A Four Step Wavelet Galerkin Method for Parabolic and Hyperbolic Problems

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Abstract: ***In this paper a four step wavelet based higher order Taylor Galerkin methods is given in this study. The method is fourth order accurate in time and $O(2^{-jp})$ give the accuracy in space.In contrast with the Taylor-Galerkin method the present four step finite element method lacks any new high order derivatives andis suitable for solving non-linear problems. The concisely supported orthogonal wavelet bases developed by Daubechies are used in Galerkin method. The behavior of the given method is applicable both parabolic and hyperbolic partial differential equation.***

Keywords: Four step, wavelets, Galerkin method, Taylor series

1. Introduction

Both theoretical and computational way is used to study numerical solution of partial differential equations by implementation of wavelet due to its attractive feature: orthogonality, arbitrary regularity, good localization. Spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization, but poor spectral localization. The advantage of both spectral and finite element basis are combined by wavelet base. In wavelet applications to the solution of partial differential equations the most frequently used wavelets are those with compact support introduced by Daubechies [1].Exploration of the utility of Daubechies wavelet to solve partial differential equations has been undertaken by numerous researches[2-5]. Jameson's adaptive finite difference method can be derived by wavelets. The accuracy achieved by using the higher order spatial discretization is partially lost in the conventional numerical approach to transient problems due to usage of low order time discretization schemes. Spatial approximation usually precedes temporal discretization. However, the reversed order of discretization can lead to better time accurate schemes with improved stability properties. The fundamental idea behind the taylor-Galerkin[10] is to incorporate more analytical information into numerical scheme is the most direct and natural way. Taylor-Galerkin approach is the substitution of space derivatives for the time derivatives in the Taylor series as used in the derivation of the Lax-Wendroff method[11], the only modification being that the process is carried out to fourth order. However, its applications are mainly to hyperbolic problems and some convection diffusion equations because too many terms are introduced in the fourth-order time derivative term, specially for non-linear multidimensional equations, and treatment of boundary integrations arising from high-order time derivative terms is too complex.

A four-step finite-element method based on a taylor series expansion in time is introduced is having simple expression as compared to proposed wavelet Taylor-Galerkin method[13].and the wavelet multilayer Taylor-Galerkin method[14]. In [13] we inquired the possibility of applying the new wavelet Taylor- Galerkin to an important application problem of the Korteweg-de Vires equation. A combination of such a time marching scheme with wavelet approximation in space can lead to simple higher-order space and time accurate numerical methods. In this paper we develop a four-step wavelet Galerkin method (F-WGM). A class of linear problems such as heat equation, the convective transport problem and non-linear Burgers equations in one dimension and heat equation in two dimensions can be handled with four step Galerkin method. This scheme hold good accuracy and stability properties of the Taylor-Galerkin method without any need to calculate higher order derivative terms.

2. Wavelet Preliminaries

Daubechies [1] has defined the class of compactly supported wavelets. Briefly, let Ψ be a solution of the scaling relation

$$\Psi(x) = \sum_k a_k \Psi(2x-k)$$

Where the a_k are a collection of coefficients that categorize the specific wavelet basis. The expression Ψ is called the scaling function. The associated wavelet function $\emptyset$ is defined by the equation

$$\emptyset(x) = \sum_k (-1)^k a_{1-k} \Psi(2x-k)$$

Normalization $\int \Psi \, dx = 1$ of the scaling function leads to the condition $\sum_k a_k = 2$

The translates of Ψ are required to be orthonormal,i.e.

$$\int \Psi(x-k)\Psi(x-m)\, dx = \delta_{k,m}$$

The scaling relation implies the condition $\sum_{k=0}^{N-1} a_k\, a_{k-2m} = \delta_{0,m}$

Where N is the order of wavelet . For coefficients verifying the above two conditions, the functions consisting of translates and dilations of the wavelet function$\emptyset(2^j x-k)$ form a complete orthogonal basis for square integrable functions on the real line $L^2(R)$.

If only a finite number of thea_k are non- zero, Ψ will have compact support. Since

$$\int \Psi(x)\emptyset(x-m)dx = \sum_k (-1)^k a_{1-k} a_{k-2m} = 0$$

The translates of the scaling function and wavelet define orthogonal subspaces

$u_j = \{2^{j/2}\Psi(2^j x - k; m = \cdots \ldots, -1,0,1 \ldots \ldots \ldots \ldots\}$

$v_j = \{2^{j/2}\emptyset(2^j x - k); m = \cdots \ldots, -1,0,1 \ldots \ldots ..\}$

The $relation\ u_{j+1} = u_j + v_j$

$Implies\ the\ Mallat\ transform$[1] $u_0 \subset u_1 \subset u_2 \subset \ldots\ldots\ldots\ldots\ldots\ldots\ldots \subset u_{j+1}$

$v_{j+1} = v_0 \oplus v_1 \oplus v_2 \oplus \ldots\ldots\ldots\ldots \oplus v_j$

Smooth scaling functions arise as a consrquence of the degree of approximation of the translates.The result that the polynomials $1, x, \ldots \ldots \ldots \ldots, x^{p-1}$ are expressed as linear combinations of the of $\Psi(x-k)$ is implied by the conditions $\sum_k (-1)^{-k} k^m a_k = 0$ for m=0,1,……..,p-1. The following are equivalent results

(a) $\{1, x, \ldots\ldots\ldots, x^{p-1}\}$ are linear combination of Ψ(x- k)

(b) $\|f - \sum c_k^j \Psi(2^j x - k)\| \leq C2^{-jp} \|f^p\|$, ,where $c_k^j = \int f(x)\Psi(2^j x - k)dx$

(c) $\int f(x)\emptyset(2^j x)dx \leq c2^{-jp}$

(d) $\int x^m \emptyset(x)\, dx = 0\ for\ m = 0, 1, \ldots \ldots \ldots \ldots \ldots, p-1$

For the Daubchies scaling/ wavelet function DN, we have p = N/2 where N is the order of wavelet. Fig. 1 shows an example of compactly supported scaling function and its associated fundamental wavelet function. For arbitrarily large even N there is Daubechies example of a fundamental scaling function defining a wavelet family with support in the interval [0, N-1][1]. For any $j, l \in Z$ we define the 1-periodic scaling function

$$\Psi_{j,l}(x) = \sum_{n=-\infty}^{\infty} \Psi_{j,l}(x+n)$$

$$= 2^{j/2} \sum_{n=-\infty}^{\infty} \Psi(2^j(x+n) - l), x \epsilon R$$

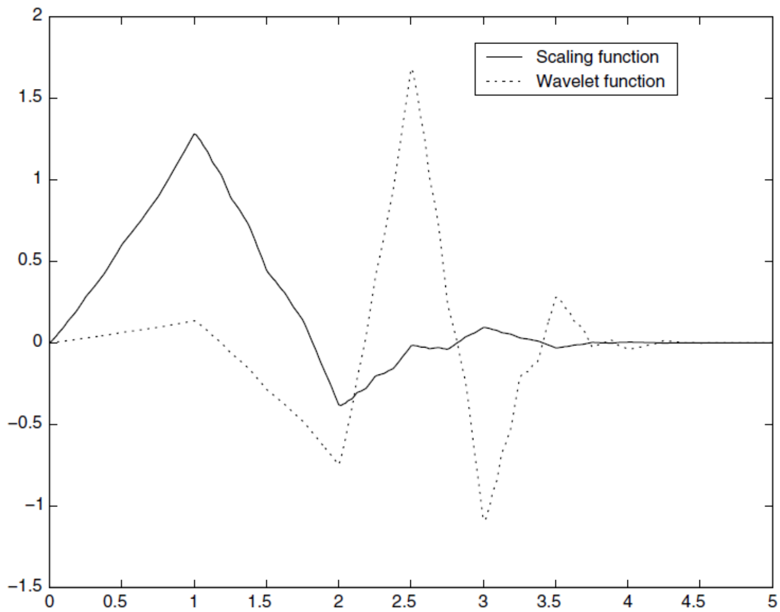

Figure1: Daubechies scaling and wavelet functions for N=6 with support on [0,5]

And the 1-periodic wavelet

$$\emptyset_{j,l}(x) = \sum_{n=-\infty}^{\infty} \Psi_{j,l}(x+n) = 2^{\frac{j}{2}} \sum_{n=-\infty}^{\infty} \Psi(2^j(x+n) - l, x \in R$$

For a PDE of the form

$$F(y, y_t, \ldots \ldots \ldots, y_x, y_{xx}, \ldots) = 0 \qquad (1)$$

The wavelet approximation is of the form

$$y_j(x,t) = \sum_{k,l=0}^{2^j-1} c_{j,k}(t)\Psi_{j,k}(x) \qquad (2)$$

where $c_{j,k}$ is an unknown coefficient of the scaling function expansion. Since we assume periodic boundary conditions, there is a periodic wrap around in(x, y) and let the period scale with the number of terms in the expansion. To determine the coefficient of expansion(2) we substitute(2) in to equation(1) and again project the resulting expression onto subspaceu_j.The projection requires $c_{j,k}$ to satisfy the equation

$$\int_{-\infty}^{\infty} \Psi_{j,m}(x) F(y_j, y_{j_t}, y_{j_x}, y_{j_{xx}}, \ldots \ldots \ldots ..)dx = 0$$

To calculate this expression we must know the coefficients of the form

$$\Lambda(l_1, l_2, \ldots\ l_n, d_1, d_2 \ldots d_n) = \Lambda_{l_1,l_2,\ldots\ldots l_n}^{d_1,d_2,\ldots\ldots d_n} = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Psi_{l_i}^{d_i}(y)dy = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Psi_{l_i}^{d_i}(y)dy$$

Since the scaling function used to define compact wavelets has a limited number of limited number of derivatives the numerical calculation of these expressions is often unstable or inaccurate. A special algorithm to calculate the connection coefficients has been devised by Latto et al.[15].

2.2 Multivariate wavelets

The easy way to obtain multivariate wavelets is to employ anisotropic or isotropic tensor products

MRA-d The multivariate wavelets are defined by

$$\emptyset_{j,l}(x) = \emptyset_{j_1,l_1}(x_1), \ldots \ldots \ldots, \emptyset_{j_d,l_d}(x_d),$$

$j = j_1, \ldots \ldots \ldots, j_d\ x, l$ analogous

(MRA) Anisotropy is avoided. The scaling functions are simply the tensor product of univariate scaling functions. A two dimensional MRA can be constructed from the following decomposition

$$u_j = u_j \otimes u_j = (u_{j-1} \oplus v_{j-1}) \otimes (u_{j-1} \oplus v_{j-1})$$
$$(v_{j-1} \otimes v_{j-1}) \oplus (v_{j-1} \otimes u_{j-1}) \oplus (u_{j-1} \otimes v_{j-1}) \oplus u_{j-1} \otimes u_{j-1}$$
$$= v_{j-1} \oplus u_{j-1}$$

Then we have $u_J = v_{J-1} \oplus \ldots\ldots\ldots\ldots\ldots \oplus v_0 \oplus u_0$ and the wavelet basis is given by

$$\{\emptyset_{j,k} \otimes \emptyset_{j,l} \emptyset_{j,k} \otimes \emptyset_{j,l} \emptyset_{j,k} \otimes \emptyset_{j,l}\}_{k,l \in Z, 0 \leq j \leq J-1}$$
$$\cup \{\psi_{0,k} \otimes \psi_{0,l}\}_{k,l \in Z}$$

We have used this MRA approach in our two dimensional problem.

3. Four-step wavelet Galerkin method

Before introducing the Four step wavelet Galerkin method(F-WGM),it is necessary to give a brief statement of the three-step Lax-Wendroff wavelet Galerkin method (L-WGM). Consider the equation

$$v_t = \mathcal{L}v + \mathcal{N}f(v) \quad (3)$$

With the initial condition

$$v(x,0) = v_0(x), 0 \leq x \leq 1 \quad (4)$$

and with suitable boundary conditions. We explicity separate equation (3)into a linear part $\mathcal{L}$v and non-linear part$\mathcal{N}f(v)$, where operations $\mathcal{L}$ and$\mathcal{N}$ are constant- coefficient differential operation that do not depend upon time t. The function $f(v)$ is non-linear. Performing a Taylor series expansion in time, we have

$$v(t+\Delta t) = v(t) + \Delta t \frac{\partial v(t)}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 v(t)}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 v(t)}{\partial t^3} + \frac{\Delta t^4}{24} \frac{\partial^4 v(t)}{\partial t^4} + o(t^5) \quad (5)$$

But approximating equation (5)to second order accuracy, the formulation of the two-step method can be derived as follow

$$v(t+\Delta t) = v(t) + \frac{\Delta t}{2} \frac{\partial v(t)}{\partial t}$$
$$v(t+\Delta t) = v(t) + \Delta t \frac{\partial v(t+\Delta t/2)}{\partial t} \quad (6)$$

Spatial discretization of equations(6) can be performed using the WGM. We will know this method L-WGM. We use approximation of equation(5)to third order accuracy, the formula of the third-step method can be derived.

$$v\left(t+\frac{\Delta t}{3}\right) = v(t) + \frac{\Delta t}{3} \frac{\partial v(t)}{\partial t}$$
$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2} \frac{\partial v(t+\Delta t/3)}{\partial t}$$
$$v(t+\Delta t) = v(t) + \Delta t \frac{\partial v(t+\Delta t/2)}{\partial t} \quad (7)$$

Spatial discretization of equation(7) can be performed using the WGM. In contrast with the Taylor-Galerkin method.

Now we introduce the Four-step wavelet Galerkin method. By approximating equation(5) up to fourth-order accuracy, the formulation of the four-step method can be written as follow

$$v\left(t+\frac{\Delta t}{4}\right) = v(t) + \frac{\Delta t}{4} \frac{\partial v(t)}{\partial t}$$
$$v\left(t+\frac{\Delta t}{3}\right) = v(t) + \frac{\Delta t}{3} \frac{\partial v(t+\Delta t/4)}{\partial t}$$
$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2} \frac{\partial v(t+\Delta t/3)}{\partial t}$$
$$v(t+\Delta t) = v(t) + \Delta t \frac{\partial v(t+\Delta t/2)}{\partial t} \quad (8)$$

Spatial discretization of equation(8) can be performed using the WGM. by use Taylor-Galerkin method, the T-WGM does not contain any new higher-order spatial derivatives. It can be easily used to solve non-linear multidimensional flows.

3.1 F-WGM for the heat equation

The heat equation

$$\mathcal{L} = w(\partial^2/\partial_x^2) \text{ and } v_t = w v_{xx} + f(x) \quad (9)$$

Where w is the positive constant .Let us first keep the spatial variable x continupus and discretize the time by the four step method.

$$v\left(t+\frac{\Delta t}{4}\right) = v(t) + \frac{\Delta t}{4}[w v_{xx}^n + f(x)]$$
$$v\left(t+\frac{\Delta t}{3}\right) = v(t) + \frac{\Delta t}{3}[w v_{xx}^{n+1/4} + f(x)]$$
$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2}[w v_{xx}^{n+1/3} + f(x)] \; v(t+\Delta t) = v(t) + \Delta t[w v_{xx}^{n+1/2} + f(x)] \quad (10)$$

In equation(10) space variable is continuous. Futher we are discrtizing variable x by wavelet Galerkin method (WEM).The Galerkin discretization scheme gives $d_v^{n+1/4} = d_v^n + \frac{\Delta t}{4}(w D^{(2)} d_v^n + d_f)$

$$d_v^{n+1/3} = d_v^n + \frac{\Delta t}{3}(w D^{(2)} d_v^{n+1/4} + d_f)$$
$$d_v^{n+1/2} = d_v^n + \frac{\Delta t}{2}(w D^{(2)} d_v^{n+1/3} + d_f)$$
$$d_v^{n+1} = d_v^n + \Delta t(w D^{(2)} d_v^{n+1/2} + d_f) \quad (11)$$

where d_v denotes the vector of the scaling function coefficients corresponding to v and d_f denotes the vector of the scaling function coefficients corresponding to f. We will refer to the matrix$D^{(d)}$ as the differentiation matrix of order d.This matrix is derived in [17]. If the function to be differentiated is periodic with period L, then $D_1^{(d)} = D^{(d)}/L^d$.

3.2 F – WGM for the convection equation

The convection equation

$$\mathcal{L} = a(\partial/\partial x), \text{ therefore } v_t = a v_x \quad (12)$$

Where a is a positive constant. Then the time discretization by the Four- step method is

$$v\left(t+\frac{\Delta t}{4}\right) = v(t) + \frac{\Delta t}{4}(w v_x^n)$$
$$v\left(t+\frac{\Delta t}{3}\right) = v(t) + \frac{\Delta t}{3}(w v_x^{n+1/4})$$
$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2}(w v_x^{n+1/3})$$
$$v(t+\Delta t) = v(t) + \Delta t(w v_x^{n+1/2}) \quad (13)$$

Spartial discretization by WGM gives

$$d_v^{n+1/4} = d_v^n + \frac{\Delta t}{4}(w D_1^{(1)} d_v^n)$$
$$d_v^{n+1/3} = d_v^n + \frac{\Delta t}{3}(w D_1^{(1)} d_v^{n+1/4})$$
$$d_v^{n+1/2} = d_v^n + \frac{\Delta t}{2}(w D_1^{(1)} d_v^{n+1/3})$$
$$d_v^{n+1} = d_v^n + \Delta t(w D_1^{(1)} d_v^{n+1/2}) \quad (14)$$

3.3 F-WGM for the Burgers equation

The equation

$$\mathcal{L}=v\,(\partial/\partial x)\text{and } v_t + vv_x = wv_{xx} \quad (15)$$

Then time discretization by the four-step method is

$$v\left(t+\frac{\Delta t}{4}\right) = v(t) + \frac{\Delta t}{4}(-v^n v_x^n + wv_{xx}^n)$$

$$v\left(t+\frac{\Delta t}{3}\right) = v(t) + \frac{\Delta t}{3}(-v^{n+1/4} v_x^{n+1/4} + wv_{xx}^{n+1/4})$$

$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2}(-v^{n+1/3} v_x^{n+1/3} + wv_{xx}^{n+1/3})\; v(t+\Delta t=vt+\Delta t(-vn+12vxn+12+wvxxn+12) \quad (16)$$

Spatial discretization by WGM give

$$d_v^{n+1/4} = d_v^n + \frac{\Delta t}{4}[-2^{3j/2}\sum_k\sum_m (c_k^j)^n (c_m^j)^n \Lambda_{lkm}^{001} + w(2^j)^2 \sum_k (c_k^j)^n \Lambda_{lk}^{02}]$$

$$d_v^{n+1/3} = d_v^n + \frac{\Delta t}{3}[-2^{3j/2}\sum_k\sum_m (c_k^j)^{n+1/4} (c_m^j)^{n+1/4} \Lambda_{lkm}^{001} + w(2^j)^2 \sum_k (c_k^j)^{n+1/4} \Lambda_{lk}^{02}]$$

$$d_v^{n+1/2} = d_v^n + \frac{\Delta t}{2}[-2^{3j/2}\sum_k\sum_m (c_k^j)^{n+1/3} (c_m^j)^{n+1/3} \Lambda_{lkm}^{001} + w(2^j)^2 \sum_k (c_k^j)^{n+1/3} \Lambda_{lk}^{02}]$$

$$d_v^{n+1} = d_v^n + \Delta t\left[-2^{3j/2}\sum_k\sum_m (c_k^j)^{n+1/2} (c_m^j)^{n+1/2} \Lambda_{lkm}^{001} + w2j2kckjn+12\Lambda lk02 \right. \quad (17)$$

4. Theoretical Stability of the Linearized Schemes

By the use of concept of asymptotic stability of a numerical method as defined in [18] for a discrete problem of the form $\frac{dV}{dt} = LV$ where L is assumed to be a diagonalizable matrix. Stability of time discretization can be determine by the most crucial property of L. We use the term stability in its ODE context if the spatial discretization is presumed.

DEFINITION: The region of absolute stability of a numerical method is defined for the scalar model problem $\frac{dV}{dt} = \lambda V$ to be the set of all $\lambda\Delta t$ such that $\|V^n\|$ is bounded as $t \to \infty$

Numerical method is asymptotically stable for particular problem if, for sufficiently small $\Delta t > 0$, the product of Δt and every eigenvalue of L lies within the region of absolute stability.

5. Result of Numerical experiments

Periodic boundary conditions and initial conditions$v_0(x)$ solve PDEs. One dimensional a heat equation, a linear convection equation and a non-linear Burgers equation are considered. All the results are presented using Daubechies D6 scaling function. The error produced by the schemes was measured against the value of the analytical solution v_e by theL_∞ norm calculated as $\|v\|_{L_\infty} = max_{k=0,1,\ldots,2^j-1}\left|v(\frac{k}{2^j})\right|$

Table 1: Accuracy of results given by the L_∞ norm

j	Δt	Lax-Wendroff	WGM	L-WGM	F-WGM
4	0.0005(0.01)	3.167×10^{-5}	3.138×10^{-7}	2.2772×10^{-10}	1.5208×10^{-10}
6	0.0005(0.01)	3.131×10^{-5}	3.138×10^{-7}	6.4610×10^{-11}	1.0188×10^{-12}
7	0.0005(0.01)	3.131×10^{-5}	3.138×10^{-7}	6.4610×10^{-11}	1.0188×10^{-12}

5.1 Heat equation

We have tested F-WGM on a heat equation with $v_0(x) = 0 \; and\; f(x) = \sin(2\pi x)$. Errors in the L_∞ norm of the solution obtained by Lax-Wendroff, WGM using Euler time stepping,L-WGM and F-WGM are compared in table 1. In the table the notation $x(y)$ means that the target time y is reached by time marching with a step size of x.

5.2 Convection equation

The accuracy of the introduced F-WGM for hyperbolic problems has been verified numerically on the classical test problem of convection with a Gaussian profile. We assume that the solution is periodic with some large period, say 4. The solutions obtained by WGM using Euler time stepping, L-WGM and F-WGM are compared in table 2. For stability analysis of L-WGM,L_j is the matrix define by

$$L_j = aD^{(1)} + \frac{a^2\Delta t}{2}D^{(2)}$$

$$L_j = aD^{(1)} + \frac{a^2\Delta t}{2}D^{(2)} + \frac{a^3\Delta t}{6}D^{(3)}$$

$$L_j = aD^{(1)} + \frac{a^2\Delta t}{2}D^{(2)} + \frac{a^4\Delta t}{6}D^{(3)} + \frac{a^4\Delta t}{24}D^{(4)}$$

For j=4 we have found by stability analysis that $\Delta t \le 0.00835$. Regions of absolute stability for F-WGM

5.3 Burgers equation

The analytical solution to equation (15) with the initial condition $v_0(x)$ after using the Cole-Hopf transformation is given by

$$v(x,t) = \frac{\int_{-\infty}^{\infty}(x-\xi)/t\,\exp\left[-\frac{(x-\xi)^2}{4wt}\right]\exp(-(2w)^{-1}\int_0^\xi v_0(\eta)d\eta]d\xi}{\int_{-\infty}^{\infty}exp\,[-\frac{(x-\xi)^2}{4wt}]\exp(-(2w)^{-1}\int_0^\xi v_0(\eta)d\eta]d\xi} \quad (18)$$

We have integrated equation(18) numerically using Gauss-Hermite equation quadrature in order to determine the accuracy of our method.

Table 2: Accuracy of results given by the L_∞ norm

j	Δt	WGM	L-WGM	F-WGM
4	0.01(0.5)	0.08601	0.0014	0.0011
6	0.001(0.5)	0.00201	2.041×10^{-4}	3.94×10^{-6}
6	0.0001(0.5)	0.0001	3.6771×10^{-6}	3.70×10^{-6}

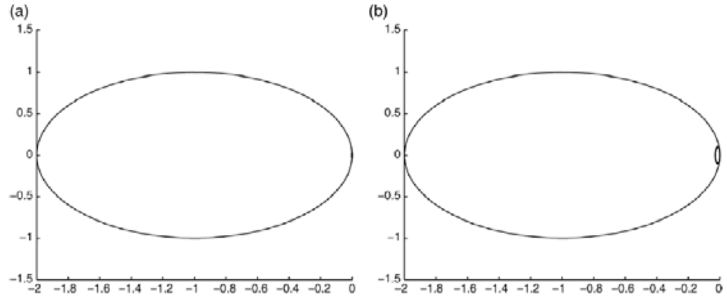

Figure 2: Absolute stability region for F-WGM and the product of Δt and the eigen values of L_j where $\Delta t = 0.001$ (a) $j = 4$ $(b) j = 6$

6. Extension to Multidimensional Problems

The schemes developed in the occurring before sections for a one-dimensional problems have demonstrated the value of F-WGM compared with WGM and L-WGM. After all, to be of practical interest F-WGM should also be applicable to multidimensional problems.

Case1. Heat conduction problem

$\frac{\partial v}{\partial t} = F(x,y)$ (19)

Where $F(x,y) = w\Delta v + f(x,y)$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Is a two-dimensional Laplacian operator and v is an unknown scalar function defined on abounded square domain with a periodic boundary condition. Time discretization by the four-step method is given by

$$v\left(t+\frac{\Delta t}{4}\right) = v(t) + \frac{\Delta t}{4}[wv_{xx}^n + wv_{yy}^n + f(x,y)]$$

$$v\left(t+\frac{\Delta t}{3}\right) = v(t) + \frac{\Delta t}{3}[wv_{xx}^{n+1/4} + wv_{yy}^{n+1/4} + f(x,y)]$$

$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2}[wv_{xx}^{n+1/3} + wv_{yy}^{n+1/3} + f(x,y)]$$

$$v(t+\Delta t) = v(t) + \Delta t[wv_{xx}^{n+1/2} + wv_{yy}^{n+1/2} + f(x,y)] \quad (20)$$

Spatial discretization of equations (20) is performed by WGM. The Galerkin discretization schemes gives

$$(c_{p,q}^j)^{n+1/4} = (c_{p,q}^j)^n + \frac{(2^j)^2\Delta t}{4}[w\sum_k \left(c_{p,q}^j\right)^n \Lambda_{pk}^{02} + w\sum_l \left(c_{p,l}^j\right)^n \Lambda_{ql}^{02} + f_{p,q}^j]$$

$$(c_{p,q}^j)^{n+1/3} = (c_{p,q}^j)^n + \frac{(2^j)^2\Delta t}{3}[w\sum_k \left(c_{p,q}^j\right)^{n+1/4} \Lambda_{p,k}^{02} + w\sum_l (c_{p,l}^j)^{n+1/4}\Lambda_{q,l}^{02} + f_{p,q}^j]$$

$$(c_{p,q}^j)^{n+1/2} = (c_{p,q}^j)^n + \frac{(2^j)^2\Delta t}{2}[w\sum_k (c_{p,q}^j)^{n+1/3}\Lambda_{p,k}^{02} + w\sum_l (c_{p,l}^j)^{n+1/3}\Lambda_{q,l}^{02} + f_{p,q}^j]$$

$(c_{p,q}^j)^{n+1} = (c_{p,q}^j)^n + (2^j)^2\Delta t[w\sum_k \left(c_{p,q}^j\right)^{n+1/2}\Lambda_{p,k}^{02} +$ +wlcp,ljn+12Λq,l02+fp,qj] (21)

Where $c_{k,l}^j$ is an unknown coefficient of scaling function expansion, $f_{p,q}^j$ is the scaling function coefficient of wavelet expansion corresponding to$f$ in two dimensions and $p,q = 0,1,\dots\dots,2^{j-1}$. Let us consider equation (19) on a doubly periodic square, with vanishing v(x,y,0)=0 initial condition and the following forcing term.

$$f(x,y) = \sin(2\pi x)\cos(2\pi y)$$

the exact analytical solution is given by

$$v(x,y,t) = \frac{1}{8\pi^2 w}(1 - e^{-8\pi^2 wt})\sin(2\pi x)\cos(2\pi y)$$

Figure3(a) shows the exact solution and figure3(b) shows the wavelet solution using Daubechies D6 scaling function on a mesh with n $=2^5$ unknown in each direction at time t=0.005.

CASE 2. Let us consider a Gaussian hill translating with a uniform velocity a and spreading isotropically with a diffusivity w

$$v_t = -a\nabla v + w\nabla^2 v \quad (22)$$

Time discretization will be same as for the one-dimensional case in the case 1 for the F-WGM scheme. The initial distribution is given in figure4 (a) and the equations are integrated until time t=0.5 is reached.

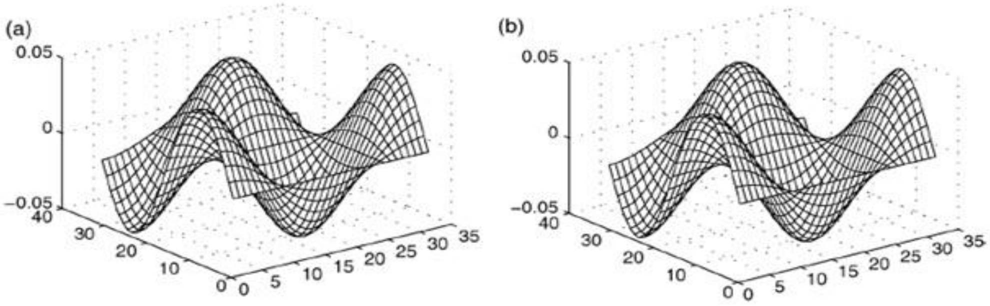

Figure 3: (a) Analytical solution. (b) Wavelet solution using the four-step method.

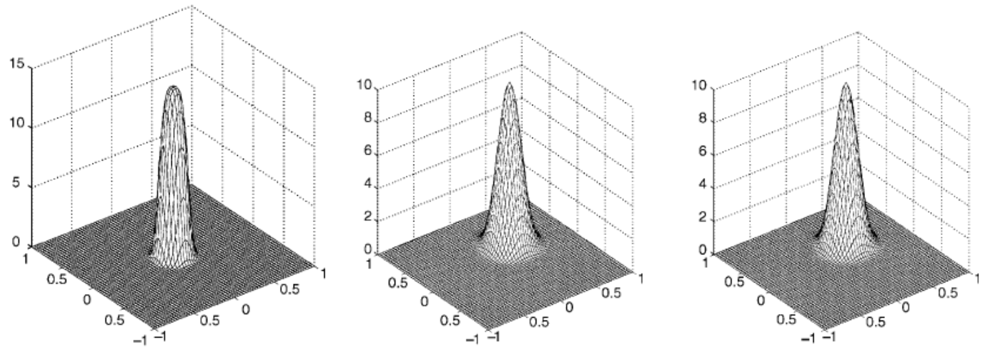

Figure 4: Initial distribution of hill (left) and solution at t=0.5 using W-FTG1 and using W-FTG2 in (right)

7. Conclusion

In the four-step wavelet Galerkin method the priority of time discretization over space discretization in conjunction with wavelet bases for expressing spatial terms renders the proposed sechemes strong and makes them space and time accurate. The Fourth-step wavelet Galerkin method holds the fourth-order accuracy and stability properties of the Taylor-Galerkin method. Since the present method is suitable for non-linear problems. Moreover,the method can be directly implemented to three-dimensional problems. The numerical results illustrates that the present method is computationally perfect.

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