Basic Structures and Development in Continuous Function in Fuzzy Topological Spaces

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Abstract: In this article on fuzzy topological spaces we studied and the further investigated different kind of continues mappings in fuzzy topological space. In this article fuzzy sets have been defined and use of fuzzy numbers in daily life. In this article we have investigated how the fuzzy topological spaces have been defined by different authors and what is its future scope.

Keywords: Fuzzy topology, fuzzy generalized closed set, fuzzy \( g^*\beta \)-continuous function, fuzzy almost contra continuous functions

1. Introduction and Literature Survey

Almost all the information that is available today about the real world is uncertain, incomplete and imprecise. The general point of view, uncertainty may include Inaccuracy and error: deviation from true values and Vagueness: imprecision in concepts used to describe the information.

Error, as one aspect of the uncertainties, represents bias from the true values. The error is 1% if 99 out of 100 events are the true values, which is a singleton value normally. Error has been tackled using probability theory ever since the 17th century. Bayesian theory is a classical model for handling error in events.

Vagueness can be the inherent nature of an object, or result from imprecise knowledge or from the method of observation. For example we want to describe today’s weather in terms of the exact percentage of cloud cover. We just say that today is a sunny day. Now this description is vague and less specific, but often it is more useful. In order for a term such as sunny to accomplish the desired introduction of vagueness, however, we cannot use it to mean precisely 0% cloud cover. A cloud cover of 100% is not sunny, but what can we say about the cloud cover of 80%. Practically it is also not sunny. Therefore we can accept certain intermediate states such as 10% or 20% of cloud cover as sunny. But the question is up to what stage we accept this? For instance any cloud cover of 25% or less is considerably sunny. Then can we say that a cloud cover of 26% is not sunny. Clearly this is not acceptable because 1% of cloud cover will not distinguish between sunny and non-sunny day. Hence we have to assume that if the day is treated as sunny and the cloud cover is increased by 1% then it will again be treated as sunny. Now according to this hypothesis we infer that all degrees of cloud cover should be treated as sunny, without bothering the fact that how gloomy the weather looks.

In order to resolve this paradox, the term sunny may introduce vagueness by allowing some sort of gradual transition from grades (degrees) of cloud cover that are considered to be sunny and those that are not. This point of view certainly offers us a new framework of set theory, called Fuzzy set theory, which form the contents of Fuzzy Mathematics.

In the order to deal with vagueness, Zadeh proposed the famous fuzzy set theory in 1965 [23]. The fuzzy set and fuzzy logic are the most powerful tool for solving these fuzzy problems.

Zadeh generalized a fuzzy set from classical set theory by allowing intermediate situation between the whole and nothing. For a fuzzy set a membership function is defined to describe the degree of membership of an element to a class. The membership values range from 0 to 1, where 0 shows that the element does not belong to a class, 1 means “belong” and the other values indicate the degree of membership to a class. The difference between fuzzy set and crisp set lies the concept that the membership function has replaced the characteristic function of a set. A fuzzy set can represent the elements in a class with a degree of membership to that class. Fuzzy set theory has been built as a natural extension of crisp set theory. It provides a means of representation and handling the vagueness of an object and imperfectly described knowledge.

When we investigate and analyse natural phenomena, we always describe them by some terminologies of human knowledge. Many terminologies express a general characteristic of an object, i.e. they possess a definite connection and cover a large extent of certain phenomena, such as “young” and “old”, “large” and “small”. Fuzzy object are those with indeterminate boundaries. The indeterminate boundary of object refers to the fact there is some degree of membership of points belonging to that object. According to the idea and explanation of fuzzy sets, fuzzy set theory is an ideal tool for handling these natural phenomena because of its capability to represent the indeterminate boundaries of these object.

Thus a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0%, 0.8 to a cloud cover of 20%, 0.4 to a cloud cover of 30%, and 0 to a cloud cover of 75% (or more). These grades signify the degree to which each percentage of cloud cover approximates our subjective concept of sunny, and the fuzzy set itself models the semantic flexibility inherent in such a common linguistic term. The full membership and full non-membership in a fuzzy set have grading 1 and 0 respectively, we can consider the concept of Fuzzy set to be a generalization of a crisp set, because in a crisp set only two grades of membership 1 and 0 are allowed. A key difference
between crisp and fuzzy sets is their membership grade. A crisp set has a unique membership grade, whereas a fuzzy set can have an infinite number of membership grades to represent it. For fuzzy sets, the uniqueness is sacrificed, but flexibility is gained because the membership grade can be adjusted to maximize the utility for a particular application.

A lot of research work has been carried out on the theory of fuzzy sets in the recent past. There are however, several types of uncertainties other than the type represented by a fuzzy set. Understanding these various types of uncertainties and their relationship with information and complexity is currently an area of active and promising research.


Various kinds of stronger and weaker forms of fuzzy continuous maps have been studied by many mathematicians and have played very important role in the development of the theory of fuzzy topological spaces. I have also defined stronger form of some continuous functions [10], [11], [12], [13], [14], [15], [16] and [17].


2. Definitions and Properties of Fuzzy sets

2.1 Definitions

Let X be a (non empty) universal set. A fuzzy set λ:X → I in X is a mapping from X into the closed unit interval I = [0,1], for all x ∈ X. The real number λ(x) is called the grade of membership of x in fuzzy set λ.

2.2 Example

Let X = {x₁, x₂} and λ: X → I be a map defined as λ(x₁) = 0.5, λ(x₂) = 0.6. Then λ is a fuzzy set in X. It is represented as

Let X = {x₁, 0.5}, (x₂, 0.6)}.

2.3 Definition

Let λ, μ:X → I be two fuzzy sets in X. Then λ is said to be subset of μ (or λ is contained in μ) denoted as λ ⊆ μ if λ(x) ≤ μ(x), ∀x ∈ X.

2.4 Definition

Let λ: X → I be a fuzzy set in X. Then complement of λ is denoted as λc or 1 − λ and is defined to be the fuzzy set λc: X → I, defined as λc(x) = 1 − λ(x), ∀x ∈ X.

2.5 Definition

The relative complement of a fuzzy set λ of X with respect to a fuzzy set μ of X denoted by μ − λ is defined by (μ − λ)(x) = μ(x) - λ(x) provided μ(x) ≥ λ(x) ∀x ∈ X.

2.6 Definition

Let λ, μ:X → I be two fuzzy sets in X. Then union and intersection of λ and μ are defined as

(λ ∪ μ)(x) = max{λ(x), μ(x)}, ∀x ∈ X,

(λ ∩ μ)(x) = min{λ(x), μ(x)}, ∀x ∈ X.

2.7 Definition

Let λⱼ: X → I, j ∈ J where J is any index set be any arbitrary collection of fuzzy sets in X. Then their union and intersection are defined as:

Uₖ∈J λⱼ(x) = sup {λⱼ(x) : j ∈ J}, ∀x ∈ X,

∩ₖ∈J λⱼ(x) = inf {λⱼ(x) : j ∈ J}, ∀x ∈ X.

2.8 Example

Let X = {x₁, x₂} and λ, μ: X → I be two fuzzy sets in X, defined as λ(x₁) = 0.4, λ(x₂) = 0.7, μ(x₁) = 0.5 and μ(x₂) = 0.6. Then (λ ∪ μ)(x₁) = 0.5, (λ ∪ μ)(x₂) = 0.7, (λ ∩ μ)(x₁) = 0.4 and (λ ∩ μ)(x₂) = 0.6.

3. Fuzzy Topological Space

3.1 Definition

Let X be a non-empty crisp set and let τ be a collection of fuzzy sets on X satisfying the following conditions:
1) 0,1 ∈ τ, where 0: X → I, denotes the null fuzzy sets and 1: X → I denotes the whole fuzzy set.
2) Arbitrary union of members of τ is a member of τ.
3) Finite intersection of members of τ is a member of τ.

Then τ is called fuzzy topology on X and the pair (X, τ) is called fuzzy topological space.
The members of $\tau$ are called fuzzy open sets and the complement of a fuzzy open set is called fuzzy closed set. The closure of a fuzzy set $\lambda$ (denoted by $\lambda$) is the intersection of all fuzzy closed which contains $\lambda$. The interior of a fuzzy set $\lambda$ (denoted by int($\lambda$)) is the union of all fuzzy open subsets of $\lambda$. A fuzzy set $\lambda$ in $X$ is fuzzy open (resp. fuzzy closed) if and only $\text{int}(\lambda) = \lambda$ (resp. $\text{cl}(\lambda) = \lambda$).

3.2 Example

Let $X = \{x_1, x_2\}$ be a set. Let $\lambda, \mu : X \to I$ be a fuzzy set in $X$ defined as $\lambda(x_1) = 0.4, \lambda(x_2) = 0.6, \mu(x_1) = 0.5$ and $\mu(x_2) = 0.3$. Then the collection $\tau = \{0, \lambda, \mu, \lambda \cup \mu, \lambda \cap \mu, 1\}$ forms a fuzzy topology on $X$ and so the pair $(X, \tau)$ is a fuzzy topological space.

4. Generalized And Specialized Form and Specialized Form Of Fuzzy Open Sets

4.1 Definition

Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $\lambda$ in the space $X$ is called
1) Fuzzy semi-open if $\lambda \leq \text{int}(\lambda)$ and Fuzzy semi-closed if $\text{cl}(\text{int}(\lambda)) \leq \lambda$ [3].
2) Fuzzy pre-open if $\lambda \leq \text{int}(\lambda)$ and Fuzzy pre-closed if $\text{cl}(\text{int}(\lambda)) \leq \lambda$ [1].
3) Fuzzy $\alpha$-open if $\lambda \leq \text{int}(\lambda)$ and Fuzzy $\alpha$-closed if $\text{cl}(\text{int}(\lambda)) \leq \lambda$ [1].
4) Fuzzy $\beta$-open if $\lambda \leq \text{int}(\lambda)$ and Fuzzy $\beta$-closed if $\text{cl}(\text{int}(\lambda)) \leq \lambda$ [7].

In a fuzzy topological space $X$ every fuzzy semi-open and fuzzy pre-open sets are fuzzy $\beta$-open. However converse may not be true.

4.2 Remark

In the above Definition 2.1 we note that a fuzzy set $\lambda$ is fuzzy semi-open (resp. pre-open, $\alpha$-open, $\beta$-open) iff $\lambda^c$ is fuzzy semi-closed (resp. pre-closed, $\alpha$-closed, $\beta$-closed). The semi-interior (resp. pre-interior, $\alpha$-interior, $\beta$-interior) of a fuzzy set $\lambda$ in $X$ is denoted by $\text{int}(\lambda)$ (resp. $\text{pre}(\lambda), \alpha \text{-int}(\lambda), \beta \text{-int}(\lambda)$) and is defined to be the union of all fuzzy semi-open (resp. pre-open, $\alpha$-open, $\beta$-open) sets in $X$. The semi-closure (resp. pre-closure, $\alpha$-closure, $\beta$-closure) of $\lambda$ is denoted by $\text{cl}(\lambda)$ (resp. $\text{pre}(\lambda), \alpha \text{-cl}(\lambda), \beta \text{-cl}(\lambda)$) and is defined to be the intersection of all fuzzy semi-closed (resp. pre-closed, $\alpha$-closed, $\beta$-closed) sets in $X$ containing $\lambda$. The fuzzy set $\lambda$ is fuzzy semi-closed (resp. pre-closed, $\alpha$-closed, $\beta$-closed) iff $\text{int}(\lambda)^c = \lambda$ (resp. $\text{pre}(\lambda)^c = \lambda, \alpha \text{-int}(\lambda)^c = \lambda, \beta \text{-int}(\lambda)^c = \lambda$) and $\lambda$ is fuzzy semi-closed (resp. pre-closed, $\alpha$-closed, $\beta$-closed) iff $\text{cl}(\lambda)^c = \lambda$ (resp. $\text{pre}(\lambda)^c = \lambda, \alpha \text{-cl}(\lambda)^c = \lambda, \beta \text{-cl}(\lambda)^c = \lambda$).

4.3 Definition

Let $X$ and $Y$ be fuzzy topological spaces and $f : X \to Y$ be a map. Then $f$ is said to be:

1. Fuzzy continuous if $f^{-1}(\lambda)$ is fuzzy open in $X$ for every fuzzy open set $\lambda$ of $Y$ [3].
2. Fuzzy semi-continuous if $f^{-1}(\lambda)$ is fuzzy semi-open in $X$ for every fuzzy open set $\lambda$ of $Y$ [3].
3. Fuzzy pre-continuous if $f^{-1}(\lambda)$ is fuzzy $\alpha$-open in $X$ for every fuzzy semi-open set $\lambda$ of $Y$ [1].
4. Fuzzy $\alpha$-continuous if $f^{-1}(\lambda)$ is fuzzy $\alpha$-open in $X$ for every fuzzy semi-open set $\lambda$ of $Y$ [7].

Every fuzzy continuous map is fuzzy $\alpha$-continuous and every fuzzy $\alpha$-continuous map is fuzzy semi-continuous and fuzzy pre-continuous map. And every fuzzy semi-continuous and fuzzy pre-continuous are fuzzy $\beta$-continuous. However converse may not be true. Since each fuzzy open set is fuzzy semi-open and fuzzy pre-open, it follows that every fuzzy pre-continuous map is fuzzy pre-irresolute and fuzzy strongly pre-continuous. However the converse may not be true.

4.4 Stronger forms of different kind of continuous maps

Let $X$ and $Y$ be fuzzy topological spaces and $f : X \to Y$ be a map. Then $f$ is said to be:

1. Fuzzy strongly semi-continuous if $f^{-1}(\lambda)$ is fuzzy open in $X$ for every fuzzy semiopen set $\lambda$ of $Y$ [5].
2. Fuzzy semi-irresolute map is $f^{-1}(\lambda)$ is fuzzy semi-open in $X$ for every fuzzy $\alpha$-open set $\lambda$ of $Y$ [5].
3. Fuzzy strongly pre-irresolute map if $f^{-1}(\lambda)$ is fuzzy preopen in $X$ for every fuzzy semi-open set $\lambda$ of $Y$ [5].
4. Fuzzy $\alpha$-preirresolute map if $f^{-1}(\lambda)$ is fuzzy preopen in $X$ for every fuzzy $\alpha$-open set $\lambda$ of $Y$ [5].
5. Fuzzy strongly $\beta$-preirresolote map if $f^{-1}(\lambda)$ is fuzzy preopen in $X$ for every fuzzy $\beta$-open set $\lambda$ of $Y$ [5].
6. Fuzzy strongly $\alpha$-irresolute map if $f^{-1}(\lambda)$ is fuzzy open in $X$ for every fuzzy $\alpha$-open set $\lambda$ of $Y$ [5].
7. Fuzzy strongly $\beta$-irresolute map if $f^{-1}(\lambda)$ is fuzzy open in $X$ for every fuzzy $\beta$-open set $\lambda$ of $Y$ [5].

4.5 Definition 2.2

Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $\lambda$ in the space $X$ is called:

1. regular open fuzzy set $\lambda = \text{int}(\text{cl}(\lambda))$ and regular closed fuzzy set if $\lambda = \text{cl}(\text{int}(\lambda))$ [1].
2. generalized closed fuzzy set (g-closed) fuzzy set if $\text{cl}(\lambda) \leq \eta$ whenever $\lambda \leq \eta$ and $\eta$ is open fuzzy set $\text{in}(X, \tau)$ [2].
3. generalized $\alpha$-closed fuzzy set (g-$\alpha$-closed) fuzzy set if $\text{acl}(\lambda) \leq \eta$ whenever $\lambda \leq \eta$ and $\eta$ is open fuzzy set $\text{in}(X, \tau)$ [2].
4. g’ - closed fuzzy set (g’-closed) fuzzy set if $\text{cl}(\lambda) \leq \eta$ whenever $\lambda \leq \eta$ and $\lambda$ is g-open fuzzy set $\text{in}(X, \tau)$ [8].
5. g’-preclosed fuzzy set (g’-p-closed) fuzzy set if $\text{pcl}(\lambda) \leq \eta$ whenever $\lambda \leq \eta$ and $\lambda$ is g-open fuzzy set $\text{in}(X, \tau)$ [3].
6. g’-semiclosed fuzzy set (g’-s-closed) fuzzy set if $\text{sc}(\lambda) \leq \eta$ whenever $\lambda \leq \eta$ and $\lambda$ is g-open fuzzy set $\text{in}(X, \tau)$ [3].
7. \( g^*\)-alpha closed fuzzy set (\( g^*\alpha\)-closed) fuzzy set if 
\[ \alpha cl(\lambda) \leq \eta \] 
whenever \( \lambda \leq \eta \) and \( \eta \) is \( g^*\)-open fuzzy set in \( (X, \tau) \) [3].

The complement of \( g\)-closed (resp. \( gp\)-closed, \( g^*\)-closed and \( gp^*\)-closed, \( g^s\)-closed, \( g^*\alpha\)-closed) fuzzy sets are called \( g\)-open (resp. \( gp\)-open, \( g^*\)-open and \( gp^*\)-open, \( g^s\)-open, \( g^*\alpha\)-open) sets in fuzzy topological spaces.

4.6 Definition

A function \( f \) from a fuzzy topological space \((X, \tau)\) to fuzzy topological space \((Y, \sigma)\) is called:
1. **fuzzy-contrac (a) continuous** if \( f^{-1}(\lambda) \) is fuzzy closed in \( X \) for every fuzzy open set \( \lambda \) of \( Y \) [6].
2. **fuzzy contra pre-continuous** (fuzzy contra \( \alpha \)-continuous [7], fuzzy contra semi-continuous) if \( f^{-1}(\lambda) \) is fuzzy pre-closed (fuzzy \( \alpha \)-closed, fuzzy semi-closed resp.) in \( X \) for every fuzzy open set \( \lambda \) of \( Y \).
3. **fuzzy \( g \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( g \)-closed in \( X \) for every fuzzy closed set \( \lambda \) of \( Y \) [2].
4. **fuzzy \( gp \)-continuous** (fuzzy \( g^s\)-continuous, fuzzy \( g^s\alpha\)-continuous) if \( f^{-1}(\lambda) \) is fuzzy \( gp\)-closed (fuzzy \( g\alpha\)-closed, fuzzy \( g^s\)-closed resp.) in \( X \) for every fuzzy closed set \( \lambda \) of \( Y \) [7].
5. **fuzzy \( g^* \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( g^*\)-open in \( X \) for every fuzzy open set \( \lambda \) of \( Y \) [8].
6. **fuzzy \( gp^* \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( gp^*\)-open in \( X \) for every fuzzy open set \( \lambda \) of \( Y \) [3].
7. **fuzzy \( g^s \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( g^s\)-open in \( X \) for every fuzzy open set \( \lambda \) of \( Y \) [23].
8. **fuzzy \( g^s\alpha \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( g^s\alpha\)-open in \( X \) for every fuzzy open set \( \lambda \) of \( Y \) [3].
9. **fuzzy contra \( gp^* \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( gp^*\)-closed set in \( X \) for every open set \( \lambda \) in \( Y \) [19].
10. **fuzzy contra \( g^s \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( g^s\)-closed set in \( X \) for every open set \( \lambda \) in \( Y \) [20].
11. **fuzzy contra \( g^s\alpha \)-continuous** if \( f^{-1}(\lambda) \) is fuzzy \( g^s\alpha\)-closed set in \( X \) for every open set \( \lambda \) in \( Y \) [21].

5. Concluding Remark

1) History of Fuzzy topological space, fuzzy numbers has been mentioned in this article.
2) Different kind of open sets have been defined by different authors.
3) We have different types of continuous mappings and relation between them.
4) Stronger form of the continuous map and further scope of research have also been given.
5) We show that the composition fuzzy \( \alpha \)-irresolute map \( \alpha \)-preirresolute is fuzzy \( \alpha \)-preirresolute map We have obtained equivalent conditions for a map from one fuzzy topological space to another to be fuzzy \( \alpha \)-preirresolute map.
6) We have established some significant properties of fuzzy \( \alpha \)-preirresolute maps.

References


Author Profile

Dr. Madhulika Shukla received the B.S. and M.S. degrees in Mathematics from R.D.V.V. University Jabalpur (M.P.) in 1997 and 1999, respectively. After that she have done Ph.D. in 2011. She has been teaching since 1999 to till date in various engineering college. Now she is working as professor in GGITS institute Jabalpur (M.P.)