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Application of Fourier-Stieltjes Transform to Partial Differential Equation

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Abstract: Mathe*matics is everywhere in every phenomenon, technology, observation, experiment etc. All we need to do is to understand the logic hidden behind. In applied mathematics and engineering the Partial Diffrential equations have great importance. Therefore it is very important to know methods to solve such partial differential equations. The Partial differential equations can be easily solved by using the method of Integral Transform. In the present paper, we have solved the some Partial Differential equations such as Wave equations, Heat flow equations and Laplace equation by using the Fourier-Stieltjes Transform.*

Keywords: Fourier Transform, Stieltjes Transform, Partial Differential equation, Fourier-Stieltjes Transform

1. Introduction

Mathematicians have long had techniques to solve lot of physical equations, Integral equation and differential equations which have many applications in all fields of sciences. But there are other equation i.e. Partial Differential equations which are solved quite easily by using the integral Fourier-Stieltjes Transform. There are number of partial differential equations which are used in all fields of Sciences and Engineering. In the present paper we have obtained the solution for the wave equation, heat flow equation and Laplace equation by using the Fourier-Stieltjes Transform. The Wave equation are found in elasticity, quantum mechanics, plasma physics, general relativity, acoustics, electromagnetic, fluid dynamics, vibrating string such as that of a musical instrument[1,2,3]. The Heat flow equations are important in probability, financial mathematics, Riemannian geometry, topology, image analysis [4] and Laplace equation notably used for image analysis, electromagnetism, astronomy, fluid dynamics [5]. A history of mathematics includes early connections with music and the basic physics of sound. As we know the Fourier Transform is applicable in Music [6] and the Stieltjes Transform also applicable in Music, since Stieltjes Transform used in Random Matrix theory (RMI) [7]. So, we can say the Fourier and Stieltjes transform are mathematically related to each other. Here, we have provided the applications of Fourier-Stieltjes Transform to partial Differential equations and extent the ideas of generalization of Fourier-Stieltjes Transform. In this present paper we have solved the wave equation, Heat flow equation and Laplace equation using the differential property of Fourier-Stieltjes Transform which is defined in our published paper [8] as-

Differentiation Property of Fourier-Stieltjes Transform

Integral Fourier-Stieltjes Transform is defined as-

$$FS\{f(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f(t,x)\, e^{-ist}(x+y)^{-p}\, dt\, dx$$

Now, differentiating w.r.t. x, we get

$$FS\{f_x(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f_x(t,x)\, e^{-ist}(x+y)^{-p}\, dt\, dx$$

Above equation become-

$FS\{f_x(t,x)\} = p\, FS\{f(t,x)\} - k,$

Where, $\int_0^\infty y^{-p} f(t,0)\, e^{-ist}\, dt = k$ (1.1)

Also, we have-

$FS\{f_n(t,x)\} = p^n\, FS\{f(t,x)\} - p^{n-1}k$ (1.2)

Similarly, if $f(t,x)$ is differentiate w.r.t. t, we get the result as follows

$FS\{f_t(t,x)\} = is\, FS\{f(t,x)\} - k$ (1.3)

$FS\{f_n(t,x)\} = (is)^n\, FS\{f(t,x)\} - (is)^{n-1}k$ (1.4)

Using this above equation (1.1), (1.2), (1.3) & (1.4), we have provided the wave equation, Heat Flow equation in the section 2, 3 & 4 respectively. The notation and terminology is given as per A.H.Zemanian [9, 10].

2. To Solve Wave Equation

2.1. Applying Fourier-Stieltjes integral Transform to wave equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$

Solution: we have,

$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$ (2.1.1)

$FS\{f_{tt}(t,x)\} = c^2 FS\{f_{xx}(t,x)\}$

Now, applying the result we have-

$(is)^2\, FS\{f(t,x)\} - isk = c^2 D_x^2\, FS\{f(t,x)\}$

$D_x^2\, FS\{f(t,x)\} = \frac{(is)^2}{c^2} FS\{f(t,x)\} - \frac{is}{c^2}k$

$\left(D_x^2 - \frac{(is)^2}{c^2}\right) FS\{f(t,x)\} = -\frac{is}{c^2}k$ (2.1.2)

This is the ordinary differential equation w.r.t. x.

To find it's complementary function, we have

It's Auxillary equation

$\left(D_x^2 - \frac{(is)^2}{c^2}\right) = 0$

$D_x^2 = \frac{(is)^2}{c^2} \Rightarrow D_x = \pm\frac{is}{c}$

Therefore,

$\text{C.F.} = c_1 \cos\frac{s}{c}x + c_2 \sin\frac{s}{c}x$ (2.1.3)

Also,

$$P.I. = \frac{1}{f(D_x)} F(x)$$

$$= \frac{1}{\left[D_x^2 - \frac{(is)^2}{c^2}\right]}\left(-\frac{is}{c^2}k\right)$$

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$$= \left(-\frac{is}{c^2}k\right)\left\{\frac{1}{\left[D_x^2-\frac{(is)^2}{c^2}\right]}e^{0t}\right\}$$

$$= \left(-\frac{is}{c^2}k\right)\left\{\frac{1}{\left[-\frac{(is)^2}{c^2}\right]}1\right\}$$

$$= \left(-\frac{is}{c^2}k\right)\left\{-\frac{c^2}{(is)^2}\right\}$$

$$P.I. = \frac{k}{is} \quad (2.1.4)$$

Therefore, It's Complete solution is

$FS\{f(t,x)\} = C.S. = C.F. + P.I.$

$$FS\{f(t,x)\} = c_1\cos\frac{s}{c}x + c_2\sin\frac{s}{c}x + \frac{k}{is} \quad (2.1.5)$$

Where, $k = \int_0^\infty f(t,0)y^{-p}\,e^{-ist}\,dt$

2.2. Example

To illustrate the use of Fourier-Stieltjes Transform solving the certain partial differential equation, we propose to find the solution $f(t,x)$ of the equation $\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$, satisfying the boundary conditions,

The initial and boundary conditions are-

a) If $x = 0$ then $f(t,0) = 0$

b) If $x = a$ then $f(t,a) = 0$

Solution: We have the wave equation $\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$ is

$$FS\{f(t,x)\} = c_1\cos\frac{s}{c}x + c_2\sin\frac{s}{c}x + \frac{k}{is} \quad (2.2.1)$$

If $x = 0$ then

$f(t,0) = c_1 + \frac{k}{is}$

And $f(t,0) = 0$ (given)

Then

$$c_1 = -\frac{k}{is} \quad (2.2.2)$$

Now,

If $x = a$ then $f(t,a) = 0 = c_1\cos\frac{s}{c}a + c_2\sin\frac{s}{c}a + \frac{k}{is}$

Then- $\left(-\frac{k}{is}\right)\cos\frac{s}{c}a + c_2\sin\frac{s}{c}a = -\frac{k}{is}$

$c_2\sin\frac{s}{c}a = \left(\frac{k}{is}\right)\cos\frac{s}{c}a - \frac{k}{is}$

$$c_2 = \frac{k}{is}\left[\cot\frac{s}{c}a - \operatorname{cosec}\frac{s}{c}a\right] \quad (2.2.3)$$

Therefore,

$FS\{f(t,x)\} = c_1\cos\frac{s}{c}x + c_2\sin\frac{s}{c}x + \frac{k}{is}$

$FS\{f(t,x)\} = -\frac{k}{is}\cos\frac{s}{c}x + \frac{k}{is}\left[\cot\frac{s}{c}a - \operatorname{cosec}\frac{s}{c}a\right]\sin\frac{s}{c}x + \frac{k}{is}$

$= \frac{k}{is}\left[\left[\cot\frac{s}{c}a - \operatorname{cosec}\frac{s}{c}a\right]\sin\frac{s}{c}x - \cos\frac{s}{c}x + 1\right]$

$$FS\{f(t,x)\} = \frac{k}{is}\left[\left[\cot\frac{s}{c}a - \operatorname{cosec}\frac{s}{c}a\right]\sin\frac{s}{c}x - \cos\frac{s}{c}x + 1\right]$$

3. To Solve the Laplace Equation

3.1. Lapalce equation in the Cartesian form solved by using Fourier-Stieltjes Transform $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = 0$

Solution: - The Laplace equation in the Cartesian form is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = 0 \quad (3.1.1)$$

The Fourier-Stieltjes integral transformation is

$FS\{f(t,x)\} = \int_0^\infty\int_0^\infty f(t,x)\,e^{-ist}\,(x+y)^{-p}\,dt\,dx$

then- $FS\{f_{xx}(t,x)\} + FS\{f_{tt}(t,x)\} = 0$

By using the result, we have

$D_x^2\,FS\{f(t,x)\} + (is)^2\,FS\{f(t,x)\} - isk = 0$

$$(D_x^2 + (is)^2)\,FS\{f(t,x)\} = isk \quad (3.1.2)$$

This equation is ordinary differential equation.

It's roots are, $m_1 = +s, m_2 = -s$

Therefore, it's

$C.F. = c_1e^{sx} + c_2e^{-sx}$

and

$P.I. = \frac{1}{f(D_x)}f(x)$

$= \frac{1}{(D_x{}^2+(is)^2)}isk$

$= isk\left[\frac{1}{(D_x{}^2+(is)^2)}e^{0t}\right]$

$= isk\left[\frac{1}{(is)^2}\right]$

$P.I. = \frac{k}{is}$

Therefore the complete solution is

$C.S. = FS\{f(t,x)\} = C.F. + P.I.$

$$\therefore FS\{f(t,x)\} = c_1e^{sx} + c_2e^{-sx} + \frac{k}{is} \quad (3.1.3)$$

Where, $k = \int_0^\infty f(t,0)\;y^{-p}\,e^{-ist}\,dt$

3.2. Example

To illustrate the use of Fourier-Stieltjes integral transformation in solving certain partial differential equation, we proposed to find solution $f(t,x)$of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = 0$ satisfying the boundary conditions

a) If $x = 0$ then $f(t,0) = 0$

b) b) If $x = a$ then $f(t,a) = 0$

Solution:-

The Solution of PDE is

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = 0$

is given by

$$FS\{f(t,x)\} = c_1e^{sx} + c_2e^{-sx} + \frac{k}{is} \quad (3.2.1)$$

a) If $x = 0$ then $f(t,0) = 0 \;\Rightarrow\; c_1 + c_2 = -\frac{k}{is}$

b) If $x = a$ then $f(t,a) = 0 \;\Rightarrow\; c_1e^{sa} + c_2e^{-sa} = -\frac{k}{is}$

$\therefore\; c_2 = \frac{k}{is}\frac{(1-e^{sa})}{(e^{sa}-e^{-sa})}$

$\therefore\; c_1 = \frac{k}{is}\frac{(1-e^{-sa})}{(e^{-sa}-e^{sa})}$

So, the required solution is-

(3.2.1) gives -

$FS\{f(t,x)\} = \frac{k}{is}\frac{(1-e^{-sa})}{(e^{-sa}-e^{sa})}e^{sx} + \frac{k}{is}\frac{(1-e^{sa})}{(e^{sa}-e^{-sa})}e^{-sx} + \frac{k}{is}$

$$FS\{f(t,x)\} = \frac{k}{is(e^{sa}-e^{-sa})}\left[(1-e^{sa})e^{-sx} - (1-e^{-sa})e^{sx} + (e^{sa}-e^{-sa})\right] \quad (3.2.2)$$

4. To Solve one dimensional Heat flow equation

4.1. The one dimensional heat flow equation is $\frac{\partial f}{\partial t} = c^2\frac{\partial^2 f}{\partial x^2}$, where $c^2 = \frac{1}{\rho s}$. By using the Fourier-Stieltjes integral transform.

Solution: We have,

The Fourier-Stieltjes integral transformation is

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$FS\{f(t,x)\}(s,p) = \int_0^\infty \int_0^\infty f(t,x)\ e^{-ist}\ (x+y)^{-p}\ dt\, dx$

Then

$FS\{f_t(t,x)\}(s,p) = c^2 FS\{f_{xx}(t,x)\}(s,p)$

$is\ FS\{f(t,x)\}(s,p) - k = c^2\ D_x^2\ FS\{f(t,x)\}(s,p)$

$$\left(D_x^2 - \frac{is}{c^2}\right) FS\{f(t,x)\}(s,p) = -\frac{k}{c^2} \qquad (4.1.1)$$

This is the ordinary differential equation in x, So, we have

It's roots are, $m_1 = +i\frac{\sqrt{s}}{\sqrt{ic}}, m_2 = -i\frac{\sqrt{s}}{\sqrt{ic}}$

$C.F. = c_1 \cos\frac{\sqrt{s}}{\sqrt{ic}}x + c_2 \sin\frac{\sqrt{s}}{\sqrt{ic}}x$

and

$P.I. = \frac{1}{f(D_x)} f(x)$

$= \frac{1}{\left(D_x^2 - \frac{is}{c^2}\right)}\left(-\frac{k}{c^2}\right)$

$= \left(-\frac{k}{c^2}\right)\left[\frac{1}{\left(D_x^2 - \frac{is}{c^2}\right)} e^{0t}\right]$

$= \left(-\frac{k}{c^2}\right)\left[\frac{1}{\left(-\frac{is}{c^2}\right)} e^{0t}\right]$

$P.I. = \frac{k}{is}$

Here, the complete solution is

$FS\{f(t,x)\}(s,p) = c_1 \cos\frac{\sqrt{s}}{\sqrt{ic}}x + c_2 \sin\frac{\sqrt{s}}{\sqrt{ic}}x + \frac{k}{is}$

Now using this above equation, we can solve an example of it as given.

4.2. Example

To illustrate the use of Fourier-Stieltjes integral transform in solving the $\frac{\partial f}{\partial x} = c^2\frac{\partial^2 f}{\partial t^2}$ certain partial differential equation, we proposed to find the solution satisfying the boundary conditions

a) If $x = 0$ then $f(t,0) = 0$

b) If $x = a$ then $f(t,a) = 0$

Solution: The partial differential equation is

$\frac{\partial f}{\partial x} = c^2\frac{\partial^2 f}{\partial t^2}$

It's solution is

$$FS\{f(t,x)\}(s,p) = c_1 \cos\frac{\sqrt{s}}{\sqrt{ic}}x + c_2 \sin\frac{\sqrt{s}}{\sqrt{ic}}x + \frac{k}{is} \qquad (4.2.1)$$

a) If $x = 0$ then $f(t,0) = 0 \quad \Rightarrow c_1 = -\frac{k}{is}$ (4.2.2)

b) If $x = a$ then $f(t,a) = 0$

$$\Rightarrow -\frac{k}{is}\cos\frac{\sqrt{s}}{\sqrt{ic}}a + c_2 \sin\frac{\sqrt{s}}{\sqrt{ic}}a + \frac{k}{is} = 0$$

$$c_2 = \frac{k}{is}\left[\cot\frac{\sqrt{s}}{\sqrt{ic}}a - \operatorname{cosec}\frac{\sqrt{s}}{\sqrt{ic}}a\right] \qquad (4.2.3)$$

Putting the values of c_1 and c_2 in equation (4.2.1) we get-

$FS\{f(t,x)\}(s,p)$

$$= \frac{k}{is}\left[\left[\cot\frac{\sqrt{s}}{\sqrt{ic}}a - \operatorname{cosec}\frac{\sqrt{s}}{\sqrt{ic}}a\right]\sin\frac{\sqrt{s}}{\sqrt{ic}}x - \cos\frac{\sqrt{s}}{\sqrt{ic}}x + 1\right]$$

5. Conclusion

In the present paper we have solved the some partial differential equations and have proved wave equation, Heat flow equation and Laplace equation using the Fourier-Stieltjes Transform.

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