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A Food Chain Model with Strong Allee Effect in Prey

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Abstract: ***In recent years, there has been an intense focus on the construction of mathematical models involving certain interesting ecological phenomena like Allee effect. Many animal and plant species suffer a decrease of the per capita rate of increase as their populations reach small sizes or low densities. In this article, a three species food chain model is proposed and analysed where prey species is affected by strong Allee effect. I have studied the stability behaviour of the system and bifurcation analysis. The system exhibits saddle-node bifurcation curve. Extensive Numerical simulation results are given to support the validity of the theoretical results. The ecological implications of our analytic and numerical findings are discussed.***

Keywords: Food chain model, Allee effect, Stability, Bifurcation, Numerical simulation

1. Introduction

Allee effect that refers to a positive relationship between individual fitness and population density provides an important conceptual framework in conservation biology. While declining Allee effect causes reduction in extinction risk in low-density population, it provides a benefit in limiting establishment success or spread of invading species. Population models that incorporated Allee effect confer the fundamental role which plays for shaping the population dynamics. In population dynamics, the Allee effect (named after the seminal works of the zoologist and ecologist Warder Clyde Allee [1]) refers to a process that reduces the growth rate for small population densities [2].

Allee effect can be either strong or weak [3]. Strong Allee effect can induce a critical density below which per capita growth rate is negative and extinction tends to occur, while weak Allee effect may result in reduced, but still positive, growth rate as population size or density decreases [4].

While a variety of species exhibit Allee effect, its underlying mechanisms still remain unclear. Because of this reason, determining which factors regulate or induce Allee effect continues to be an important subject in the field of both theoretical and experimental research [5-8]. Allee effect can arise from many ecological processes under various interactions of functional and aggregative responses. The previous work revealed that increased risk of predation at low population density [9-12], sexual selection [13,14], reduced mating efficiency [15] and reduced foraging efficiency [16] can give rise to Allee effect.

My objective here is not to reveal which species has Allee effect or which factor can cause Allee effect, but to evoke the importance of Allee dynamics and its potential consequences in population dynamical models. I analyze the stability of three-species food chain model with strong Allee effect.

The rest of the article has been organized as follows. In Section 2, I state the formulation of the model under consideration and its assumptions. Section 3 contains some preliminary results. Then in Section 4 the model is analyzed, identifying its equilibrium points, giving conditions for their feasibility, stability and bifurcation. Numerical simulation has been carried out in Section 5. The article concludes with a discussion of the results obtained.

2. The Mathematical Model

The general food chain model system interaction is represented by the system of following differential equations

$$\frac{DX}{DT} = RXg(X) - F_1(X,Y)Y \quad \ldots\ldots\ldots. \quad (2.1a)$$

$$\frac{DY}{DT} = E_1F_1(X,Y)Y - D_1Y - F_2(Y,Z)Z \quad \ldots\ldots\ldots.. \quad (2.1b)$$

$$\frac{DZ}{DT} = E_2F_2(Y,Z)Z - D_2Z \quad \ldots\ldots\ldots \quad (2.1c)$$

$$X(0) > 0, Y(0) > 0, Z(0) > 0,$$

where X, Y, Z are the population densities of prey, predator and top-predator respectively; $R, E_1, E_2, D_1, D_2, F_1, F_2$ are positive constants that stands for prey intrinsic growth rate, conversion factors, death rates and functional responses for representing predation process of respective predator.

In this paper, an attempt has been made to update the food chain model (2.1) incorporating Allee effect in the prey species. Thus, I obtain the following model

$$\frac{DX}{DT} = RXg(X)(X-L) - F_1(X,Y)Y \quad \ldots\ldots. \quad (2.2a)$$

$$\frac{DY}{DT} = E_1F_1(X,Y)Y - D_1Y - F_2(Y,Z)Z \quad \ldots\ldots \quad (2.2b)$$

$$\frac{DZ}{DT} = E_2F_2(Y,Z)Z - D_2Z \quad \ldots\ldots.. \quad (2.2c)$$

$$X(0) > 0, Y(0) > 0, Z(0) > 0,$$

where L is the survival threshold of the prey and other parameters are defined in the previous model (2.1). Here, we consider the predation functions $F_1(X,Y) = A_1X$, $F_2(Y,Z) = A_2Y$, that is , the Holling type-I functional responses where the constants A_1, A_2 are two parameters characterising the functional response.

Using the transformation $x = \frac{X}{K}, y = \frac{Y}{KE_1}, z = \frac{Z}{KE_1E_2}, t = KRT$, the model (2.2) tales the following form

$$\frac{dx}{dt} = x(1-x)(x-m) - a_1xy \quad \ldots.. \quad (2.3a)$$

$$\frac{dy}{dt} = a_1xy - d_1y - a_2yz \quad \ldots.... \quad (2.3b)$$

$$\frac{dz}{dt} = a_2yz - d_2z \quad \ldots. \quad (2.3c)$$

$$x(0) > 0, y(0) > 0, z(0) > 0, \quad (2.3d)$$

where $m = \frac{L}{K}, a_1 = \frac{A_1E_1}{R}, a_2 = \frac{A_2E_1E_2}{R}, d_1 = \frac{D_1}{KR}, d_2 = \frac{D_2}{KR}$.

3. Preliminary Results

3.1. Existence and positive invariance

Theorem: Every solution of system (2.3) with initial conditions (2.3d) exists in the interval $[0, \infty)$ and $x(t) \geq 0, y(t) \geq 0, z(t) \geq 0$ for all $t > 0$.

Proof. Since the right hand side of system (2.3) is completely continuous and locally Lipschitzian on C, the solution of (2.3) with initial conditions (2.3d) exists and is unique on $[0, \eta)$, where $0 < \eta \leq \infty$ [3]. From system (2.3) with initial conditions (2.3d), we have

$$x(t) = x(0)\exp\left[\int_0^t \{(1-x(\theta))(x(\theta)-m) - a_1y(\theta)\}d\theta\right] \geq 0,$$

$$y(t) = y(0)\exp\left[\int_0^t \{a_1x(\theta) - d_1 - a_2z(\theta)\}d\theta\right] \geq 0,$$

$$z(t) = z(0)\exp\left[\int_0^t \{a_2y(\theta) - d_2\}d\theta\right] \geq 0,$$

which completes the proof.

3.2. Equilibrium points and their feasibility

The system (2.3) has the following positive equilibrium points:

i) The trivial equilibrium point $E_0(0,0,0)$, the origin, exists always;

ii) The predators free equilibrium points $E_m(m,0,0), E_1(1,0,0)$ exist provided $m > 0$;

iii) The top-predator free equilibrium point $E_2(x_2, y_2, 0)$ provided $m \leq \frac{a_1}{d_1} \leq 1$; where $x_2 = \frac{a_1}{d_1}, y_2 = \frac{(d_1-a_1)(a_1-md_1)}{a_1d_1^2}$.

iv) The system has two distinct co-existence equilibrium points $E_3(x_3, y_3, z_3), E_4(x_4, y_4, z_4)$, where

$x_3 = \frac{1+m+\sqrt{(1-m)^2-4a_1y_3}}{2}, y_3 = \frac{a_2}{d_2}, z_3 = \frac{a_1x_3-d_1}{a_2}, x_4 = \frac{1+m-\sqrt{(1-m)^2-4a_1y_4}}{2}, y_4 = \frac{a_2}{d_2}, z_4 = \frac{a_1x_4-d_1}{a_2}$.

4. Statement of the Main Results

The main properties and qualitative behaviour of the system (2.3) are given in this section. Local stability of equilibrium points are determined by the Jacobian matrix:

$$J(x,y,z) = \begin{bmatrix} (1-x)(x-m) + x(1-x) - x(x-m) - a_1y & -a_1x & 0 \\ a_1y & a_1x - d_1 - a_2z & -a_2y \\ 0 & a_2z & a_2y - d_2 \end{bmatrix}$$

4.1. E_0 is always stable in nature.

Proof. The jacobian matrix J_0 of the system (2.3) at E_0 is given by

$$J_0 = \begin{bmatrix} -m & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}.$$

The eigen values of J_0 are $-m, -d_1, -d_2$. Hence, the equilibrium point E_0 is always stable in nature.

4.2. E_m is unstable in nature.

Proof. The jacobian matrix J_m of the system (2.3) at E_m is given by

$$J_m = \begin{bmatrix} m(1-m) & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}.$$

The eigen values of J_m are $m(1-m), -d_1, -d_2$. Since, $0 \leq m < 1$, the first eigenvalue is always positive. Hence, the equilibrium point E_m is unstable in nature.

4.3. E_1 is unstable in nature.

Proof. The jacobian matrix J_1 of the system (2.3) at E_1 is given by

$$J_1 = \begin{bmatrix} 1-m & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}.$$

The eigen values of J_1 are $1-m, -d_1, -d_2$. Since, $0 \leq m < 1$, the first eigenvalue is always positive. Hence, the equilibrium point E_1 is unstable in nature.

4.4. E_2 is stable if $\theta = x_2(1-x_2) - x_2(x_2-m) < 0$ and $a_2y_2 - d_2 < 0$.

Proof. The jacobian matrix J_2 of the system (2.3) at E_2 is given by

$$J_2 = \begin{bmatrix} x_2(1-x_2) - x_2(x_2-m) & -a_1x_2 & 0 \\ a_1y_2 & 0 & -a_2y_2 \\ 0 & 0 & a_2y_2 - d_2 \end{bmatrix}.$$

The eigen values of J_2 are $\lambda_{1,2} = \frac{(\theta) \quad \sqrt{\{(\theta)^2 - 4a_1^2x_2y_2\}}}{2}$, $a_2y_2 - d_2$, where $\theta = x_2(1-x_2) - x_2(x_2-m)$. If $\theta < 0, \lambda_3 = a_2y_2 - d_2 < 0$, then, the equilibrium point E_2 will be locally asymptotically stable in nature.

4.5. The system experiences Hopf-bifurcation around E_2 for $a_1 = a_1^{[hb]}$ where $a_1^{[hb]} = \frac{d_1(1+m)}{2}$.

Proof. From sub-section 4.4, we see that λ_3 is real and $\lambda_{1,2}$ will be purely imaginary if and only if there is a $a_1 = a_1^{[hb]}$

such that $a_1^{[hb]} = \frac{d_1(1+m)}{2}$. Now for $= 1, 2$; $Re\left[\frac{d\lambda_i}{da_1}\right]_{a_1=a_1^{[hb]}} \neq 0$. Therefore, the system experiences Hopf-bifurcation around E_2 for $a_1 = a_1^{[hb]}$.

4.6. E_3 is locally asymptotically stable if $c_{11} = x_3(1 -$ *x3−x3x3−m<0.*

Proof. The jacobian matrix J_3 of the system (2.3) at E_3 is given by

$$J_3 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

where $c_{11} = x_3(1 - x_3) - x_3(x_3 - m), c_{12} = -a_1 x_3,\ c_{13} = 0, c_{21} = a_1 y_3, c_{22} = 0,\ c_{23} = -a_2 y_3, c_{31} = 0, c_{32} = a_2 z_3, c_{33} = 0.$

The characteristic equation of J_3 is $\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$, where $B_1 = -c_{11},\ B_2 = -c_{23}c_{32} - c_{12}c_{21}, B_3 = c_{11}c_{23}c_{32}$.
Therefore, $B_1B_2 - B_3 = c_{11}c_{12}c_{21}$. Now, we choose $c_{11} < 0$, then $B_1 > 0$, $B_3 > 0$ and $B_1B_2 - B_3 > 0$. Hence, the proof.

4.7. E_4 is locally asymptotically stable if $x_4(1 - x_4) - x_4(x_4 - m) < 0$.
The proof is similar with the proof of 4.5.

4.8. System (2.3) undergoes a saddle-node bifurcation around interior equilibrium point E_3 with respect to bifurcation parameter m with bifurcation threshold $a_1^{[sn]}$ if
Proof: One eigenvalue of the Jacobian matrix J_3 will be zero if $\det(J_3) = 0$ which gives $a_1^{[sn]} = \frac{d_2(1-m)^2}{4a_2}$. The other two eigenvalues of J_3 are evaluated at $a_1^{[sn]}$ and one of them must be negative in order to get a saddle-node bifurcation. Let u and v are the eigenvectors corresponding to the eigenvalue 0 of the matrix J_3 and its transpose respectively. We obtain that $u = (u_1, u_2, u_3)^T$ and $v = (v_1, v_2, v_3)^T$ where $u_2 = 0, u_3 = -\frac{c_{21}}{c_{23}}u_1, v_2 = 0, v_3 = -\frac{c_{12}}{c_{32}}v_2$ in which u_1 and v_1 are any two real numbers. Since $v^T\left[F_{a_1}\left(E_3, a_1^{[sn]}\right)\right] = -x_3 y_3 v_1 \neq 0$, $v^T\left[D^2F\left(E_3, a_1^{[sn]}\right)(u,u)\right] \neq 0$, then the system experiences a saddle-node bifurcation around E_3 $a_1 = a_1^{[sn]}$ [17].

5. Numerical Simulation

Analytical results can never be completed without numerical justification of the derived results. In this section, we present computer simulations of some solutions of the system (2.3). Beside justification of our analytical findings, these numerical simulations are very important from practical point of view.

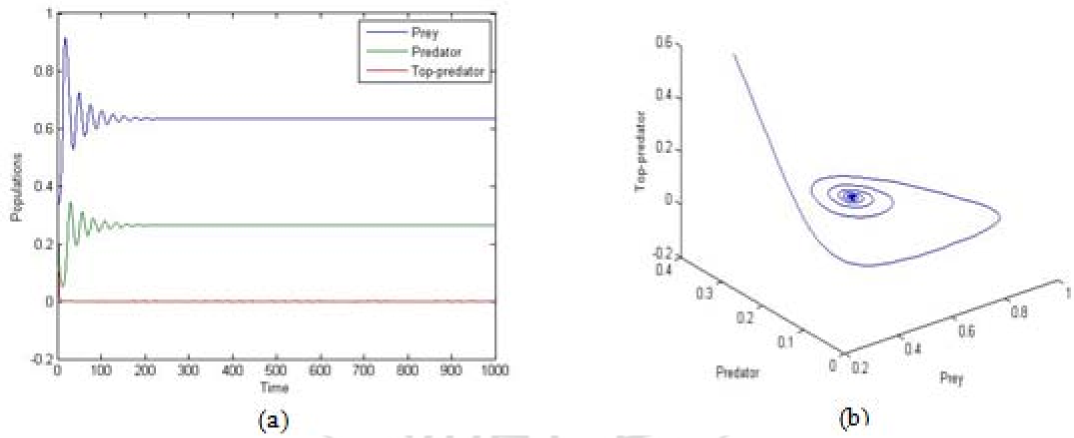

Figure 1: Locally asymptotically stable behaviour of the system (2.3) around boundary equilibrium point E_2 for the set of parameters $a_1 = 0.6, a_2 = 0.5, d_1 = 0.38, d_2 = 0.8, m = 0.2$.

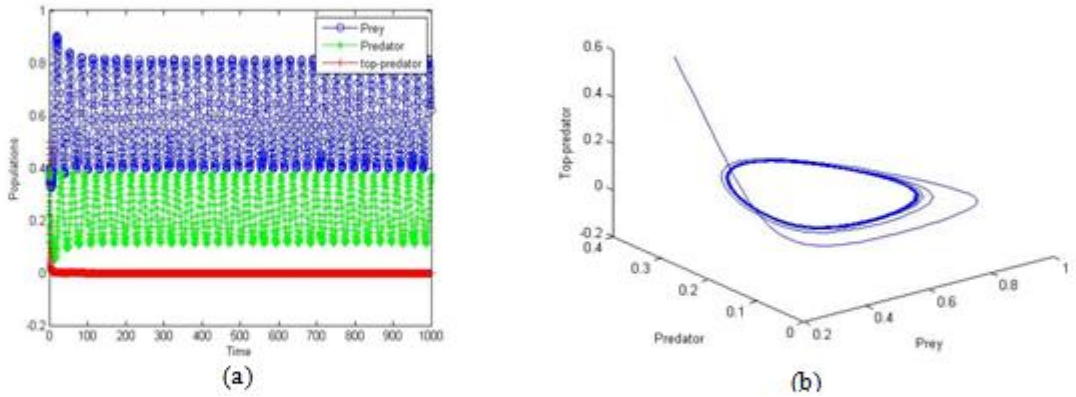

Figure 2: Hopf-bifurcation behaviour of the system (2.3) around boundary equilibrium point E_2 for the set of parameters $a_1 = 0.6$, $a_2 = 0.5$, $d_1 = 0.35$, $d_2 = 0.8$, $m = 0.2$.

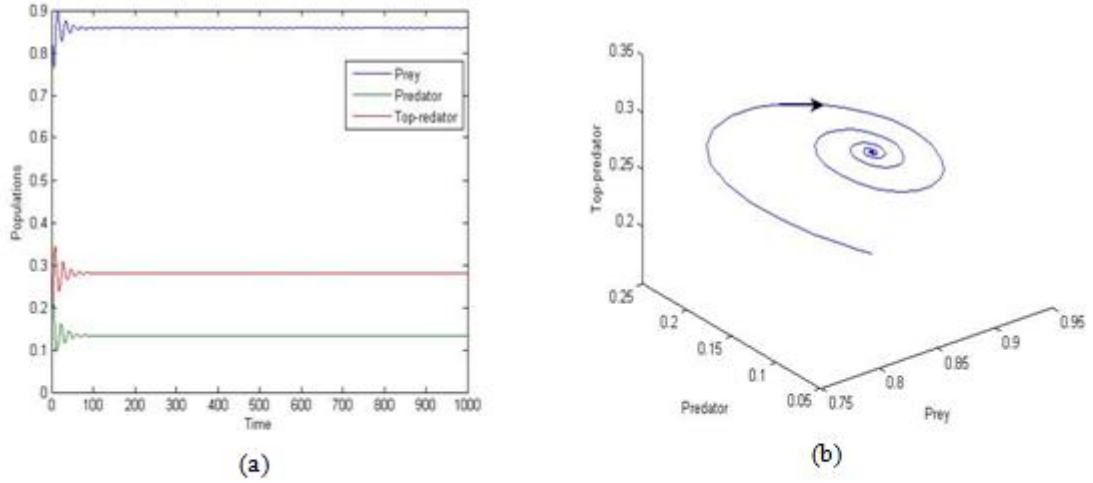

Figure 3: Locally asymptotically stable behaviour of the system (2.3) around interior equilibrium point for the set of parameters $a_1 = 0.7, a_2 = 1.5, d_1 = 0.18, d_2 = 0.2, m = 0.2$.

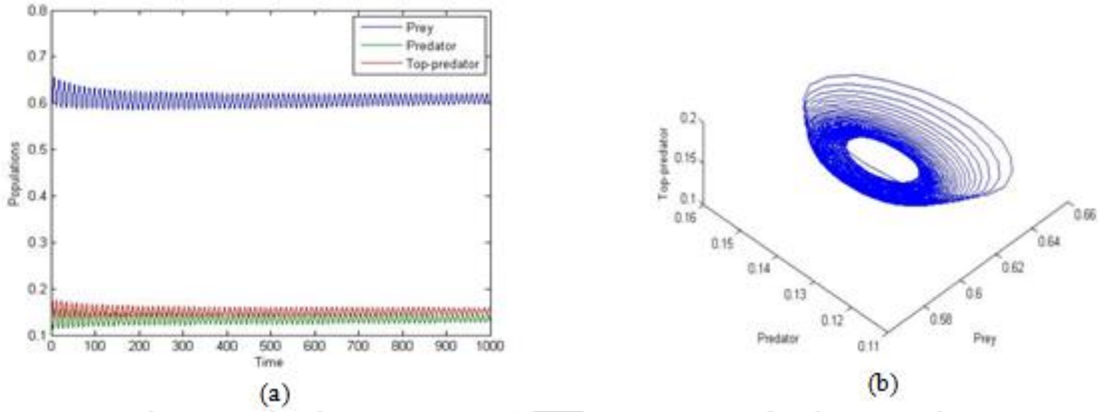

Figure 4: Periodic behaviour of the system (2.3) around boundary equilibrium point E_3 for the set of parameters $a_1 = 1.178765, a_2 = 2.95, d_1 = 0.27, d_2 = 0.4, m = 0.2$.

6. Conclusion and Comments

In this paper, three species food chain model where prey species is affected by strong Allee effect is analyzed and possible dynamical behaviour of this system investigated at equilibrium points. It has been shown that, the solutions posses Saddle-node and Hopf-bifurcations. Both analytically and numerical simulation shown that in certain regions of the parameter space, three species food chain model is sensitively depending on the parameter values.

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