Variance Reduction Technique for Multi Network of Markovian Queue

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Abstract: Variance reduction technique that is particularly well suited for simulating rare events and, more specifically, estimating rare event probabilities. Properly applied, it often results in tremendous efficiency improvements compared to direct simulation schemes. Markovian process is a process of decision making through Poisson arrivals and exponential service to facilitate easy analyzing and get usable results. It representing the number of customers in a system is known as a birth and death process, which is widely used in population models. Birth death terminology is used to represent increase and decrease in the population size. The corresponding events in queuing systems are arrivals and departures. In the present paper we consider the performance analysis of network of queues using variance reduction technique. A variance reduction technique essentially transform the underlying simulation model into a related one; the letter permitting more accurate estimation of the parameter of interest. The most variance reduction techniques typically reduce the variance by a constant factor and the exception to this rule is the importance sampling approach, which lead to dramatic variance reduction in many non-trivial cases of interest.

Keywords: Point-estimation, confidence interval, finite horizon and steady-state simulation

1. Introduction

System with more queues or more servers where the customer may be served more than once is said to be network of queues. Variance reduction technique can be viewed as a mean of utilizing known information of the models in order to obtain more accurate estimators of its performance. In fact, variance reduction technique cannot be achieved without prior knowledge of a system implies zero variance and obviates the need for simulation. Generally the more we know about the system, the more effective the variance reduction. One way of gaining this knowledge is through an initial crude simulation of the model. Result from this used simulation can then be used to formulate variance reduction technique that will be subsequently improving the accuracy of the estimator in the second simulation stage. Variance reduction techniques is basically include the control variates, stratified sampling and importance sampling, common and antithetic variates methods, conditional Monte Carlo method (for instance Fishman [7]). Rubinstein et al [21] discussed the solution for corresponding minimal variance problem, which show that if L_1 & L_2 are monotonic in the same direction in each component of vectors X = (X_1, ..., X_n) and Y = (Y_1, ..., Y_n), respectively, and if dependence is permitted only between like components, then the vectors of CRVs (common random variables) are optimal. For additional references on common and antithetic variables, see Pflug [19], Kleijnen [14], [15]. The common variates method is one of the most widely used variance reduction technique. A sampling and its applications may be found Asmussen and Rubinstein [2], Lavernag and Welch [17] and Wilson [23]. Dussault at el [6] discussed combining the stochastic counterpart and stochastic approximation, discrete events dynamic system, theory and application. L’ Ecuyer [16] defined the convergence rate for steady-state derivative estimators; Marti [18] considered structural design for stochastic optimization methods. Gal et al [8] defined on the optimality and efficiency of common random numbers, in queuing networks. Rubinstein [21] discussed Monte Carlo optimization simulation and sensitivity of queuing networks. Cao [3] defined Convergence of parameter sensitivity estimates in a stochastic environment, Chung [5] discussed on a stochastic approximation method. Glynn and Whitt [9] discussed estimating the asymptotic variance with batch mean while Glynn [10] considered Likelihood ratio gradient estimation for the sensitivity analysis for network of Markovian queue.

2. Preliminaries

The motivation of the use of common and antithetic variates in simulation, consider the example as follows: Let X and Y be variates with cumulative distribution function F_1 & F_2 respectively, then estimation of the expected value of their difference, E[X-Y], with minimal variance

\[
Var \{X-Y\} = Var \{X\} - Var \{Y\} - 2 Cov \{X,Y\} \tag{2.1}
\]

where marginal cumulative density function (cdf) of X and Y have been prescribed, it follows that the variance X-Y is minimized by maximized the covariance in (2.1). Let X and Y both are generated by the inverse transformation method as

\[
X = F_1^{-1}(U_1) = \inf \{x : F_1(x) \geq U_1\}; Y = F_2^{-1}(U_2) = \inf \{x : F_2(x) \geq U_2\} \tag{2.2}
\]

where U_1 and U_2 are uniformly distributed on (0, 1).

Definition 1.1[1] The common random variable (CRVs) are used when U_2 = U_1 and antithetic random variables are used when U_2 = 1- U_1 where U_1 ~ U(0, 1) implies U_2 = 1- U_1 ~ U(0, 1).

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Since both $F^{-1}_1$ and $F^{-1}_2$ are monotonic non-decreasing function of $U$, it is readily seen that CRVs implies
\[ \text{Cov}(F^{-1}_1(U), F^{-1}_2(U)) \geq 0; \] and consequently, variance reduction is achieved in the sense that the estimator
\[ F^{-1}_1(U) - F^{-1}_2(U) \] of $E[X-Y]$ has smaller variance, then crud Monte Carlo (CMC) estimator $X-Y = F^{-1}_1(U_1) - F^{-1}_2(U_2)$ which may be seen Whitt [22] that using CRVs does, in fact, max cov $(X, Y)$, so that var $(X-Y)$ is minimized as well, similarly var $(X+Y)$ is minimized when ARVs are used.

3. Main Results

Let $X$ and $Y$ are unidimensional variates with known marginal cdf's $F_1$ and $F_2$ respectively, and functions $L_1$ and $L_2$ are real valued monotonic function Chen and Chen [4].

**Theorem 1:** Within the set $F_2$ of all two dimensional joints cdf's of random variable pairs, $(X_1, X_2)$ there exist two dimensional distribution function $F^*(say)$ so as to
\[ \min \{L_1(X)−L_2(Y)\} \] subject to the prescribed marginal cdf's $F_1$ and $F_2$.

**Proof:** Let $L_1$ and $L_2$ are real valued monotonic functions in the same directions; the statement is proved by Gal et al [8],
\[ \min \{L_1(X)−L_2(Y)\} = \min \{L_1\{F^{-1}_1(U)\}\} \quad (3.2) \]

(3.2) it follows the use of CRVs, i.e. $U_2 = U_1 = U$, lead again to optimal variance reduction. Proof of (2.2) uses the fact that if $F^{-1}(U)$ is monotonic function, then $L\{F^{-1}(U)\}$ is monotonic as well since $F^{-1}(U)$ is. By symmetry, if $L_1$ and $L_2$ are monotonic in opposite directions, then the use of ARVs, i.e. $U_2 = 1 - U_1$ is optimal for the variance minimizing problem (3.1). Extension of variance minimizing problem of (3.1) to multidimensional case is as follows.

**Theorem 2:** Within the set of all $2n$ dimensional joint cdf's $F \in F_{2n}$ of random vector pairs, $(X, Y)$, where each random vector has independent components, then there exist a $2n$ dimensional distribution function such that $F^* \in F_{2n}$ so as to
\[ \min \{L_1(X)−L_2(Y)\} \]
subject to the prescribed $n$-dimensional distributions
\[ F_1(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} F_1(x_i) \quad \text{and} \quad F_2(y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} F_2(y_i) \]

Where $X = F^{-1}_1(U_1) = \{F^{-1}_1(U_{11}), \ldots, F^{-1}_1(U_{1n})\}$, and $Y = F^{-1}_2(U_2) = \{F^{-1}_2(U_{21}), \ldots, F^{-1}_2(U_{2n})\}$.

**Proof:** Assume in addition that within the set of all cdf's $F \in F_{2n}$ on $\mathbb{R}^2$, we permit dependence only between like components of the vectors $X$ and $Y$. The solution of corresponding minimal variance problem is defined by Rubinstein [21], which shows that if $L_1$ and $L_2$ are monotonic in the same directions in each component of the vectors $X=(X_1, X_2, \ldots, X_n)$ and $Y=(Y_1, Y_2, \ldots, Y_n)$ respectively, and if dependence is permitted only between like components, then
\[ \min_{F \in F_{2n}} \{L_1(X)−L_2(Y)\} = \min_{F \in F_{2n}} \{L_1\{F^{-1}_1(U)\}\} \quad (3.4) \]

(3.4) is attained as the solution of (3.3), when $U_2 = U_1 = U$, (here and elsewhere, vector equality is component wise). The vector of CRVs $U$ is optimal again. If $L_1$ and $L_2$ are monotonic in opposite directions, then $U_2 = 1 - U_1$ (the vector of ARVs) is optimal where $1$ is a vector of $1$'s. Finally, if $L_1$ and $L_2$ are monotonic increasing with respect to some components and decreasing with respect to the others, then one can find a proper combination of common and antithetic variates, which is again optimal for the statement (3.3). Many researchers discussed, the components of random vectors $X$ and $Y$ either dependent or the sample performance functions, $L_1(x)$ and $L_2(y)$, are not monotonic (or both). The use of CRVs (ARVs) for the case of dependent components of $X$ and $Y$ are strictly monotonic functions, $L_1$ and $L_2$ was described in Rubinstein et al [21], while the use of CRVs (ARVs) for piecewise monotonic functions, $L_1$ and $L_2$ was treated in Rubinstein [20]. For additional use of common and antithetic variates, see Pflug [19] and Kleijnen [14].

**Theorem 3:** If $L_1$ and $L_2$ are monotonic functions and that the components of the random vectors $X$ and $Y$ are independent, then CRVs (ARVs) are more accurate than CRVs (ARVs) are more accurate then crud Monte Carlo (CMC), estimator and an efficiency measure $E$ may be calculated as
\[ E = \frac{-\ell_N(1)}{-\ell_N(2)}, \] for a bridge network of queue.

**Proof:** Let $L(X)$ be sample performance defined as
\[ L(X) = \min_{j=1,2,m \in S_j} \sum_{i=1}^{n} X_i \]
(3.5)

where $S_j$ is the $j^{th}$ complete path from source to the sink of the network; $X_j: i=1, \ldots, n$, are the duration of the activities (links); and $m$ is the number of complete path in the network. Assuming that $L(X)$ is non-decreasing in each component of vector $x$. The coherent life function $L(X)$ can be written as
\[ L(X) = \max_{j=1,2,m} \min_{i \in \mathcal{S}_j} X_i \]
(3.6)

where $\mathcal{S}_j$ and $X_i$ are analogous to their counterpart in (3.6), and a coherent life function, $L(X)$, is non-decreasing in each component of the vector $x$. Let we seek to estimate the $\ell = E\{L(X)\}$, $X$ is random vector with independent
component and sample functions, (3.5) and (3.6), are monotonic function in each component of X.

An unbiased estimator of $\ell$ is the crude Monte Carlo (CMC) estimator is given by

$$\tilde{\ell}_N = \frac{1}{N} \sum_{i=1}^{N} L(X_i), \quad (3.7)$$

where $X_1, \ldots, X_N$ is an independent identically distributed (iid) sample from the cumulative distribution function (cdf) $F(x)$. An alternative unbiased estimator of $\ell$, for $N$ even, is

$$\tilde{\ell}_{N(a)} = \frac{1}{N} \sum_{i=1}^{N/2} \left(L(X_i) + L(X_i^{(a)})\right), \quad (3.8)$$

where $X_i = F^{-1}(U_i)$ and $X_i^{(a)} = F^{-1}(1-U_i)$ are generated by (2.2). The estimator $\tilde{\ell}_N$ is called an antithetic estimator of $\ell$.

Since $L(X) + L(X^{(a)})$ is a particular case of $L_1(X) - L_2(Y)$ in (3.4) [with $L_2(Y)$ replaced by $-L(X^{(a)})$], one immediately obtains $\text{var}\left[\tilde{\ell}_{N(a)}\right] \leq \text{var}\left[\tilde{\ell}_N\right].$

That is, the ARVs estimator $\tilde{\ell}_{N(a)}$, is more accurate then the CMC estimator $\tilde{\ell}_N$. Assume now that one seeks to estimates the expected value of the difference of a pair of the function $L_1(X)$ and $L_2(Y)$ that is

$$\ell = E[L_1(X) - L_2(Y)], \quad (X,Y) \in \mathbb{R}^{2n}$$

The CMC estimator of $\ell$ is

$$\tilde{\ell}_N = \frac{1}{N} \sum_{i=1}^{N} \left[L_1(X_i) - L_2(Y_i)\right], \quad (3.9)$$

while the CRV estimator of $\ell$ is

$$\tilde{\ell}_{N(c)} = \frac{1}{N} \sum_{i=1}^{N} \left[L_1(X_i) - L_2(Y_i^{(c)})\right], \quad (3.10)$$

where $X_i = F_1^{-1}(U_i)$ and $Y_i^{(c)} = F_2^{-1}(U_i)$, that is $X_i$ and $Y_i^{(c)}$ are generated by using the same uniform random vector $U_i$.

Again, from (3.4), where $X_i = \text{var}\left[\tilde{\ell}_{N(c)}\right] \leq \text{var}\left[\tilde{\ell}_{N(a)}\right]$, so that the CRV estimator $\tilde{\ell}_{N(c)}$, is more accurate then CMC estimator $\tilde{\ell}_{N(a)}$, provided that both X and Y have independent components, and both $L_1$ and $L_2$ are monotonic functions in the same direction.

Let $\tilde{\ell}_N(1)$ and $\tilde{\ell}_N(2)$ be two estimator of $\ell$. To compare $\tilde{\ell}_N(1)$ with $\tilde{\ell}_N(2)$, define an efficiency measure

$$E = \frac{t_1}{t_2} \text{var}\left[\tilde{\ell}_N(1)\right] \text{var}\left[\tilde{\ell}_N(2)\right], \quad (3.11)$$

where $t_1$ and $t_2$ are the CPU times consumed in calculating the estimators $\tilde{\ell}_N(1)$ and $\tilde{\ell}_N(2)$, respectively. To fix the ideas, let $t_1$ and $t_2$ be the CPU times required to calculate the CMC and the ARV estimators respectively. Since the ARV estimators $\tilde{\ell}_N$, needs only half as many random numbers as its CMC counterpart, $\tilde{\ell}_{N(a)}$, it is readily seen that $t_2 \leq t_1$.

Ignoring this advantage of $\tilde{\ell}_{N(a)}$, the efficiency measure (3.11) reduce to

$$E = \frac{\text{var}\left[\tilde{\ell}_N(1)\right]}{\text{var}\left[\tilde{\ell}_N(2)\right]}, \quad (3.12)$$

The following table (3.1) displays the efficiency $E$ of the estimator $\tilde{\ell}_N$ for various combinations of $\lambda_1$ in $\text{Exp}(\lambda_1)$ and $\lambda_Y$ in $\text{Exp}(\lambda_Y)$, resulting from estimating the expected sample performance $\ell$ for the bridge network of queue in figure (3.1).

<table>
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<th>$\lambda_1$</th>
<th>$\lambda_Y = 1$</th>
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<th>$\lambda_Y = 3$</th>
<th>$\lambda_Y = 4$</th>
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<td>2.62</td>
<td>4.07</td>
<td>3.24</td>
</tr>
</tbody>
</table>

**Figure 3.2:** Bridge network of queues

**Remark:** The bridge network of queue (3.5) and (3.6) become

$$L(X) = \min(X_1 + X_2, X_1 + X_3 + X_4, X_1 + X_4) \quad (3.13)$$

and

$$L(X) = \max \{ \min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_1, X_4) \} \quad (3.14)$$

Respectively. The component of five dimensional vectors $X$ and $Y$ are independent and that each of their components is distributed exponentially with parameters $\lambda_1$ and $\lambda_Y$, $i = 1, \ldots, 5$, respectively. The table (3.1) is obtained from simulation runs of 500 replications each, and the results are self-explanatory.

Example (3.3): GI/GI/1 queue: From Lindley’s equation (Gross and Harris, [11])

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\[ L_{t+1} = \max\{0, L_t + U_t\}, \quad L_0 = 0 \]

For the waiting time of the \((t+1)^{th}\) customer in a GI/GI/1 queue. Here \(U_t = Y_{1j} - Y_{2(j+1)}\); \(Y_{1j}\) and \(Y_{2j}\) are the service and interarrival times of the \(j^{th}\) customer, respectively. \(Y_{2j} = A_j - A_{j-1}\) for \(j \geq 2\); \(Y_{21} = 0\) for \(j = 1\); and \(A_j\) is the arrival time of the \(j^{th}\) customer. Since \(L_t = L(Y_{1t}, Y_{2t})\) is monotonic in each component of the two-dimensional vector, \(X_t = (Y_{1t}, Y_{2t})\), one can obtain variance reduction relative to the CMC method by using the ARV estimator (3.8).

### 4. Conclusion

In the present paper we discussed the performance analysis of network of queue via variance reduction technique. The variance-reducing techniques provide a much more precise estimator of the mean than straightforward simulation (the crude Monte Carlo technique). We consider that variance reduction technique as a mean of utilizing known information of the model in order to obtain more accurate estimator of its performance. Using table (3.1) the efficiency measure is discussed from estimating the expected sample performance for the bridge network of the queue. In comparison with straightforward simulation, these techniques (including several more complicated ones not presented here) do indeed provide a much more precise estimator with the same amount of computer time, or they provide an equally precise estimator with much less computer time. Although the example was particularly simple, it is often possible, though more difficult, to apply these techniques to much more complex problems.

### References