The Entry Formula for a Mixed Resultant Matrix of Three-Variable Polynomial System

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Abstract: ***Resultant matrix is one of important tools in elimination theory . one of many resultant matrices is mixed Cayley-Sylvester resultant matrix. This paper starts from the definition of mixed Cayley-Sylvester resultant matrices, for a three-variable polynomial system, and use the property of determinant to simplify the main steps to avoid polynomial division and to improve the computational efficiency of mixed Cayley-Sylvester resultant.***

Keywords: resultant matrices, elimination theory, mixed Cayley-Sylvester matrice

1. Introduction

In the multivariable polynomial ring on a number field F, a resultant for a polynomial system is an irreducible polynomial consisted of coefficients of polynomials . Resultant equals to zero means there exists at least a common root in F for the original polynomial system. Resultant is a powerful tool in elimination theory. One to find resultant method is matrix method. Currently ,there is a kind of mixed resultant matrix, known as the Cayley-Sylvester matrix. Its construction process combines Sylvester dialytic step and Dixon quotient step. As a fusion of two resultant matrices, it inherits a block structure from Sylvester dialytic step. Furthermore, a block structure means that we can restore the entire matrix only based on the discussion of a part of the matrix. This paper starts from the definition of mixed resultant matrix, and converts polynomial division to the expression of entry formula. We can use mathematical induction to prove the entry formula. Finally we can reconstruct the matrix according to the block structure.

This paper's main structure is as follows: first, we define the mixed resultant matrix, and then introduce a simplified notation to represent our main conclusion --- the entry formula of a mixed resultant matrix in three-variable polynomial system. In the fourth section, we use mathematical induction to prove the main conclusion and give a concrete example in the last section.

2. The Mixed Resultant Matrix

Giving a system of four 3-variable polynomials as follows, we set the highest degrees of each polynomial in x, y, z be respectively m, n, t , where m, n, t are non-negative integers. The monomial order here is lexicographic $(x \prec y \prec z)$:

$$\begin{cases} f(\mathrm{x,y,z}) = \sum_{i=0}^{t}\sum_{q=0}^{n}\sum_{p=0}^{m} a_{pqi} x^p y^q z^i \\ g(\mathrm{x,y,z}) = \sum_{i=0}^{t}\sum_{q=0}^{n}\sum_{p=0}^{m} b_{pqi} x^p y^q z^i \\ e(\mathrm{x,y,z}) = \sum_{i=0}^{t}\sum_{q=0}^{n}\sum_{p=0}^{m} c_{pqi} x^p y^q z^i \\ h(\mathrm{x,y,z}) = \sum_{i=0}^{t}\sum_{q=0}^{n}\sum_{p=0}^{m} d_{pqi} x^p y^q z^i \end{cases} . \ (\mathrm{a}_{pqi}, \mathrm{b}_{pqi}, \mathrm{c}_{pqi} \in \mathrm{F})$$

For the system as defined above, we firstly consider the polynomial:

$$\phi(f,g) = \frac{1}{z-\overline{z}} \begin{vmatrix} f(\mathrm{x,y,z}) & g(\mathrm{x,y,z}) \\ f(\mathrm{x,y,\overline{z}}) & g(\mathrm{x,y,\overline{z}}) \end{vmatrix},$$

where $f(\mathrm{x,y,\overline{z}})$ represents a new polynomial after a replacement of argument z by $\overline{z}$ in original polynomial. By the property of determinant , $\phi(f,g)$ is a polynomial in variables $x, y, z, \overline{z}$, and the highest degrees of four arguments are respectively $2m, 2n, t-1, t-1$. Taking $\overline{z}$ as a key argument, and gathering the coefficients of monomials 1, $\overline{z}$,……, $\overline{z}^{t-1}$, we can get t polynomials in variables x, y, z . Similarly, we can deal $\phi(f,e), \phi(f,h), \phi(g,e), \phi(g,h), \phi(e,h)$ with above process , and finally we will get a total of $6t$ polynomials in $x, y, z,$ and the highest degrees of these polynomials in $x, y, z,$ are $2m, 2n, t-1$. Secondly, multiply these polynomials by monomials $\{1, x, ..., x^{m-1} y^{2n-1}\}$, and we can get $12mnt$ polynomials in $x, y, z,$ and the degrees of which is $3m-1, 4n-1, t-1$. Rewrite these polynomials in matrix form,

$$M \square \begin{bmatrix} 1 & x & ..... & x^{3m-1} y^{4n-1} z^{t-1} \end{bmatrix}$$

The coefficient matrix M of size $12mnt \times 12mnt$ is known as the Cayley-Sylvester resultant matrix.

3. Main Conclusions

For a three-variable polynomial system(2.1), we will rewrite them in an other way.

$$\begin{cases} f(x,y,z)=\sum_{i=0}^{t}\alpha_i z^i \\ g(x,y,z)=\sum_{i=0}^{t}\beta^i z^i \\ e(x,y,z)=\sum_{i=0}^{t}\gamma^i z^i \\ h(x,y,z)=\sum_{i=0}^{t}\eta^i z^i \end{cases} \begin{cases} \alpha_i=\sum_{p=0}^{m}\sum_{q=0}^{n}a_{ipq}x^p y^q \\ \beta_i=\sum_{p=0}^{m}\sum_{q=0}^{n}b_{ipq}x^p y^q \\ \gamma_i=\sum_{p=0}^{m}\sum_{q=0}^{n}c_{ipq}x^p y^q \\ \eta_i=\sum_{p=0}^{m}\sum_{q=0}^{n}d_{ipq}x^p y^q \end{cases} \quad (3.1)$$

In order to be convenient, we introduce a notation:

$$|ij|_{\alpha\beta}=\begin{vmatrix}\alpha_i & \beta_i \\ \alpha_j & \beta_j\end{vmatrix}$$

Lemma3.1. By properties of determinant, we can get:

$|ij|_{\alpha\beta}=\sum_{0\le p\le m}\sum_{0\le q\le n}\begin{vmatrix}a_{ipq} & b_{ipq} \\ a_{jpq} & b_{jpq}\end{vmatrix}x^{2p}y^{2q}$, Similarly we can get the expression of $|ij|_{\alpha\gamma}, |ij|_{\alpha\eta}, |ij|_{\beta\gamma}, |ij|_{\beta\eta}, |ij|_{\gamma\eta}$

Theorem3.1. $\phi(\mathrm{f},\mathrm{g})$ can be expressed in the following expression:

$$\phi(f,g)=\left(\sum_{\substack{i+j=1 \\ j=0,1\le i\le t}}^{t}|ij|_{\alpha\beta}z^{i+j-1}\right)\bar{z}^0+\left(\sum_{\substack{i+j=2 \\ 0\le j\le 1,2\le i\le t}}^{t+1}|ij|_{\alpha\beta}z^{i+j-2}\right)\bar{z}+\ldots+\left(\sum_{\substack{i+j=t \\ 0\le j\le t-1,i=t}}^{2t-1}|ij|_{\alpha\beta}z^{i+j-t}\right)\bar{z}^{t-1} \quad (3.2)$$

4. The Proof of Theorem 3.1

In this section , we will prove the formula of $\phi(\mathrm{f},\mathrm{g})$ by mathematical induction, the other proofs for $\phi(\mathrm{f},\mathrm{e}),\phi(\mathrm{f},\mathrm{h}),\phi(\mathrm{g},\mathrm{e}),\phi(\mathrm{g},\mathrm{h}),\phi(\mathrm{e},\mathrm{h})$ are similar.

(i)When the highest degree of z is1, because

$$(z-\bar{z})\phi_1(f,g)=\begin{vmatrix}\alpha_0+\alpha_1 z & \beta_0+\beta_1 z \\ \alpha_0+\alpha_1\bar{z} & \beta_0+\beta_1\bar{z}\end{vmatrix}=\begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_0+\alpha_1\bar{z} & \beta_0+\beta_1\bar{z}\end{vmatrix}(z-\bar{z})$$

We can get

$$\phi_1(f,g)=\begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_0 & \beta_0\end{vmatrix}+\begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_1 & \beta_1\end{vmatrix}\bar{z}=\begin{vmatrix}\alpha_1 & \beta_1 \\ \alpha_0 & \beta_0\end{vmatrix}\bar{z}^0$$

So the statement holds.

(ii)Suppose the formula (3.2) holds, when the highest degree of z is $t-1$, namely

$$\phi_{t-1}(f,g)=\left(\sum_{\substack{i+j=1 \\ j=0,1\le i\le t-1}}^{t-1}|ij|_{\alpha\beta}z^{i+j-1}\right)\bar{z}^0+\left(\sum_{\substack{i+j=2 \\ 0\le j\le 1,2\le i\le t-1}}^{t}|ij|_{\alpha\beta}z^{i+j-2}\right)\bar{z}+\ldots+\left(\sum_{\substack{i+j=t-1 \\ 0\le j\le t-2,i=t-1}}^{2t-3}|ij|_{\alpha\beta}z^{i+j-t}\right)\bar{z}^{t-2}$$

Consider the following expression when the highest degree of z is t.

$$(z-\bar{z})\phi(f,g)=\begin{vmatrix}\sum_{i=0}^{t}\alpha_i z^i & \sum_{i=0}^{t}\beta_i z^i \\ \sum_{i=0}^{t}\alpha_i\bar{z}^i & \sum_{i=0}^{t}\beta_i\bar{z}^i\end{vmatrix}=\begin{vmatrix}\sum_{i=0}^{t-1}\alpha_i z^i & \sum_{i=0}^{t-1}\beta_i z^i \\ \sum_{i=0}^{t-1}\alpha_i\bar{z}^i & \sum_{i=0}^{t-1}\beta_i\bar{z}^i\end{vmatrix}+\begin{vmatrix}\alpha_t z^t & \beta_t z^t \\ \sum_{i=0}^{t-1}\alpha_i\bar{z}^i & \sum_{i=0}^{t-1}\beta_i\bar{z}^i\end{vmatrix}+\begin{vmatrix}\sum_{i=0}^{t-1}\alpha_i z^i & \sum_{i=0}^{t-1}\beta_i z^i \\ \alpha_t\bar{z}^t & \beta_t\bar{z}^t\end{vmatrix}+\begin{vmatrix}\alpha_t z^t & \beta_t z^t \\ \alpha_t\bar{z}^t & \beta_t\bar{z}^t\end{vmatrix}$$

In the above expression, there are four parts. Among them,

$$\begin{vmatrix}\sum_{i=0}^{t-1}\alpha_i z^i & \sum_{i=0}^{t-1}\beta_i z^i \\ \sum_{i=0}^{t-1}\alpha_i\bar{z}^i & \sum_{i=0}^{t-1}\beta_i\bar{z}^i\end{vmatrix}=(z-\bar{z})\phi_{t-1}(f,g)$$

$$\phi(f,g)=\phi_{t-1}(f,g)+\sum_{i=0}^{t-1}|ti|_{\alpha\beta}z^i\bar{z}^i\sum_{j=0}^{t-i-1}z^j\bar{z}^{t-i-1-j}$$

So we can only consider the other two parts as follows

$$\begin{vmatrix} \alpha_t z^t & \beta_t z^t \\ \sum_{i=0}^{t-1}\alpha_i \overline{z}^i & \sum_{i=0}^{t-1}\beta_i \overline{z}^i \end{vmatrix} + \begin{vmatrix} \sum_{i=0}^{t-1}\alpha_i z^i & \sum_{i=0}^{t-1}\beta_i z^i \\ \alpha_t \overline{z}^t & \beta_t \overline{z}^t \end{vmatrix}$$

$$= \sum_{i=0}^{t-1} |ti|_{\alpha\beta} z^t \overline{z}^i + \sum_{i=0}^{t-1} |it|_{\alpha\beta} z^i \overline{z}^t$$

$$= \sum_{i=0}^{t-1} |ti|_{\alpha\beta} \left(z^t \overline{z}^i - z^i \overline{z}^t \right)$$

So

$$\left(z^t \overline{z}^i - z^i \overline{z}^t \right) = z^i \overline{z}^i \left(z^{t-i} - \overline{z}^{t-i} \right)$$

$$= z^i \overline{z}^i \left(z - \overline{z} \right) \sum_{j=0}^{t-i-1} z^j \overline{z}^{t-i-1-j}$$

We can get

$$\phi(f,g) = \phi_{t-1}(f,\text{g}) + \sum_{i=0}^{t-1} |ti|_{\alpha\beta} z^i \overline{z}^i \sum_{j=0}^{t-i-1} z^j \overline{z}^{t-i-1-j}$$

Hence, by the induction hypothesis of $\phi_{t-1}(f,g)$ and simplifying the above expression, we can prove theorem.

5. Example

Give a polynomial system in $m=1$, $n=1$, $t=2$ as follows:

$$\begin{cases} f(x,y,z) = 1 + x + 2y + xy + 3z + xz + 2yz + xyz + 3z^2 \ldots \\ g(x,y,z) = 2 + x + y + 3xy + z + xz + yz + 2xyz + z^2 + \ldots \\ e(x,y,z) = 3 + 2x + y + xy + z + 2xz + yz + xyz + 2z^2 + \ldots \\ h(x,y,z) = 2 + 3x + 2y + 2xy + z + xz + yz + 3xyz + z^2 + \ldots \end{cases}$$

$$\phi(f,g) = \frac{\begin{vmatrix} 1+x+2y+xy+3z+xz+2yz+xyz+3z^2+xz^2+yz^2+xyz^2 & \ldots \\ 1+x+2y+xy+3\overline{z}+x\overline{z}+2y\overline{z}+xy\overline{z}+3\overline{z}^2+x\overline{z}^2+y\overline{z}^2+xy\overline{z} & \ldots \end{vmatrix}}{z - \ldots}$$

We take $\overline{z}$ as a key argument after expansion and elimination denominator, and arrange it,

$$\phi(f,\text{g}) = \left(5+3x+4y+8xy+x^2y+2xy^2+x^2y^2+5z+3xz+2yz+7xyz+x^2yz-y^2z-xy^2z+x^2y^2z\right)\overline{z}^0 + \left(5+3x+2y+7xy+x^2y-y^2-xy^2+x^2y^2-yz-xyz-y^2z-2xy^2z\right)\overline{z}$$

The other calculations for $\phi(f,e),\phi(f,h),\phi(g,e),\phi(g,h),\phi(e,h)$ are similar.

$$\phi(f,e) = \left(8+5x-2x^2+6y+3xy-x^2y+7z+4xz+yz+2xyz-xy^2z\right)\overline{z}^0 + \left(7+4x+y+2xy-y^2-xy^2-3z+2xz+x^2z-3yz-xyz+x^2yz-y^2z-xy^2z\right)\overline{z}$$

$$\phi(f,h) = \left(5+9x+2x^2+7y+9xy+x^2y+2y^2-xy^2-x^2y^2+5z+9xz+2x^2z+4yz+6xyz+2x^2yz-2y^2z-2xy^2z\right)\overline{z}^0 + \left(5+9x+2x^2+4y+6xy+2x^2y-2y^2-2xy^2-4yz+xyz+x^2yz-3y^2z-2xy^2z+x^2y^2z\right)\overline{z}$$

$$\phi(g,e) = \left(1+y+2xy-2x^2y-xy^2-x^2y^2+xz+x^2z+x^2yz-xy^2z-x^2y^2z\right)\overline{z}^0 + \left(-1+x+x^2+x^2y-xy^2-x^2y^2-z+x^2z-yz-xyz+2x^2yz\right)\overline{z}$$

$$\phi(g,h) = \left(2x+2x^2+y+2x^2y+y^2-5x^2y^2+2xz+2x^2z-yz+xyz-2xy^2z-2x^2y^2z\right)\overline{z}^0 + \left(2x+2x^2-y+xy+3x^2y-2xy^2-2x^2y^2-yz+x^2yz-y^2z-xy^2z+2x^2y^2z\right)\overline{z}$$

$$\phi(e,h) = \left(-1+2x+4x^2-2xy+y^2-x^2y^2+z+3xz+x^2z-yz-xyz\right)\overline{z}^0 + \left(1+3x+x^2-y-xy+z-x^2z+xyz-x^2yz-y^2z+x^2y^2z\right)\overline{z}$$

We multiply the above six polynomials by monomials $\{1,y\}$, propose coefficients of the polynomials in $\overline{z}$, and write the form of a matrix, thus

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$$\begin{bmatrix}
5&3&0&4&8&1&0&2&1&0&0&0&5&3&0&2&7&1&-1&-1&1&0&0&0\\
5&3&0&2&7&1&-1&-1&1&0&0&0&0&0&0&-1&-1&0&-1&-2&0&0&0&0\\
0&0&0&5&3&0&4&8&1&0&2&1&0&0&0&5&3&0&2&7&1&-1&-1&1\\
0&0&0&5&3&0&2&7&1&-1&-1&1&0&0&0&0&0&0&-1&-1&0&-1&-2&0\\
8&5&-2&6&3&-1&0&0&0&0&0&0&7&4&0&1&2&0&0&-1&0&0&0&0\\
7&4&0&1&2&0&-1&-1&0&0&0&0&-3&2&1&-3&-1&1&-1&-1&0&0&0&0\\
0&0&0&8&5&2&6&3&-1&0&0&0&0&0&0&7&4&0&1&2&0&0&-1&0\\
0&0&0&7&4&0&1&2&0&-1&-1&0&0&0&0&-3&2&1&-3&-1&1&-1&-1&0\\
5&9&2&7&9&1&2&-1&-1&0&0&0&5&9&2&4&6&2&-2&-2&0&0&0&0\\
5&9&2&4&6&2&-2&-2&0&0&0&0&0&0&0&-4&1&1&-3&-2&1&0&0&0\\
0&0&0&5&9&2&7&9&1&2&-1&-1&0&0&0&5&9&2&4&6&2&-2&-2&0\\
0&0&0&5&9&2&4&6&2&-2&-2&0&0&0&0&0&0&0&-4&1&1&-3&-2&1\\
1&0&0&1&2&-2&0&-1&-1&0&0&0&0&1&1&0&0&1&0&-1&-1&0&0&0\\
-1&1&1&0&0&1&0&-1&-1&0&0&0&-1&0&1&-1&-1&2&0&0&0&0&0&0\\
0&0&0&1&0&0&1&2&-2&0&-1&-1&0&0&0&0&1&1&0&0&1&0&-1&-1\\
0&0&0&-1&1&1&0&0&1&0&-1&-1&0&0&0&-1&0&1&-1&-1&2&0&0&0\\
0&2&2&1&0&2&1&0&-5&0&0&0&0&2&2&-1&1&0&0&-2&-2&0&0&0\\
0&2&2&-1&1&3&0&-2&-2&0&0&0&0&0&0&-1&0&1&-1&-1&2&0&0&0\\
0&0&0&0&2&2&1&0&2&1&0&-5&0&0&0&0&2&2&-1&1&0&0&-2&-2\\
0&0&0&0&2&2&-1&1&3&0&-2&-2&0&0&0&0&0&0&-1&0&1&-1&-1&2\\
-1&2&4&0&2&0&1&0&-1&0&0&0&1&3&1&-1&-1&0&0&0&0&0&0&0\\
1&3&1&-1&-1&0&0&0&0&0&0&0&1&0&-1&0&1&-1&-1&0&1&0&0&0\\
0&0&0&-1&2&4&0&2&0&1&0&-1&0&0&0&1&3&1&-1&-1&0&0&0&0\\
0&0&0&1&3&1&-1&-1&0&0&0&0&0&0&0&1&0&-1&0&1&-1&-1&0&1
\end{bmatrix} \square v^T$$

$$v=\begin{bmatrix}1 & x & x^2 & y & xy & x^2y & y^2 & xy^2 & x^2y^2 & y^3 & xy^3 & x^2y^3 & z & xz & x^2z & yz & xyz & x^2yz & y^2z & xy^2z & x^2y^2z & y^3z & xy^3z & x^2y^3z\end{bmatrix}$$

The coefficient matrix of 24 order is mixed Cayley-Sylvester resultant matrix of the original polynomial system. Using the formula in lemma3.1, we can get

$$|10|_{\alpha\beta} = 5+3x+4y+8xy+x^2y+2xy^2+x^2y^2$$

$$|20|_{\alpha\beta} = 5+3x+2y+7xy+x^2y-y^2-xy^2+x^2y^2$$

$|21|_{\alpha\beta} = -y-xy-y^2-2xy^2$ where $t=2$, thus

$$\phi(f,g)=\left(|10|_{\alpha\beta}+|20|_{\alpha\beta}\,z\right)\overline{z}^0+\left(|20|_{\alpha\beta}+|21|_{\alpha\beta}\,z\right)\overline{z},$$

in which the coefficients of $\overline{z}$ are respectively $|10|_{\alpha\beta}+|20|_{\alpha\beta}\,z$ and $|20|_{\alpha\beta}+|21|_{\alpha\beta}\,z$. Arranging them on the basis of given monomial order and picking up the coefficients can obtain front two row of mixed resultant matrix. If using $c\left(|10|_{\alpha\beta}\right)$, $c\left(|20|_{\alpha\beta}\right)$ and $c\left(|21|_{\alpha\beta}\right)$ represents respectively coefficient vectors , Then the mixed matrix can be expressed as:

$$\begin{bmatrix}
c(|10|_{\alpha\beta}) & c(|20|_{\alpha\beta}) & c(|10|_{\alpha\gamma}) & c(|20|_{\alpha\gamma}) & c(|10|_{\alpha\eta}) & c(|20|_{\alpha\eta}) & c(|10|_{\beta\gamma}) & c(|20|_{\beta\gamma}) & c(|10|_{\beta\eta}) & c(|20|_{\beta\eta}) & c(|10|_{\gamma\eta}) & c(|20|_{\gamma\eta}) \\
c(|20|_{\alpha\beta}) & c(|21|_{\alpha\beta}) & c(|20|_{\alpha\gamma}) & c(|21|_{\alpha\gamma}) & c(|20|_{\alpha\eta}) & c(|21|_{\alpha\eta}) & c(|20|_{\beta\gamma}) & c(|21|_{\beta\gamma}) & c(|20|_{\beta\eta}) & c(|21|_{\beta\eta}) & c(|20|_{\gamma\eta}) & c(|21|_{\gamma\eta})
\end{bmatrix}^T$$

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