On Non- Homogeneous Biquadratic Diophantine Equation $7(x^2+y^2) - 13xy = 31z^4$

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Abstract: ***Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $7(x^2 + y^2) - 13xy = 31z^4$ are determined. Introducing the linear transformations $x = u + v$, $y = u - v$, $u \neq v \neq 0$ in $7(x^2 + y^2) - 13xy = 31z^4$, it reduces to $u^2 + 27v^2 = 31z^4$. We are solved the above equation through various choices and the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed***

Keywords: Quadratic, non-homogenous, integer solutions, special numbers, polygonal, and pyramidal numbers

2010 Mathematics Subject Classification: 11D09

Notations used

$T_{m,n}$: Polygonal number of rank n with sides m.

p_n^m : Pyramidal number of rank m with side n

G_n: Gnomonic number of rank n

$f_{4,3}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Triangle

$f_{4,4}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Square

$f_{4,5}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Pentagon

$f_{4,6}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Hexagon

$f_{4,7}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Heptagon

$f_{4,8}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Octagon.

1. Introduction

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1-12]. In 2014, Jayakumar. P, Sangeetha. K, [12] have published a paper in finding the integer solutions of the homogeneous Biquaratic Diophantine equation $(x^3 - y^3) z = (W^2 - P^2)R^4$. In 2015, Jayakumar. P, Meena.J [14, 15] published two papers in finding integer solutions of the homogeneous Biquaratic Diophantine equation $(x^4 - y^4) = 26 (z^2 - w^2) R^2$ and $(x^4 - y^4) = 40 (z^2 - w^2) R^2$.Inspired by these, In this work, we are observed another interesting five different methods of the non-zero integral solutions of the non- homogeneous biquadratic Diophantine equation with three unknowns$7(x^2 + y^2) -13xy = 31z^4$. Further, some elegant properties among the special numbers and the solutions are observed.

2. Description of Method

Consider the bi - quadratic Diophantine equation

$$7(x^2 + y^2) - 13xy = 31z^4 \qquad (1)$$

We introduce the linear transformations

$$x = u + v,\ y = u - v,\ u \neq v \neq 0 \qquad (2)$$

Using (2) in (1), it gives to $u^2 + 27v^2 = 31z^4$ (3)

We solved (3) through various choices and the different methods of solutions of (1) are obtained as follows.

2.1 Method: I

Consider (3) as $u^2 + 27v^2 = 27z^4 + 4z^4$
and write it as in the form of ratio

$$\frac{u - 2z^2}{27(z^2 - v)} = \frac{z^2 + v}{u + 2z^2} = \frac{a}{b},\ b \neq 0 \qquad (4)$$

(4) is equivalent to the system of equations

$$bu - 27av + (2b - 27a)z^2 = 0 \qquad (5)$$

$$- au - bv + (b + 2a)z^2 = 0 \qquad (6)$$

By the cross multiplication method, the above equations yields as

$$\left.\begin{array}{l} u = 54a^2 - 2b^2 + 54ab \\ z^2 = 27a^2 + b^2 \end{array}\right\} v = -27a^2 + b^2 + 4ab \qquad (7)$$

Putting $a = 2pq$, $b = 27p^2 - q^2$ in (7) and using (2), it gives us

$x = x (p, q) = - 729p^4 - q^4 + 162 p^2q^2 + 3132p^3q - 116 pq^3$

$y = y (p, q) = - 2187p^4 - 3q^4 + 486p^2q^2 - 2700 p^3q + 100pq^3$

$z = 27p^2 + q^2$,

This gives us the non- zero different integer values to (1)

Observations:-

1. $x(p,1)+8748 f^{p}_{4,4} - 12096 p^{5}_{p} + 2241T_{4,p} - G_{617p} = 0$.
2. $x(1,p) +24 f^{p}_{4,3} +220 p^{5}_{p} -283T_{4,p} - G_{1569p} \equiv 0 (Mod 2)$
3. $y(1,p)+72 f^{p}_{4,5} -6T_{4,p^2} -254 p^{5}_{p} -389T_{4,p}+G_{1347p} \equiv 0 (Mod 2)$
4. $y(p, 1)+13122 f^{p}_{4,6} -7722 p^{5}_{p} - 999T_{4,p} - G_{50p} +2 = 0$
5. $x(1, p)-y(1, p)- 48 f^{p}_{4,8} +10 T_{4,p^2} +496 p^{5}_{p} +88T_{4,p} - G_{2920p} \equiv 1 (Mod 2)$
6. $x(p,1) + y(p,1) +69984 f^{p}_{4.7} -11664 T_{4,p^2} -8251 p^{5}_{p} +20196T_{4,p} + G_{2924p} \equiv 0 (Mod 5)$
7. $\frac{1}{3}z(2,0)$ is a perfect square.
8. $\frac{1}{9}z(1,0)$ is a cubic integer.
9. z(1, 6) is a woodall number.
10. z(1,10) is a jacobsthal lucas number.

2.2 Method: II

In place of (4), let us take the form of ratio as

$$\frac{u+2z^2}{z^2-v} = \frac{27(z^2+v)}{u-2z^2} = \frac{a}{b}, b \neq 0 \tag{8}$$

The following techniques is similar as in the method - I,
The relating integer values to (1) are found as

$x = x (p, q) =-58320p^4-80q^4 +4332p^2q^2 +2700p^3q - 100pq^3$
$y = y (p, q) =-20412p^4-28q^4 +151p^2q^2 + 3132p^3q -116pq^3$
$z = 27p^2 + q^2$

Observations:-

1. $x (p, 1) + 349920 f^{p}_{4,6} - 355320 p^{5}_{p} +56688T_{4,p} +G_{50p} \equiv 0 (Mod 3)$
2. $x (1, p) +1920 f^{p}_{4,5} -160 T_{4,p^2} -1400 p^{5}_{p} - 4352T_{4,p} - G_{1430p} \equiv 29 (Mod 2011)$
3. $y(1, p) +672 f^{p}_{4,8} - 140 T_{4,p^2} - 664 p^{5}_{p} +164 T_{4,p} - G_{1510p} \equiv 31 (Mod 101)$
4. $x(1, p) - y(1, p) +1248 f^{p}_{4.7} -208 T_{4,p^2} -1488 p^{5}_{p} - 2436T_{4,p} + G_{268p} \equiv 7 (Mod 12634)$
5. $y(p,1)+489888 f^{p}_{4,3} - 251208 p^{5}_{p} - 100444T_{4,p} - G_{61178p} \equiv 0 (Mod 3)$
6. $x (p,1)+ y(p,1) +472392 f^{p}_{4,6} -484056 p^{5}_{p} +78716T_{4,p} + G_{108p} \equiv 7 (Mod 17)$
7. $\frac{1}{8}z(4,0)$ is a Nasty number
8. z (1,3) is a perfect square.
9. z (5,6) is a cubic integer
10. $\frac{1}{3}z(4,9)$ is a woodall number.
11. $\frac{1}{2}z(1,9)$ is a Nasty number

2.3 Method: III

$$\text{Take 31 as } 31 = (2 + i\sqrt{27})(2 - i\sqrt{27}) \tag{9}$$
$$\text{Write z as } z = z(a, b) = a^2 + 27b^2 \tag{10}$$

Using (9) and (10) is (3) and applying the factorization process, define $(u + i\sqrt{27}v) = (2+ i\sqrt{27})(a + i\sqrt{27}b)^4$ This give us $u = 2a^4+1458b^4 -324a^2b^2 - 108a^3b +2916ab^3$

$$v = a^4 + 729b^4 -162a^2b^2 + 8a^3b -216ab^3 \tag{11}$$

Using (11) in (2), the relating integer values of (1) are furnished by

$$\left.\begin{aligned} x &= x(a, b) = 3a^4 + 2187b^4 - 486a^2b^2 - 100ab^3 + 2700a^3b \\ y &= y(a, b) = a^4 + 729b^4 - 162a^2b^2 - 108ab + 3132a^3b \\ z &= z(a, b) = a^2 + 27b^2 \end{aligned}\right\}$$

Observations:

1. $x (A, 1) -72 f^{A}_{4,5} +6 T_{4,A^2} +260 p^{5}_{A} +374T_{4,A} - G_{1347A} \equiv 0 (Mod 2)$
2. $y (A,1) -12 f^{A}_{4,4} +224 p^{5}_{A} + 55T_{4,A} - G_{1565A} \equiv 0 (Mod 5)$
3. $x (A,1)- y(A, 1) -48 f^{A}_{4,8} +10 T_{4,A^2} +48 p^{5}_{A} +312T_{4,A} + G_{212p} \equiv 31 (Mod 47)$
4. $x(A, 1) + y(A, 1) -4 T_{4,A^2} + 416 p^{5}_{A} +440T_{4,A}+ G_{216A} \equiv 0 (Mod 5)$
5. $x(A, 1) + y(A, 1) +z(A,1) -24 f^{A}_{4,6} + 440 p^{5}_{A} +435T_{4,A} + G_{216A} \equiv 0 (Mod 2)$
6. $x (1, 1) + y (1, 1) \equiv 0 (Mod 2)$
7. $x (1, A)-52488 f^{A}_{4.7} +8748 T_{4,A^2} + 55836 p^{5}_{A}$ $12123T_{4,A} - G_{21372A} \equiv 0 (Mod 2)$
8. $\frac{1}{7}z(5,5)$ is a perfect square
9. $\frac{1}{49}z(0,7)$ is a cubic integer
10. $\frac{1}{2}z(0,4)$ is a Nasty number

2.4 Method: IV

In place of (9) take 31 as

$$31 = \frac{(33+i\sqrt{27})(33-i\sqrt{27})}{36} \tag{12}$$

The following techniques is same as in the method-III, the relating integer values of (1) are found as

$x = x (A, B) = 7344A^4 +5353776B^4 -1189728A^2B^2 -139968AB^3 + 5184A^3B$
$y = y (A, B) = 6912A^4+ 5038848B^4- 1119744A^2B^2 - 51840A^3 B + 1399680AB^3$
$z = z (A, B) = 16A^2 + 4563B^2$

Observations:

1. $x (A, 1) -88128 f^{A}_{4,4} + 48384 p^{5}_{A} + 1202256T_{4,A} +G_{77328A} \equiv 0 (Mod 5)$

2. $y (A, 1) -165888 f^{A}_{4,3} +185904 p^{5}_{A} + 1102824T_{4,A} - G_{679104A} \equiv 11 (Mod 719834)$

3. x (A, 1)+ y(A, 1) - 85536 $f^{A}_{4,6}$ + 178848 p^{5}_{A} +2248560$T_{4,A}$ – $G_{629856A}$ ≡ 0 (Mod 5)

4. x (1,A) - 321226584 $f^{A}_{4,6}$ +321506520 p^{5}_{A} - 52488004$T_{4,A}$ - G_{2592A} ≡ 0 (Mod 5)

5. y (1, A)-120932352 $f^{A}_{4,5}$ +10077696 T_{4,A^2} +97977600 p^{5}_{A} - 2519424$T_{4,A}$+ $G_{5064768A}$ ≡ 3(mod628)

6. x (1, A)- y(1, A)-7558272 $f^{A}_{4,7}$ +1259712 T_{4,A^2} +11897280 p^{5}_{A} - 3674160$T_{4,A}$ –$G_{343440A}$ = star number

7. z (1,0) is a perfect square.

8. $\frac{1}{1521}$ z (0, 1) is a cubic integer

9. $\frac{1}{364}$ z (1, 1) ≡ 0 (Mod 13)

10. z(5,0)-2 is a kynea number.

2.5 Method V

Let us\take (3) as $u^2 + 27v^2 = 31z^4 * 1$ (13)

Take 1 as $1 = \frac{(13+i\sqrt{27})(13-i\sqrt{27})}{196}$ (14)

Using (9), (10) and (14) in (13) and applying the factorization process, define (u + i$\sqrt{27}$ v) = (2+ i$\sqrt{27}$) (a + i $\sqrt{27}$ b)^4 $\frac{(13+i\sqrt{27})}{14}$ This gives us

$$u = \frac{1}{14}[-a^4 - 729b^4 + 162a^2b^2 + 1180980ab^3 - 1620a^3b] \quad (14)$$

$$v = \frac{1}{14}[15a^4 + 10935b^4 - 2430a^2b^2 + 2916ab^3 - 4a^3b] \quad (15)$$

In sight of (2), the values of x, and y are

$$x = \frac{1}{14}[14a^4 + 10206b^4 - 2268a^2b^2 + 1183896ab^3 - 1624a^3b] \quad (16)$$

$$y = \frac{1}{14}[-16a^4 - 1164b^4 + 2592a^2b^2 + 1178064ab^3 - 1616a^3b] \quad (17)$$

As our intension is to find integer solutions, taking a as 5A and b as 5B in (4), (16) and (17), the relating parametric integer values of (1) are found as

x = x (A, B) = $625A^4 + 225625B^4 - 71250A^2B^2 + 997500AB^3 - 52500A^3B$ y = y (A, B) = $-750A^4 - 270750B^4 + 85500A^2B^2 + 94500AB^3 - 52000A^3B$ z = z (A, B) = $25A^2 + 475B^2$

Observations:

1. z (A, A) -500$T_{4,A}$ = 0 2. z (A, 0) – 25 $T_{4,A}$ = 0

3. z (0, B)- 475 $T_{4,B}$ = 0

4. $\frac{1}{5}$ z (1,1) is a perfect square

5. 6 x (A, 1) + 5 y (A, 1) + 1150000 p^{5}_{A} - 57500$T_{4,A}$ - $G_{3228750A}$+ 1 = 0

6. 6 x (A, 1) + 5 y (A, 1) = 0

7. x (1, 0) is a perfect square

8. x (A, 1) -300 $f^{A}_{4,7}$ + 108410 p^{5}_{A} - 17875$T_{4,A}$ – $G_{498875A}$≡ 0(Mod 2)

Each of the following is a nasty number

9. $\frac{6}{5}$ z (1, 0), $\frac{3}{50}$ z (1,1), $\frac{6}{125}$ x (1, 0), -$\frac{1}{25}$ y (1, 0)

3. Conclusion

In this work, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous bi-quadratic Diophantine equation $5(x^2 + y^2) - 9xy = 23z^4$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

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