Threshold Leverage Stochastic Volatility Model with Jump

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Abstract: ***This paper extends a triple-threshold leverage stochastic volatility model in Wu and Zhou (2015) by incorporating jump in mean equation and volatility equation. In this paper the volatility is treated as observable, in particular, the realized volatility is considered in the paper to be a proxy for the latent true volatility.***

Keywords: Threshold, Leverage, Stochastic volatility, Jump.

1. Introduction

As one of the important indexes of financial risk, volatility has always been a hot spot in financial research in the past two decades, and now is widely used in the risk management, portfolio selection and pricing of financial assets. However, the volatility of financial asset in the market cannot be observed directly and needs to adopt some certain methods to estimate.

With the development of financial econometrics analysis technology, there have been various volatility models. The most popular models are Autoregressive Condition Heteroskedasticity (ARCH)-type models and Stochastic volatility(SV)model. ARCH model was firstly proposed by Engle to describe the volatility in the British inflation rate [1]. SV model derived from Mathematical Finance and Financial Econometrics. Taylor proposed a SV model in discrete time [2], Hull and White proposed SV model in continuous time [3]. Different from ARCH-type model, SV model specifies volatility process as a separate random process. Theoretically, the SV model embeds a new innovation term in the latent volatility process, so it is more flexible than the ARCH model.

The leverage effect, jump component and heavy-tailed errors in asset returns are well-known to be important in most of literature (e.g., Chib et al. [4]; Jacquier et al. [5]; Yu [6]; Omori et al. [7];Berg et al. [8]).As well known, asset returns data have heavier tails than those of normal distributions. Eraker et al. incorporated the jump component to explain the heavy-tail behavior [9]. This paper incorporates the jump component to fit to the behavior of asset returns based on the model of Wu and Zhou [10].

Andersen and Bollerslev [11], Barndorff-Nielsen and Shephard [12] proposed the construction of the realized volatility, specified by the sum of squared intra-day returns over a certain interval. Particularly, let $p_{d,t}$ denotes the logarithmic price at a certain sampling frequency interval on day *t*. Then, the continuously compounded return with D observations on day *t* can be written as, $r_{d,t} = 100(p_{d,t} - p_{d-1,t})$, where $d = 1,2,\ldots,D$, $t = 1,2,\ldots,T$. Therefore this simple estimator of the daily volatility, denoted as RV, can be written as

$$(RV)_t = \sum_{d=1}^{D} r_{d,t}^2 .$$

Because the rapid development in the computer technology, the financial transaction data is visible. This paper treats the volatility as observable.

2. Model

In the basic discrete-time SV model, two stochastic processes are used to describe the dynamics of return and volatility. The SV model proposed by Taylor (1986) can be written as

$$r_t = \exp(h_t/2)\varepsilon_t \qquad \varepsilon_t i.i.d \sim N(0,1)$$

$$h_{t+1} = \beta + \alpha h_t + \eta_{t+1} \qquad \eta_t i.i.d \sim N(0,\sigma^2)$$

where r_t is the time series of return, h_t characterizes the logarithmic conditional variance at time t, ε_t and η_{t+1} are stochastically independent white noise processes.

Wu and Zhou proposed a Double Threshold Leverage SV (DTLSV) model; the correlation coefficient is state-dependent and switched by the threshold. The DTLSV model can be written as follows:

$$r_t = \exp(h_t/2)\varepsilon_t ,$$

$$h_{t+1} = \beta_{s_{t+1}} + \alpha_{s_{t+1}} h_t + \eta_{s_{t+1},t+1} ,$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_{s_{t+1},t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{s_{t+1}}\sigma_{s_{t+1}} \\ \rho_{s_{t+1}}\sigma_{s_{t+1}} & \sigma^2_{s_{t+1}} \end{pmatrix} \right),$$

where s_{t+1} is a state variable defined by

$$s_{t+1} = \begin{cases} 0, & r_t \le 0 \\ 1, & r_t > 0 \end{cases} .$$

This paper has mentioned that the jump components in asset returns are important. Therefore, this paper extends the DTLSV model by simultaneously incorporating the jump component in mean equation and volatility equation.

3. Double Threshold Leverage SV Model with Jump

Incorporating the jump component in the DTLSV model to get the double threshold leverage SV model with jump (DTLSVJ), the model structure is given by

$$r_t = \lambda_{t1} k_1 + \exp(h_t/2)\varepsilon_t \quad (1)$$

$$h_{t+1} = \beta_{s_{t+1}} + \alpha_{s_{t+1}} h_t + \eta_{s_{t+1},t+1} + \lambda_{t2} k_2 \quad (2)$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_{s_{t+1},t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{s_{t+1}}\sigma_{s_{t+1}} \\ \rho_{s_{t+1}}\sigma_{s_{t+1}} & \sigma_{s_{t+1}}^2 \end{pmatrix} \right) \quad (3)$$

where $k_1\lambda_{t1}, k_2\lambda_{t2}$ represent jump component in mean equation and volatility equation respectively .The $k_1 \sim N(\mu_{k1}, \sigma_{k1})$ is a jump size and the λ_{t1} is a jump flag defined as a Bernoulli random variable as follows:

$\pi(\lambda_{t1}=1)=\kappa_1, \pi(\lambda_{t1}=0)=1-\kappa_1, 0<\kappa_1<1.$

The $k_2 \sim N(\mu_{k2}, \sigma_{k2})$ is a jump size and the λ_{t2} is a jump flag defined as a Bernoulli random variable as follows:

$\pi(\lambda_{t2}=1)=\kappa_2, \pi(\lambda_{t2}=0)=1-\kappa_2, 0<\kappa_2<1.$

s_{t+1} is a state variable specified by

$$s_{t+1} = \begin{cases} 0, & r_t \le 0 \\ 1, & r_t > 0 \end{cases} \quad (4)$$

σ_0^2 and σ_1^2 are the variances of the innovations in the two volatility regimes. ρ_0 and ρ_1 capture the asymmetric correlations between the return and volatility in the two regimes, respectively. Then, the unknown parameter vector to be estimated is defined as follows:

$$\Theta = (\beta_0, \beta_1, \alpha_0, \alpha_1, \sigma_0, \sigma_1, \rho_0, \rho_1, \mu_{k1}, \mu_{k2}, \sigma_{k1}, \sigma_{k2}, \kappa_1, \kappa_2)'.$$

The likelihood function is as follows,

$$\begin{aligned} L(\Theta; r, h) &= L_{s_T}(r_T, h_{T+1} \mid r_{T-1}, h_T, \Theta) \\ &\times L_{s_{T-1}}(r_{T-1}, h_T \mid r_{T-2}, h_{T-1}, \Theta) \times \cdots \\ &\times L_{s_2}(r_2, h_3 \mid r_1, h_2, \Theta) \times L_{s_1}(r_1, h_2 \mid r_0, \Theta) \end{aligned} \quad (5)$$

The likelihood function consists of a sequence of the conditional densities, and at each time t,

$L_{s_t} = f(r_t \mid r_{t-1}, h_t, \Theta) \times f(h_{t+1} \mid r_t, h_t, \Theta)$.Given the state determined from the previous observation, under the condition of (3),

$$f(r_t \mid r_{t-1}, h_t, \Theta) = \int_{-\infty}^{+\infty} \frac{\kappa_1}{\sqrt{2\pi}\cdot\sigma_{k1}} \cdot e^{-\frac{(\omega_1-\mu_{k1})^2}{2\sigma_{k1}^2}} \cdot \frac{1}{\sqrt{2\pi}\cdot e^{h_t/2}} \cdot e^{-\frac{(r_t-\omega_1)^2}{2e^{h_t/2}}} d\omega_1,$$

$$f(h_{t+1} \mid r_t, h_t, \Theta) = \int_{-\infty}^{+\infty} \left(\frac{\kappa_2}{\sqrt{2\pi}\cdot\sigma_{k2}} \cdot e^{-\frac{(\omega_2-\mu_{k2})^2}{2\sigma_{k2}^2}} \cdot \frac{1}{\sqrt{2\pi(1-\rho_{s_t}^2)}\sigma_{s_t}} e^{-\frac{(h_{t+1}-\omega_2-\beta_{s_t}-\alpha_{s_t}-\sigma_{s_t}\rho_{s_t}\varepsilon_t)^2}{2\sigma_{s_t}^2(1-\rho_{s_t}^2)}} \right) d\omega_2.$$

At each time t,the conditional density can be given as follows,

$$L_{s_t} = \int_{-\infty}^{+\infty} \frac{\kappa_1}{\sqrt{2\pi}\cdot\sigma_{k1}} e^{-\frac{(\omega_1-\mu_{k1})^2}{2\sigma_{k1}^2}} \cdot \frac{1}{\sqrt{2\pi}\cdot e^{h_t/2}} \cdot e^{-\frac{(r_t-\omega_1)^2}{2e^{h_t/2}}} d\omega_1 \times \int_{-\infty}^{+\infty} \frac{\kappa_2 \cdot e^{-\frac{(\omega_2-\mu_{k2})^2}{2\sigma_{k2}^2}}}{\sqrt{2\pi}\cdot\sigma_{k2}} \cdot \frac{1}{\sqrt{2\pi(1-\rho_0^2)}\sigma_0} e^{-\frac{(h_{t+1}-\omega_2-\beta_0-\alpha_0-\sigma_0\rho_0\varepsilon_t)^2}{2\sigma_0^2(1-\rho_0^2)}} d\omega_2$$

when $s_t = 0$,

$$L_{s_t} = \int_{-\infty}^{+\infty} \frac{\kappa_1}{\sqrt{2\pi}\cdot\sigma_{k1}} e^{-\frac{(\omega_1-\mu_{k1})^2}{2\sigma_{k1}^2}} \cdot \frac{1}{\sqrt{2\pi}\cdot e^{h_t/2}} \cdot e^{-\frac{(r_t-\omega_1)^2}{2e^{h_t/2}}} d\omega_1 \times \int_{-\infty}^{+\infty} \frac{\kappa_2 \cdot e^{-\frac{(\omega_2-\mu_{k2})^2}{2\sigma_{k2}^2}}}{\sqrt{2\pi}\cdot\sigma_{k2}} \cdot \frac{1}{\sqrt{2\pi(1-\rho_1^2)}\sigma_1} e^{-\frac{(h_{t+1}-\omega_2-\beta_1-\alpha_1-\sigma_1\rho_1\varepsilon_t)^2}{2\sigma_1^2(1-\rho_1^2)}} d\omega_2$$

when $s_t = 1$.

In the likelihood function above, the volatility is treated as observable. This paper uses the realized volatility as a proxy for the true volatility.

4. Conclusion

This paper incorporates the jump component in the DTLSV model to generalize the DTLSV model. The jump component in asset returns is well-known to be important, so the model proposed in this paper is more suitable to some features of financial market.

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