} \]

**Theorem 2.8** Helm graph \( H_n \) is \( I \)-cordial.

**Proof.** Let \( H_n = G \), then \( p = 2n + 1 \) and \( q = 3n \). Let \( v \) be the apex vertex, \( v_1, v_2, \ldots, v_n \) be the rim vertices of the cycle and \( u_1, u_2, \ldots, u_n \) be the pendant vertices corresponding to \( v_i \)’s. Suppose \( n = 2m \). We define \( f : V \rightarrow \{ -(2m + 1), (2m + 1) \} \) as \( f(v) = 0; f(v_i) = -i, 1 \leq i \leq m; f(v_{m+i}) = i, 1 \leq i \leq m \).

Let us consider the vertices \( v_{m+1} \) and \( v_{m+2} \). We have \( f(v_{m+1}) = m \) and \( f(v_{m+2}) = 0 \). Also, \( f(v_{m+i}) = m, 1 \leq i \leq m \).

Now let us consider the pendant vertices \( u_i \)’s. Here \( f(u_i) = m + 1 \) for all \( i = 1, 2, 3, \ldots, m \) and \( f(u_{m+i}) = 1 \). Thus \( |ef(0) - ef(1)| = 1 \). From all the cases, \( |ef(0) - ef(1)| = 1 \).

\[ \text{Figure 4: } H_7 \text{ is } I\text{-cordial} \]

**Theorem 2.9** The closed helm graph \( CH_n \) is \( I \)-cordial.

**Proof.** Let \( CH_n = G \). Then let \( p = 2n + 1 \) and \( q = 4n \). Let \( v \) be the apex vertex, \( v_1, v_2, \ldots, v_n \) be vertices of the inner cycle and \( u_1, u_2, \ldots, u_n \) be the rim vertices of the outer cycle. The case when \( n \) is even follows from Theorem 2.7. Now suppose \( n \) is odd and \( n = 2m + 1 \).

Since the apex vertex \( v \), is labeled with 0, \( f(v) = 0; f(v_i) = -i, 1 \leq i \leq m; f(v_{m+i}) = i, 1 \leq i \leq m; f(u_i) = m + 1; f(u_{m+i}) = i, 1 \leq i \leq m \). Thus \( |ef(0) - ef(1)| = 1 \). From all the cases, \( |ef(0) - ef(1)| = 1 \).

Thus \( G \) is \( I \)-cordial.

\[ \text{Figure 5: } CH_6 \text{ is } I\text{-cordial} \]
Theorem 2.10 Friendship graph $F_n$, $n > 1$ is $I$-cordial.

**Proof.** Let $v_0$ be the central vertices of $n$ triangles, consecutively of $F_n$. We note that $p = 2n + 1$ and $q = 3n$. We consider two cases.

**CASE 1.** $n$ is even

Let $n = 2m$. Define $f: V \rightarrow [-2m \ldots 2m]$ as $f(v_i) = i$, $1 \leq i \leq 2m$, $i$ is odd; $f(v_{2m+i}) = -i$, $1 \leq i \leq 2m$, $i$ is odd; so that the edges of triangles $C_i$, $i = 1, 2, \ldots, m$ are all $> 0$ and the edges of triangles $C_i$, $i = m + 1, \ldots, 2m$ are all $< 0$. Hence, $3m$ edges shares positive and negative labels. That is, $e_0(0) = e_1(1)$.

**CASE 2.** $n$ is odd.

Let $n = 2m + 1$. We consider $f: V \rightarrow [-2m \ldots 2m]$ as $f(v_i) = i$, $1 \leq i \leq 2m + 1$, $i$ is odd; $f(v_{2m+1+i}) = -i$, $1 \leq i \leq 2m + 1$, $i$ is odd; so that the edges of triangles $C_i$, $i = 1, 2, \ldots, m$ are all $> 0$ and the edges of triangles $C_i$, $i = m + 2, \ldots, 2m + 1$ are all $< 0$. Also in the triangle, $C_m$ the edges $f(v_{2m+1}v_0) > 0$, $f(v_{2m+1}v_{2m+2}) > 0$ and $f(v_0v_{2m+2}) < 0$. Hence, $e_0(0) = 1, e_1(1) + 2$.

Thus, $|e_0(0) - e_1(1)| = 1$. Therefore, $F_n$ is $I$-cordial.

Figure 6: Fan graph $F_6$ is $I$-cordial

Theorem 2.11 The fan $f_n$, $n \geq 3$ is $I$-cordial if and only if $n$ is even.

**Proof.** $f_n$ has $n + 1$ vertices and $2n - 3$ edges. Let $u$ be the apex vertex with degree $n$. Let $v_1, v_2, v_3, \ldots, v_n$ denote the path vertices adjacent to $u$ in $f_n$.

**Case 1.** $n$ is even

Let $n = 2m$. Define $f: V \rightarrow [-m \ldots m]$ as $f(v_i) = i$, $1 \leq i \leq m$ and $f(v_{2m+i}) = -i$; $1 \leq i \leq m$. Then $f(v_i) < 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+i}) > 0$ for all $i = m + 1, m + 2, \ldots, 2m$. Thus $|e_0(0) - e_1(1)| = 0$. Similarly, $f(v_{2m+1+v_i}) < 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+1}v_i) > 0$ for all $i = m + 1, m + 2, \ldots, 2m$. Thus $|e_0(0) - e_1(1)| = 1$. Therefore from the above cases $e_0(0) - e_1(1) = 1$. Hence, $F_n$, $n \geq 3$ is $I$-cordial.

When $n$ is odd, by Theorem 2.2.3 $f_n$ is not $I$-cordial.

Figure 7: $f_6$ is $I$-cordial

Theorem 2.12 The double fan $Df_n$, $n \geq 2$ is $I$-cordial.

**Proof.** $Df_n$ has $n + 2$ vertices and $3n - 1$ edges. Let $a$ and $b$ denote the apex vertices of degree $n$ and $v_1, v_2, \ldots, v_n$ be the path vertices adjacent to $a$ and $b$ in $Df_n$. Then $E(Df_n) = A \cup B \cup C$ where $A = \{av_i\}_{i=1}^n$; $B = \{bv_i\}_{i=1}^n$ and $C = \{v_iv_{i+1}\}_{i=1}^n$.

We consider two cases:

**CASE 1.** $n$ is even

Let $n = 2m$. We define $f: V \rightarrow \{-m - 1, \ldots, m + 1\}$ as $f(a) = m + 1$, $f(b) = - (m + 1)$; $f(v_i) = i$, $1 \leq i \leq m$ and $f(v_{2m+i}) = -i$, $1 \leq i \leq m$. Consider $f(v_{2m+1}v_0) > 0$, $f(v_{2m+1}v_{2m+2}) > 0$ and $f(v_0v_{2m+2}) < 0$. Hence, $e_0(0) = 1, e_1(1) + 2$.

Now, $f(v_i) < 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+1}) < 0$ for all $i = m + 1, m + 2, \ldots, 2m - 1$.

Thus $|e_0(0) - e_1(1)| = 1$.

**CASE 2.** $n$ is odd

Let $n = 2m + 1$. We define $f: V \rightarrow \{-m, \ldots, m\}$ as $f(a) = m + 1$, $f(b) = - (m + 1)$; $f(v_i) = i$, $1 \leq i \leq m$ and $f(v_{2m+i}) = -i$, $1 \leq i \leq m$. Let us consider, $f(v_{2m+1}v_0) > 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+1}v_{2m+2}) > 0$ and $f(v_0v_{2m+2}) < 0$. Hence, $e_0(0) = 1, e_1(1) + 2$.

Now, $f(v_i) < 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+1}) < 0$ for all $i = m + 1, m + 2, \ldots, 2m$.

Thus $|e_0(0) - e_1(1)| = 0$ for all $e \in A \cup B$.

Case 2. $n$ is odd

Let $n = 2m + 1$. We define $f: V \rightarrow \{-m - 1, \ldots, m\}$ as $f(a) = m + 1$, $f(b) = - (m + 1)$; $f(v_i) = i$, $1 \leq i \leq m$ and $f(v_{2m+i}) = -i$, $1 \leq i \leq m$. Let us consider, $f(v_{2m+1}v_0) > 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+1}v_{2m+2}) > 0$ and $f(v_0v_{2m+2}) < 0$. Hence, $e_0(0) = 1, e_1(1) + 2$.

Now, $f(v_i) < 0$ for all $i = 1, 2, \ldots, m$ and $f(v_{2m+1}) < 0$ for all $i = m + 1, m + 2, \ldots, 2m$.

Thus $|e_0(0) - e_1(1)| = 0$ for all $e \in C$. Hence from all the cases $|e_0(0) - e_1(1)| \leq 1$.

Therefore double fan $Df_n$, $n \geq 2$ is $I$-cordial.

Figure 8: $Df_6$ is $I$-cordial.
Theorem 2.13 Double Wheel $DW_n$, $n > 2$ is $I$-cordial.

Proof. Let $G = DW_n$ be the double wheel. Let $v_0, v_1, \ldots, v_n$ be the inner rim vertices and $v_1', v_2', \ldots, v_n'$ be the outer rim vertices of $DW_n$. Then $p = 2n + 1$ and $q = 4n$.

We define $f : V \rightarrow [-n \ldots n]$ as, $f(v_0) = 0$, $f(v_i) = i$, $1 \leq i \leq n$ and $f(v_i') = -i$, $1 \leq i \leq n$ so that $f(v_i v_{i+1}) > 0$ for all $i = 1, 2, \ldots, n - 1$ and $f(v_i' v_{i+1}') < 0$ for all $i = 1, 2, \ldots, n - 1$. Here $n$ edges equally shares negative and positive integers. Since, $f(v_0) = 0$ then $f(v_0 v_i) > 0$ for all $i = 1, 2, \ldots, n$ and $f(v_0 v_i') < 0$ for all $i = 1, 2, \ldots, n$. Here also $n$ edges shares negative and positive integers. That is, $e_f(0) = e_f(1)$. Hence, $|e_f(0) - e_f(1)| = 0$.

Figure 9: $DW_4$ is $I$-cordial.

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