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Random Fixed Point Theorems for Contraction Mappings in Metric Space

Suhas Patil[1], Uttam Dolhare[2]

[1]Research Scholar, SRTMU, Nanded (M.S), India,

[2]Associate Professor, Department of Mathematics, D.S.M. College, Jintur (M.S), India

Abstract: ***"In this article we use the concept of contraction and Banach contraction principle and prove some Random fixed point theorems with the help of Lipchitz's mapping and new type of contraction called asymptotic contraction. The article includes some important remark and examples the main result in the article is generalizations of Banach Contraction principle where Lipchitz's constant replaced by real valued contraction function".***

Keywords: Contraction Mappings, Asymptotic contractions, Lipchitzan mapping, Complete Metric Space.

1. Introduction

Fixed point theorems concern maps of a set *X* into itself under certain conditions, admit a fixed point i.e. a point *x* in *X* such that *f(x) = x*. In 1922 the famous mathematician stated & proved his important result called Banach contraction principle [7], the study of random fixed point of mapping satisfying certain contractive conditions [1] which has been very interesting fields in mathematics. This theorem formed a technique for solving a variety of applied problems (applications) in many branches of Engineering and mathematics.

In this paper we have to see some asymptotic fixed point theorems by using contraction mappings and for mappings which are more general than contraction mappings in metric spaces. W.A. Kirk [4] first introduced the notion of asymptotic contractions and proved the fixed point theorem for this class of mappings. In the first part of this article we establish a fundamental definitions, remarks and asymptotic fixed point theorem in which we know the Banach contraction principle [7] and further we give its generalizations in metric spaces.

And in second part we have to see some generalization of Banach contraction principle by taking Boyd & Wong's fixed point theorem as a special case [2]. In [1] Meir and Keeler extended the Boyd–Wong result [2], Boyd and Wong also show in [2] that if the space X is metrically convex, then the upper semi-continuity assumption on σ can be dropped. Matkowski has extended this fact even further in [3].

Also the important generalization of Boyd & Wong's fixed point theorem given by Matkowski [3] fixed point theorem a special case using the mapping that we called asymptotic contractions. By an asymptotic fixed point theorem for the mappings it means that a theorem that guarantees the existence of a fixed points of S if the iterative S^k having certain properties.

2. Preliminaries

Through the article R denotes the set of all real numbers & R^+ is the set of all positive real numbers (*X*, δ) be a metric space and N is the set of all natural numbers.

Definition 2.1 Let X be a metric space with d be the distance then a map $S : X \to X$ is said to be Lipschitz continuous if there is $\sigma \geq 0$ such that

$$d[S(x), S(y)] \leq \sigma d(x, y) \quad \text{, for all } x,y \text{ in } X.$$

Definition 2.2 Let Lip(x) denote the class of mappings in a metric space *(X, δ)* then $S : X \to X$ such that

$$\rho(S^k) = Sup \left\{ d\left(S^k x,\ S^k y\right) \ / \ d\left(x, y\right) \ : \ x, y \in X, x \neq y \right\} < \infty$$

for all k ∈ N.

$\rho\,(S^k)$ is called Lipchitz constant of S^k and the members of Lip(x) are called Lipschitzian mappings

Remark 2.1 For two Lipschitzian mappings $S, T : X \to X$ we have T(X) subset of domain of S such that $\rho(SoT) \leq \rho(S)\rho(T)$. It is clear that the mapping S belongs to *Lip(X)* there exists a constant $Z_k \geq 0$ such that $d\left(S^k x,\ S^k y\right) \leq Z_k d(x, y)$ for all *x,y* belongs to X and *k* ∈ *N.*

Definition 2.3 A Lipschitzian mapping $S : X \to X$ is said to be uniformly L - Lipschitzian if $Z_k = Z$ for all *k*∈ *N.*

Remark 2.2 Lipschitzian mapping is non-expansive if ρ(S) < 1.

Definition 2.4[4] A mapping $S : X \to X$ in a metric space *(x, δ)* is said to be an asymptotic contraction if $d\left(S^k x,\ S^k y\right) \leq \sigma_k d(x, y)$ for all x, y belongs to *X* & *k* ∈ *N*, where $\sigma_k : R^+ \to R^+$ and σ_k tends to σ belongs to μ uniformly on the range of *d*.

The following theorems are very useful for proving some existence theorems in metric space

Theorem 2.1 Let *(x, δ)* complete metric space and a bounded below, lower semi-continuous function $\sigma : X \to (-\infty, \infty]$ suppose that $\{u_k\}$ is a sequence in X such that

$$d(u_k, u_{k+1}) \leq \sigma(u_k) - \sigma(u_{k+1}) \text{ for all } k \in N_0,$$

then $\{u_k\}$ converges to a point $q \epsilon X$

and

$$d(u_k, q) \leq \sigma(u_k) - \sigma(q) \text{ for all } k \in N_0$$

Theorem 2.2 Let be a proper, lower semi-continuous and bounded below function in a complete metric space *(x, δ)*, suppose for each *p* belongs to *X* with *inf x∈ X* $\sigma(x) < \sigma(u)$ there exist q$\in X$ such that,

$$\sigma(p,q) \leq \sigma(p) - \sigma(q)$$

and $p \neq q$ there exists an $x_0 \in X$. such that $\sigma(x_0) = \inf x \in X \sigma(x)$.

<u>Banach Contraction Principle</u>

Theorem 2.3[5] Let $S : X \to X$ be a contraction mapping in a complete metric space *(x, δ)* with lipschitz's constant *k* $\in (0,1)$, then we have

1) $d(u_k, q) \leq (1-n)^{-1} n^k d(u_0, u_1)$ for all $k \in N_0$.
2) For arbitrary $u_0 \in X$, then by Picard iteration process $u_{k+1} = Su_k$, k $\in$ N_0 converges to q.
3) There exist a unique fixed point q ∈ X.

Following are some examples on contraction mapping.

Example 2.1 Let a mapping c , X = [c, d] & S is differentiable at every x belongs to *(c, d)* therefore $|S^1(x)| \leq k < 1$, then by mean value theorem if *x, y* ∈ *X* there exist a point ξ between *x* & *y* such that

$$Sx - Sy = S^1(\xi)\ (x-y)$$

Hence $|Sx - Sy| = |S^1(\xi)|\ |x-y| \leq k\ |x-y|$, therefore S is contraction and it has unique fixed point.

Example 2.2 Let $S : [0,1] \to [0,1]$ be a mapping and *X = [0,1]* then the mapping is $Sx = 1 - x$ For *x ∈ [0,1]* then T has a unique fixed point 1/2 but T is not a contraction.
The above example shows that there exists a mapping which is not a contraction but it has a unique fixed point.

Definition 2.5 A mapping $S : X \to X$ is said to be contractive in a metric space *(x, δ)* if it satisfied the following condition

$$d(Sx, Sy) < d(x, y) \text{ for all x, y} \in \text{X and } x \neq y$$

Remark 2.3 The class of contractive mapping have been in between the class of contraction mappings and that of non-expansive mappings.

Remark 2.4 Contractive mapping having at most one fixed point. Boundedness and completeness do not ensure the existence of fixed point of contractive mappings in a metric space, but contractive mappings always have fixed point in a compact metric space.

Theorem 2.4[9] Let $S : X \to X$ be a contractive mapping in a compact metric space *X*, then *S* has a unique fixed point *q* in *X*, for each *x* in *X* the sequence $\{S^k x\}$ of iterates converges to *q*.

We also see even in a Hilbert space for contractive mappings, we can't have $S^k x \to X_0$ for every $x \in V_x$ and $X_0 = Sx_0$.

3. Main Result

In the main result we see some useful generalizations of the Banach Contraction Principle in which the Lipchitzs constant k is replaced by some real valued contraction function.

Theorem 3.1 Let $\sigma : X \to (-\infty, \infty]$ be a mapping which is bounded below, proper and lower semi-continuous functions in a complete metric space *(x, δ)*, let for each p$\in$X with *inf x ∈X* $\sigma(x) < \sigma(u)$ there exist p $\in X$, such that $\sigma(p,q) \leq \sigma(p) - \sigma(q)$ for $p \neq q$, ∃ an $x_0 \in X$ such that $\sigma(x_0) = \inf x \in X \sigma(x)$.

Theorem 3.2[3] Let *(x, δ)* be a complete metric space and $\sigma : X \to (-\infty, \infty]$ be a proper, lower semi-continuous and bounded below mapping and let $S : X \to X$ be a mapping such that $d(x, Sx) \leq \sigma(x) - \sigma(Sx)$ for all *x* in *X*, then there exist a point $q \in X$ such that $q = Sq$ and $\sigma(q) < \infty$.

Proof let σ is a proper mapping then there exists p ∈ X such that $\sigma(p) < \infty$, and let

$$J = \{x \in X : d(p, x) \leq \sigma(p) - \sigma(x)$$

Then J is nonempty closed subset of X, further we have to show that c is invariant under S, then for each x ϵ J we have,

$$d(p, x) \leq \sigma(p) - \sigma(x)$$

and from result we get,

$$\sigma(Sx) \leq \sigma(x) - d(x, Sx)$$
$$\leq \sigma(x) - d(x, Sx) + \sigma(p) - \sigma(x) - d(p, x)$$
$$\leq \sigma(p) - [d(x, Sx) + d(p, x)]$$
$$\leq \sigma(p) - d(p, Sx)$$

Hence from these it follows it follows that, $Sx \in J$

For a contraction suppose that *x* is a not equal to *Sx* for all *x* in *J*, then for each *x* in *J* there exist *γ* belongs to *J* such that

$$d(x, \gamma) \leq \sigma(x) - \sigma(\gamma), \text{ for } x \neq \chi$$

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Hence from above theorem (3.1) there exist an $X_0 \epsilon\ J$ with $\sigma(x_0) = \inf x \in J, \sigma(x)$.

And for $X_0 \in J$ we get,

$$0 < d(X_0, SX_0) \le \sigma(X_0) - \sigma(TX_0)$$
$$\le \sigma(SX_0) - \sigma(SX_0)$$
$$= 0$$

This proved a contradiction.

Hence the complete the proof of the theorem.

Remark 3.1 The fixed point of the mappings S in the above theorem need not be unique.

In this theorem we see the generalization of Banach contraction principle, and we take some real valued control function in place of the Lipschitz constant k.

Theorem 3.4[3] Let the mapping in a complete metric space X satisfies

$$d(Sx, Sy) \le \mu d(x, y) \text{ for all } x, y \in X,$$

where the upper semi-continuous function $\mu : R^+ \to R^+$ from right such that

$\mu(t) < t$ for each t > 0,

then S has a unique fixed point q ϵ X, for each x ∈ X , $\lim_{k\to\infty} S^n x = q$.

Proof Let a sequence $\{x_k\}$ in X and $x \epsilon X$ defined by, $x_k = S^k x = q$.

And let the set $d_k = d(x_k, x_{k+1})$ then we have the proof in following case

Part (I):- let $\lim_{k\to\infty} d_k = 0$

$$\therefore d_{k+1} = d(x_{k+1}, x_{k+2}) = d(sx_k, sx_{k+1})$$
$$\le \mu(d_k) \text{ for } k \in N_0$$

Hence sequence d_k is monotonic decreasing, bounded below & the limit exists

Let $\lim_{k\to\infty} d_k = \omega \ge 0$.

Assume that **ω** > 0 then by right continuity of μ we get,

$$\omega = \lim_{k\to\infty} d_{k+1} \le \lim_{k\to\infty} \mu(d_k) \le \mu(\omega) < \omega$$

$$\therefore \boldsymbol{\omega} = 0.$$

Part (II):- let $\{x_k\}$ is a Cauchy sequence,

Assume that $\{x_k\}$ is not Cauchy sequence then there exist ϑ > 0 and integers u_n, v_n ϵ N_0, such that $u_n > v_n \ge n$ and $d(xu_n, xv_n) \ge \vartheta$ for $n = 0,1,2,....$and u_n consider as small as possible, then it may be assumed that $d(xu_{n-1}, xv_n) < \vartheta$

For each n ϵ N_0 we get,

$$\vartheta \le d(xu_n, xv_n) \le d(xu_n, xu_{n-1}) + d(xu_{n-1}, xv_n)$$
$$\le d(xu_n - 1, xu_n) + \in$$
$$= dxu_n - 1 + \in$$

Then $\lim_{n\to\infty} d(xu_n, xv_n) = \in$ as $dxu_n \to 0$ observe that,

$$d(xu_n xv_n) \le d(xu_n, xu_{n+1}) + d(xu_{n+1}, xv_{n+1}) + d(xv_{n+1}, xv_n)$$

$$\le du_n + \mu[d(xu_n xv_n) + du_n$$

Using the upper semi-continuity of μ from right and taking $n \to \infty$ we get,

$$\vartheta = \lim_{n\to\infty} d(xu_n, xv_n) \le \mu\, [\lim_{n\to\infty} d(xu_n, xv_n)] \le \mu\, (\vartheta)$$

which is a contradiction. Hence $\{x_k\}$ is a Cauchy sequence in X.

Part (III):- In this Part we prove the existence and uniqueness of fixed points.

We have $\{x_k\}$ is a Cauchy sequence in X and is complete, then

$\lim_{k\to\infty} x_k = q \in X$, by continuity of S we have $q = Sq$.

Uniqueness of q easily follows from condition in the statement.

Let φ denotes the class of all mappings $\mu : R^+ \to R^+$ satisfying μ is continuous and

$\mu(t) < t$ for all $t > 0$.

Hence complete proof of the theorem.

Corollary3.1 Let $S : X \to X$ be a mapping in a complete metric space (x, δ) that satisfies

$d(Sx, Sy) \le \sigma[d(x, y)]$ for all x, y ∈ X.

where $\sigma \in \rho$, then S has a unique fixed point $q \in X$ such that

$\lim_{k\to\infty} S^k x = q$.

In the following theorem we show asymptotic contractions which having unique fixed point.

Theorem 3.4[10] Let $S : X \to X$ be a mapping in a complete metric space (x, δ) which is a continuous and asymptotic contraction, in which the mapping σ_k in

$d(S^k x, S^k y) \le \sigma_k d(x, y)$ ∀ x, y ∈ X are also continuous.

Let S is same bounded orbit then S has a unique fixed point q ∈ X and for each x ∈ X the sequence $\{S^k x\}$ converges to q.

Proof Let the sequence $\{\sigma_t\}$ is uniformly convergent then σ is continuous,

for any x, y ∈ X, x≠y we have

$$\lim_{k\to\infty} \sup d(S^k x, S^k y) \le \lim_{k\to\infty} \sup \sigma_k d(x, y)$$
$$= \sigma d(x, y) < d(x, y)$$

If there exist $x, y \in X$ and $\gamma > 0$, such that

$$\lim_{k\to\infty} \sup d(S^k x, S^k y) = \in$$

Then there exist $n \in N$ such that because σ is continuous and σ(γ) ,then we get,

$$\lim_{k\to\infty} \sup d\left(S^k x,\ S^k y\right) = \lim_{k\to\infty}$$

$$\sup d[S^k(S^n x),\ S^k(T^n y)]$$

$$\leq \sup \sigma_k[d(S^n x,\ S^n y)$$

$$= \sigma[d(S^n x,\ S^n y) < \in$$

This is a contradiction,

Hence $\lim_{k\to\infty} \sup d\left(S^k x,\ S^k y\right) = 0$, for any x, y ∈ X.

Hence all sequences defines by T are equi-convergent & bounded by Picard iterates.

Let $g_0 \in X$ be an arbitrary, { g_k } be a sequence of Picard iterates of S at the point g_0.

∴ D = { g_k } and J_k = { $x \in D : d(x, S^k x) \leq 1/k$, for $k=1,2,3,\ldots$ }

Because { g_k } is bounded. Hence D is bounded, from the above limit J_k is non empty because S is continuous then we get J_k is closed for any k.

Also we have J_{k+1} subset of J_k.

Let { x_k } and { y_k } be two arbitrary sequence such that $x_k,\ y_k \in J_k$ and let { k_i } be a sequence of integers such that

$$\lim_{i\to\infty} d(xk_i, yk_i) \leq \lim_{k\to\infty}$$

$$[d(xk_i, S^{k_i} xk_i) + d(S^{k_i} xk_i, S^{k_i} yk_i) + d(yk_i, S^{k_i} yk_i)]$$

$$= \lim_{i\to\infty} \sigma k_i [d(xk_i, yk_i)]$$

$$= \sigma[\lim_{i\to\infty} d(xk_i, yk_i)]$$

Hence $\lim_{i\to\infty} d(xk_i, yk_i) = \sigma[\lim_{i\to\infty} d(xk_i, yk_i)] = 0$

Because D is bounded.

Thus $\lim_{k\to\infty} \sup d(x_k,\ y_k) = 0$ and

Hence $\lim_{k\to\infty} d(x_k,\ y_k) = 0$

This we get $\lim_{k\to\infty} diam\,(J_k) = 0.$

By completeness of D, it follows that there exist $q \in X$, such that

$$\bigcap_{k=1}^{\infty} J_k = \{q\}.$$

Hence we get $\lim_{k\to\infty} d(S^k x, q) = 0$ for any $x \in X$.

This is the complete proof of the theorem.

In the following theorem we see an important generalization of Boyd and Wong's fixed point theorem in which the function σ is extended in different direction i.e. continuity condition on σ is replaced by $\lim_{k\to\infty} \sigma^k(t) = 0$ for all t > 0.

Theorem 3.7[5] Let $S : X \to X$ be a mapping in complete metric space (x, δ) satisfies

$$d(Sx, Sy) \leq \alpha[d(x, y)] \text{ for all x, y} \in \text{X}.$$

where $\alpha : (0, \infty) \to (0, \infty)$ is non-decreasing and satisfies

$\lim_{k\to\infty} \alpha^k(t) = q\ \forall$ t > 0.

then S has a unique fixed point q ∈ X and for each x ∈ X, $\lim_{k\to\infty} S^k(t) = q$.

Proof Consider $x_0 \in X$ and let $x_k = S^k x_0$, k belongs to N then

$$0 \leq \lim_{k\to\infty} \sup d(x_k,\ x_{k+1}) \leq \lim_{k\to\infty}$$

$$\sup \alpha^k[d(x_0,\ x_1)] = 0$$

Hence $\lim_{k\to\infty} \sup d(x_k,\ x_{k+1}) = 0$ as $\alpha^k(t)$ tends to 0 for $t > 0$,

$\alpha(T) < T$ any $T > 0$,

thus $\lim_{k\to\infty} d(x_k,\ x_{k+1}) = 0$ any γ >0

then it is possible to choose n such that,

$$d(x_{k+1}, x_k) \leq \gamma - \alpha(\gamma)$$

For J belongs to $C\gamma(x_k) = \{x \in X : d(x, x_k) \leq \in\}$, we have

$$d(s_J, x_k) \leq d(s_J, sx_k) + d(sx_k, x_k)$$

$$\leq \alpha[d(J, x_k)] + d(x_{k+1}, x_k)$$

$$\leq \alpha(\gamma) + [\gamma - \alpha(\gamma)] = \gamma$$

$$\therefore\ S : C\gamma[x_k] \to C\gamma[x_k]$$

such that $d(x_u, x_v) < \gamma$ for all $u \geq v$.

Hence { x_k } is Cauchy sequence.

This completes the proof of theorem.

4. Conclusion

Thus we have to study some common fixed point theorems in complete metric space by lipschitz condition (mapping) and asymptotic contraction by taking Banach contraction principle by generalizing it prove some random fixed point theorem as a special case of Boyd and Wong's fixed point theorem by asymptotic contraction.

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Author Profile

Mr. Suhas S. Patil had completed M.Sc, B.Ed and M.Phil in Mathematics with distinction. And he is having Twelve years of teaching experience of Mathematics and pursuing Ph.D in Mathematics from Swami Ramanand Teerth Marthwada University, Nanded (MH), India.

Dr. Uttam P. Dolhare working as Associate Professor & Head, Department of Mathematics, D.S.M. College, Jintur (M.S), India