Variance Reduction Technique for Multi Network of Markovian Queue

Ankit Kaparwan[1], V. S. Singh[2], Jagdish C. Purohit[3]

[1]Research Scholar, Department of Statistics, HNB Garhwal University, Srinagar Garhwal

[2]Prof. Department of Statistics, S.R.T. Campus HNB Garhwal University, Srinagar Garhwal

[3]Research Scholar, Department of Statistics, HNB Garhwal University, Srinagar Garhwal

Abstract: ***Variance reduction technique that is particularly well suited for simulating rare events and, more specifically, estimating rare event probabilities. Properly applied, it often results in tremendous efficiency improvements compared to direct simulation schemes. Markovian process is a process of decision making through Poisson arrivals and exponential service to facilitate easy analyzing and get usable results. It representing the number of customers in a system is known as a birth and death process, which is widely used in population models. The birth death terminology is used to represent increase and decrease in the population size. The corresponding events in queueing systems are arrivals and departures. In the present paper we consider the performance analysis of network of queues using variance reduction technique. A variance reduction technique essentially transform the underlying simulation model into a related one; the letter permitting more accurate estimation of the parameter of interest. The most variance reduction techniques typically reduce the variance by a constant factor and the exception to this rule is the importance sampling approach, which lead to dramatic variance reduction in many non-trivial cases of interest.***

Keywords: Point-estimation, confidence interval, finite horizon and steady-state simulation

1. Introduction

System with more queues or more servers where the customer may be served more than once is said to be network of queues. Variance reduction technique can be viewed as a mean of utilizing known information of the models in order to obtain more accurate estimators of its performance. In fact, variance reduction technique cannot be achieved without prior knowledge of a system implies zero variance and obviates the need for simulation. Generally the more, we know about the system, the more affective the variance reduction. One way of gaining this knowledge is through an initial crud simulation of the model. Result from this used simulation can then be used to formulate variance reduction technique that will be subsequently improving the accuracy of the estimator in the second simulation stage. Variance reduction techniques is basically include the control variates, stratified sampling and importance sampling, common and antithetic variates methods, conditional Monte Carlo method (for instance Fishman [7]). Rubinstein et al [21] discussed the solution for corresponding minimal variance problem, which show that if L_1 & L_2 are monotonic in the same direction in each component of vectors $X = (X_1,\ldots.. X_n)$ and $Y = (Y_1,\ldots Y_n)$, respectively, and if dependence is permitted only between like components, then the vectors of CRVs (common random variables) are optimal. For additional references on common and antithetic variables, see Pflug [19], Kleijnen [14], [15]. The common variates method is one of the most widely used variance reduction technique. A sampling and its applications may be found Asmussen and Rubinstein [2], Lavenberg and Welch [17] and Wilson [23]. Dussault at el [6] discussed combining the stochastic counterpart and stochastic approximation, discrete events dynamic system, theory and application. L' Ecuyer [16] defined the convergence rate for steady-state derivative estimators; Marti [18] considered structural design for stochastic optimization methods. Gal et al [8] defined on the optimality and efficiency of common random numbers, in queueing networks. Rubinstein [21] discussed Monte Carlo optimization simulation and sensitivity of queueing networks. Cao [3] defined Convergence of parameter sensitivity estimates in a stochastic environment, Chung [5] discussed on a stochastic approximation method. Glynn and Whitt [9] discussed estimating the asymptotic variance with batch mean while Glynn [10] considered Likelihood ratio gradient estimation for the sensitivity analysis for network of Markovian queue.

2. Preliminaries

The motivation of the use of common and antithetic variates in simulation, consider the example as follows:
Let X and Y be variates with cumulative distribution function F_1 & F_2 respectively, then estimation of the expected value of their difference, E[X-Y], with minimal variance

$$\text{Var}\{X-Y\} = \text{Var}\{X\} - \text{Var}\{Y\} - 2\,\text{Cov}\{X,Y\} \quad (2.1)$$

where marginal cumulative density function (cdf) of X and Y have been prescribed, it follows that the variance X-Y is minimized by maximized the covariance in (2.1). Let X and Y both are generated by the inverse transformation method as

$$X = F_1^{-1}(U_1) = \inf\{x : F_1(x) \geq U_1\};\ Y = F_2^{-1}(U_2) = \inf\{x : F_2(x) \geq U_2\} \quad (2.2)$$

where U_1 and U_2 are uniformly distributed on (0, 1).

Definition 1.[1] The common random variable (CRVs) are used when $U_2 = U_1$ and antithetic random variables are used when $U_2 = 1- U_1$ where $U_1 \sim U(0, 1)$ implies $U_2= 1- U_1 \sim U(0, 1)$.

Since both F_1^{-1} and F_2^{-1} are monotonic non decreasing function of U, it is readily seen that CRVs implies $\mathrm{Cov}\{F_1^{-1}(U),\ F_2^{-1}(U)\} \geq 0$; and consequently, variance reduction is achieved in the sense that the estimator $F_1^{-1}(U) - F_2^{-1}(U)$ of E[X-Y] has smaller variance, then crud Monte Carlo (CMC) estimator X -Y = $F_1^{-1}(U_1) - F_2^{-1}(U_2)$ which may be seen Whitt [22] that using CRVs does, in fact, max cov {X, Y}, so that var {X-Y} is minimized as well, similarly var {X+Y} is minimized when ARVs are used.

3. Main Results

Let X and Y are unidimensional variates with known marginal cdf's F_1 and F_2 respectively, and functions L_1 and L_2 are real valued monotonic function Chen and Chen [4],

Theorem 1: Within the set $\mathcal{F}_2$ of all two dimensional joints cdf's of random variable pairs, (X_1, X_2) there exist two dimensional distribution function F^* (say) so as to

$$\min \mathrm{var}\,\{L_1(X) - L_2(Y)\} \qquad (3.1)$$

subject to the prescribed marginal cdf's F_1 and F_2.

Proof: Let L_1 and L_2 are real valued monotonic functions in the same directions; the statement is proofed by Gal et al [8], then

$$\min_{F\in \mathcal{F}_{2n}} \mathrm{var}\,\{L_1(X) - L_2(Y)\} = \mathrm{var}\,[L_1\{F_1^{-1}(U_1)\} - L_2\{F_2^{-1}(U_2)\}] \qquad (3.2)$$

from (3.2) it follows the use of CRVs, i.e. $U_2 = U_1 = U$, lead again to optimal variance reduction. Proof of (2.2) uses the fact that if L(U) is monotonic function, then $L\{F^{-1}(U)\}$ is monotonic as well since $F^{-1}(U)$ is. By symmetry, if L_1 and L_2 are monotonic in opposite directions, then the use of ARVs, i.e. $U_2 = 1 - U_1$ is optimal for the variance minimizing problem (3.1). Extension of variance minimizing problem of (3.1) to multidimensional case is as follows.

Theorem 2: Within the set of all 2n dimensional joint cdf's $F \in \mathcal{F}_{2n}$ of random vector pairs, (X,Y), where each random vector has independent components, then there exist a 2n dimensional distribution function such that $F^* \in \mathcal{F}_{2n}$ so as to

$$\text{minimize var}\,\{L_1(X) - L_2(Y)\} \qquad (3.3)$$

subject to the prescribed n- dimensional distributions

$F_1(x_1, x_2 \ldots\ldots x_n) = \prod_{i=1}^{n} F_{1i}(x_i)$ and $F_2(y_1, y_2 \ldots\ldots y_n) = \prod_{i=1}^{n} F_{2i}(y_i)$

Where $X = F_1^{-1}(U_1) = \{F_{11}^{-1}(U_{11}) \ldots\ldots\ldots\ldots F_{1n}^{-1}(U_{1n})\}$,
and $Y = F_2^{-1}(U_2) = \{F_{21}^{-1}(U_{21}) \ldots\ldots\ldots\ldots F_{2n}^{-1}(U_{2n})\}$.

Proof: Assume in addition that within the set of all cdf's $F \in \mathcal{F}_{2n}$ on R^{2n}, we permit dependence only between like components of the vectors X and Y. The solution of corresponding minimal variance problem is defined by Rubinstein [21], which shows that if L_1 and L_2 are monotonic in the same directions in each component of the vectors $X = (X_1, X_2, \ldots\ldots\ X_n)$ and $Y = (Y_1, Y_2, \ldots\ Y_n)$ respectively, and if dependence is permitted only between like components, then

$$\min_{F\in \mathcal{F}_{2n}} \mathrm{var}\,\{L_1(X) - L_2(Y)\} = \mathrm{var}\,[L_1\{F_1^{-1}(U)\} - L_2\{F_2^{-1}(U)\}] \qquad (3.4)$$

is attained as the solution of (3.3), when $U_2 = U_1 = U$, (here and elsewhere, vector equality is component wise). The vector of CRVs U is optimal again. If L_1 and L_2 are monotonic in opposite directions, then $U_2 = 1 - U_1$ (the vector of ARVs) is optimal where 1 is a vector of 1's. Finally, if L_1 and L_2 are monotonic increasing with respect to some components and decreasing with respect to the others, then one can find a proper combination of common and antithetic variates, which is again optimal for the statement (3.3). Many researchers discussed, the components of random vectors X and Y either dependent or the sample performance functions, $L_1(x)$ and $L_2(y)$, are not monotonic (or both). The use of CRVs (ARVs) for the case of dependent components of X and Y are strictly monotonic functions, L_1 and L_2 was described in Rubinstein et al [21], while the use of CRVs (ARVs) for piecewise monotonic functions, L_1 and L_2 was treated in Rubinstein [20]. For additional use of common and antithetic variates, see Pflug [19] and Kleijnen [14].

Theorem 3: If L_1 and L_2 are monotonic functions and that the components of the random vectors X and Y are independent, then CRVs (ARVs) are more accurate then crud Monte Carlo (CMC), estimator and an efficiency measure $\mathcal{E}$ may be calculated as

$$\mathcal{E} = \frac{\mathrm{var}\left[\bar{\ell}_N(1)\right]}{\mathrm{var}\left[\bar{\ell}_N(2)\right]}$$, for a bridge network of queue.

Proof: Let L(X) be sample performance defined as

$$L(X) = \min_{j=1,2..m} \sum_{i\in\Im_j} X_i \qquad (3.5)$$

where $\Im_j$ is the j^{th} complete path from source to the sink of the network; X_i: i = 1, …n, are the duration of the activities (links); and m is the number of complete path in the network. Assuming that L(x) is non-decreasing in each component of vector x. The coherent life function L(X) can be written as

$$L(X) = \max_{j=1,2..m}\ \min_{i\in\Im_j} \sum X_i \qquad (3.6)$$

where $\Im_j$ and X_i are analogous to their counterpart in (3.6), and a coherent life function, L(X), is non-decreasing in each component of the vector x. Let we seek to estimate the $\ell = E\{L(X)\}$, X is random vector with independent component and sample functions, (3.5) and (3.6), are monotonic function in each component of X.

An unbiased estimator of ℓ is the crud Monte Carlo (CMC) estimator is given by

$$\bar{\ell}_N = \frac{1}{N}\sum_{i=1}^{N} L(X_i), \quad (3.7)$$

where $X_1, \ldots\ldots X_N$ is an independent identically distributed (iid) sample from the cumulative distribution function (cdf) F(x). An alternative unbiased estimator of ℓ, for N even, is

$$\bar{\ell}_N^{(a)} = \frac{1}{N}\sum_{i=1}^{N/2}\left\{L(X_i) + L(X_i^{(a)})\right\}, \quad (3.8)$$

where $X_i = F^{-1}(U_i)$ and $X_i^{(a)} = F^{-1}(1-U_i)$ are generated by (2.2). The estimator $\bar{\ell}_N^{(a)}$ is called an antithetic estimator of ℓ.

Since $L(X)+L(X^{(a)})$ is a particular case of $L_1(X) - L_2(Y)$ in (3.4) [with $L_2(Y)$ replaced by $-L(X^{(a)})$], one immediately obtains $\mathrm{var}\left\{\bar{\ell}_N^{(a)}\right\} \le \mathrm{var}\left\{\bar{\ell}_N\right\}$.

That is, the ARVs estimator $\bar{\ell}_N^{(a)}$, is more accurate then the CMC estimator $\bar{\ell}_N$. Assume now that one seeks to estimates the expected value of the difference of a pair of the function $L_1(X)$ and $L_2(Y)$ that is

$$\ell = E\{L_1(X) - L_2(Y)\}, (X, Y) \in R^{2n}$$

The CMC estimator of ℓ is

$$\bar{\ell}_N = \frac{1}{N}\sum_{i=1}^{N}\{L_1(X_i) - L_2(Y_i)\} \quad (3.9)$$

while the CRV estimator of ℓ is

$$\bar{\ell}_N^{(c)} = \frac{1}{N}\sum_{i=1}^{N}\left\{L_1(X_i) - L_2(Y_i^{(c)})\right\} \quad (3.10)$$

where $X_i = F_1^{-1}(U_i)$ and $Y_i^{(c)} = F_2^{-1}(U_i)$, that is X_i and $Y_i^{(C)}$ are generated by using the same uniform random vector U_i.

Again, from (3.4), where $X_i = \mathrm{var}\left\{\bar{\ell}_N^{(c)}\right\} \le \mathrm{var}\left\{\bar{\ell}_N\right\}$, so that the CRV estimator $\bar{\ell}_N^{(c)}$, is more accurate then CMC estimator, $\bar{\ell}_N$, provided that both X and Y have independent components, and both L_1 and L_2 are monotonic functions in the same direction.

Let $\bar{\ell}_N(1)$ and $\bar{\ell}_N(2)$ be two estimator of ℓ. To compare $\bar{\ell}_N(1)$ with $\bar{\ell}_N(2)$, define an efficiency measure

$$\mathcal{E} = \frac{t_1 \cdot}{t_2 \cdot}\frac{\mathrm{var}\left\{\bar{\ell}_N(1)\right\}}{\mathrm{var}\left\{\bar{\ell}_N(2)\right\}}, \quad (3.11)$$

where t_1 and t_2 are the CPU times consumed in calculating the estimators $\bar{\ell}_N(1)$ and $\bar{\ell}_N(2)$, respectively. To fix the ideas, let t_1 and t_2 be the CPU times required to calculate the CMC and the ARV estimators respectively. Since the ARV estimators $\bar{\ell}_N^{(a)}$, needs only half as many random numbers as its CMC counterpart, $\bar{\ell}_N$, it is readily seen that $t_2 \le t_1$. Ignoring this advantage of $\bar{\ell}_N^{(a)}$, the efficiency measure (3.11) reduce to

$$\mathcal{E} = \frac{\mathrm{var}\left\{\bar{\ell}_N(1)\right\}}{\mathrm{var}\left\{\bar{\ell}_N(2)\right\}}, \quad (3.12)$$

The following table (3.1) displays the efficiency $\mathcal{E}$ of the estimator $\bar{\ell}_N^{(a)}$ for various combinations of λ_X in $\mathrm{Exp}(\lambda_X)$ and λ_Y in $\mathrm{Exp}(\lambda_Y)$, resulting from estimating the expected sample performance ℓ for the bridge network of queue in figure (3.1).

Table (3.1): Efficiency $\mathcal{E}$ for the bridge network of queues

λ_X	$\lambda_Y = 1$	$\lambda_Y = 2$	$\lambda_Y = 3$	$\lambda_Y = 4$	$\lambda_Y = 5$
1	2.01	1.24	1.01	1.19	1.11
2	2.93	2.95	1.72	1.59	1.26
3	1.20	3.02	4.29	1.63	1.33
4	1.06	1.89	4.01	2.40	2.28
5	1.01	1.52	2.62	4.07	3.24

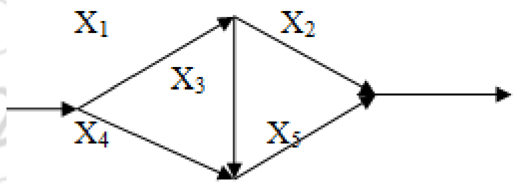

Figure 3.2: Bridge network of queues

Remark: The bridge network of queue (3.5) and (3.6) become

$$L(X) = \min(X_1 + X_2,\ X_1 + X_5 + X_4, X_3 + X_4) \quad (3.13)$$

and

$$L(X) = \max\{\min(X_1, X_2), \min(X_1, X_5, X_4), \min(X_3, X_4)\} \quad (3.14)$$

Respectively. The component of five dimensional vectors X and Y are independent and that each of their components is distributed exponentially with parameters λ_{1i} and λ_{2i}, i = 1, ….5, respectively. The table (3.1) is obtained from simulation runs of 500 replications each, and the results are self-explanatory.

Example (3.3): GI/GI/1 queue: From Lindley's equation (Gross and Harris, [11])

$$L_{t+1} = \max\{0, L_t + U_t\}, \qquad L_0 = 0$$

For the waiting time of the $(t+1)^{th}$ customer in a GI/GI/1 queue. Here $U_j = Y_{1j} - Y_{2((j+1)}$; Y_{1j} and Y_{2j} are the service and the interarrival times of the j^{th} customer, respectively.

$Y_{2j} = A_j - A_{j-1}$ for $j \geq 2$, $Y_{2j} = 0$ for $j = 1$; and A_j is the arrival time of the j^{th} customer. Since $L_t = L(Y_{1t}, Y_{2t})$ is monotonic in each component of the two dimensional vector, $X_t = (Y_{1t}, Y_{2t})$, one can obtain variance reduction relative to the CMC method by using the ARV estimator (3.8).

4. Conclusion

In the present paper we discussed the performance analysis of network of queue via variance reduction technique. The variance-reducing techniques provide a much more precise estimator of the mean than straightforward simulation (the crude Monte Carlo technique). We consider that variance reduction technique as a mean of utilizing known information of the model in order to obtain more accurate estimator of its performance. Using table (3.1) the efficiency measure is discussed from estimating the expected sample performance for the bridge network of the queue. In comparison with straightforward simulation, these techniques (including several more complicated ones not presented here) do indeed provide a much more precise estimator with the same amount of computer time, or they provide an equally precise estimator with much less computer time. Although the example was particularly simple, it is often possible, though more difficult, to apply these techniques to much more complex problems.

References

[1] Abate, J. and Whitt, W.: Transient behavior of regulated Brownian motion starting at origin, Adv. Appl. Prob., 19 (1987), 560-598.
[2] Asmussen, S. and Rubinstein, R.Y.: The efficiency and heavy traffic properties of the score function method in sensitivity analysis of queueing models, Adv. Prob., 24(1), (1992), 172-201.
[3] Cao, X. R.: Convergence of parameter sensitivity estimates in a stochastic environment, IEEE Trans. autom. Cont. 30, (1985), 845-853.
[4] Chen, L. and Chen, C.L.: A Fast Simulation Approach for Tandem Queueing Systems, Proc., Winter Simul. Conf., (1990), 539–546.
[5] Chung, K.L.: On a stochastic approximation method, Ann. Math. Statist., 25, (1954), 463–483.
[6] Dussault, J.P., Labrecque, D., L'Ecuyer, P. and Rubinstein, R. Y.: Combining the stochastic counterpart and stochastic approximation, Discrete Event Dynamic System, Theory and Application, 7(1), (1997), 5-28.
[7] Fishman, G.S.: Monte Carlo: Concepts, algorithms and applications, Springer, New York, 1996.
[8] Gal, S., Rubinstein, R.Y. and Ziv, A.: On the optimality and efficiency of common random numbers, Math. Comp. Simu., 26, (1984), 502-512.
[9] Glynn, P.W. and Whitt, W.: Estimating the Asymptotic variance with batch mean, Oper. Res. Lett., 10 (1991), 431-435.
[10] Glynn, P.W.: Likelihood ratio gradient estimation: an overview, Proc. Winter Simul. Conf., (1987), 366–374.
[11] Gross, D. and Carl M. Harris: Fundamentals of Queuing Theory, Second Edition, John Wiley & Sons, New York, 1985.
[12] Haddock, J. and Bengu, G.: Application of a simulation optimization system for a continuous review inventory model, Proc. Winter Simul. Conf., (1987), 383–390.
[13] Ho, Y.C.: Performance Evaluation and Perturbation Analysis of Discrete Event Dynamic Systems, IEEE Trans. Automat. Control, AC-32(7), (1987), 463–572.
[14] Kleijnen, J.P.C.: Analysis of simulation with common random number: A note on Heikes et. al., Simuletter, 11, (1976), 7-13.
[15] Kleijnen, J.P.C.: Statistical techniques in simulation, Pt. I, Marcel Dekker, New York, 1974.
[16] L' Ecuyer, P.L.: Convergence rates for steady-state derivative estimators, Ann. Oper. Res., 39, (1992), 121-137.
[17] Lavenberg, S.S. and Welch, P.W.: A perspective on the use of control variables with application to queueing network simulation, Oper. Res., 30, (1981), 322-335.
[18] Marti, K.: Stochastic optimization methods of structural design, ZAMM, 4, (1990), 7742-7745.
[19] Pflug, G. C.: Simulation and Optimization: The Interface Between Simulation and Optimization, Kluwer, 1995.
[20] Rubinstein, R. Y.: Sensitivity analysis of computer simulation models via the score function, Ops Res. 37, (1989), 72-81.
[21] Rubinstein, R.Y., and Marcus, R.: Efficiency of multivariate control variate in Monte Carlo simulation, Oper. Res., 33, (1996), 661-667.
[22] Whitt, W.: Bivariate distribution with given marginal, Annal of Stat., 4, (1976), 1280-1289.
[23] Wilson, J.R.: Variance reduction technique for digital simulation, J. Manag. Sci., 4, (1984), 277-312.