

# K-Even Sequential Harmonious Labeling of Some Cycle Related Graphs

B. Gayathri<sup>1</sup>, D. Muthuramakrishnan<sup>2</sup>

<sup>1</sup>Periyar E.V.R. College, Tiruchirappalli, India

<sup>2</sup>National College, Tiruchirappalli, India

**Abstract:** *Graham and Sloane [6] introduced the harmonious graphs and Singh & Varkey [7] introduced the odd sequential graphs. Gayathri & Hemalatha [2] introduced even sequential harmonious labeling of graphs. We studied even sequential harmonious labeling of trees in [3]. In [4] we have extended this notion to k-even sequential harmonious labeling graphs. It is further studied in [5]. Here, we investigate the k-even sequential harmonious labeling of some cycle related graphs.*

**Keywords:** *k*-ESHL, *k*-ESHG.

**AMS Subject Classification:** 05C78.

## 1. Introduction

All graphs in this paper are finite, simple and undirected. The symbols  $V(G)$  and  $E(G)$  denote the vertex set and the edge set of a graph  $G$ .

The cardinality of the vertex set is called the order of  $G$ . The cardinality of the edge set is called the size of  $G$ . A graph with  $p$  vertices and  $q$  edges is called a  $(p, q)$  graph.

A **graph labeling** is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices or edges then the labeling is called **vertex or edge labeling**.

Graph labeling was first introduced in late 1960's. In the recent years, dozens of graph labeling techniques have been studied in over 1200 papers.

Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing etc. [7 Gallian]

We say that a labeling is an *k*-even sequential harmonious labeling if there exists an injection  $f$  from the vertex set  $V$  to  $\{k-1, k+1, \dots, k+2q-1\}$  such that the induced mapping  $f^*$  from the edge set  $E$  to  $\{2k, 2k+2, 2k+4, \dots, 2k+2q-2\}$

defined by  $f^*(uv) = \begin{cases} f(u)+f(v), & \text{if } f(u)+f(v) \text{ is even} \\ f(u)+f(v)+1, & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$  are distinct.

A graph  $G$  is said to be an *k*-even sequential harmonious graph if it admits an *k*-even sequential harmonious labeling. In this paper, we investigate some results on *k*-even sequential harmonious labeling of some cycle related graphs. Throughout this paper,  $k$  denote any positive integer  $\geq 1$ . For brevity, we use *k*-ESHL for *k*-even sequential harmonious labeling.

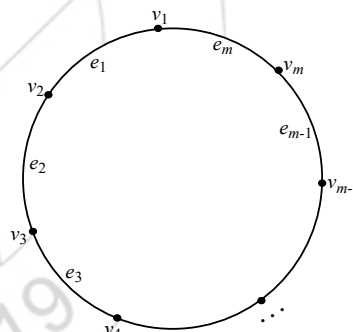
## 2. Main Results

### Theorem 3.2

The cycle  $C_m$  is a *k*-even sequential harmonious graph for  $m \geq 3$ .

### Proof

Let  $\{v_1, v_2, \dots, v_m\}$  be the vertices and  $\{e_1, e_2, \dots, e_m\}$  be the edges of  $C_m$  which are denoted as in Fig. 3.1.



**Figure 3.1:**  $C_n$  with ordinary labeling

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k+1, \dots, k+2q-1\}$  by

$$f(v_i) = \begin{cases} k+2i-3 & 1 \leq i \leq \frac{m+1}{2}, \text{ if } m \text{ is odd} \\ k+2m-2(i-1) & 1 \leq i \leq \frac{m+2}{2}, \text{ if } m \text{ is even} \\ \frac{m+3}{2} \leq i \leq m, \text{ if } m \text{ is odd} \\ \frac{m+4}{2} \leq i \leq m, \text{ if } m \text{ is even} \end{cases}$$

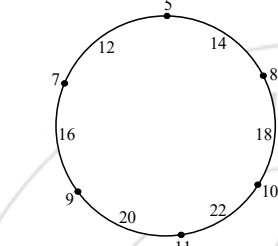
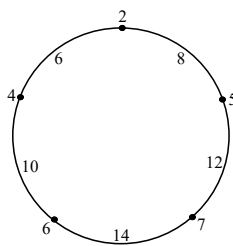
Then the induced edge labels are:

$$f^+(e_i) = \begin{cases} 2k + 4(i-1) & 1 \leq i \leq \frac{m+1}{2}, \text{ if } m \text{ is odd} \\ 2k + 4(i-1) & 1 \leq i \leq \frac{m}{2}, \text{ if } m \text{ is even} \\ 2k + 4(m-i) + 2 & \frac{m+3}{2} \leq i \leq m, \text{ if } m \text{ is odd} \\ 2k + 4(m-i) + 2 & \frac{m+2}{2} \leq i \leq m, \text{ if } m \text{ is even} \end{cases}$$

Therefore,  $f^+(E) = \{2k, 2k + 2, \dots, 2k + 2q - 2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph  $C_m$  ( $m \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Illustration 3.3**

3-ESHL of  $C_5$  and 6-ESHL of  $C_6$  are shown in Fig. 3.2(a) and Fig. 3.2(b) respectively.



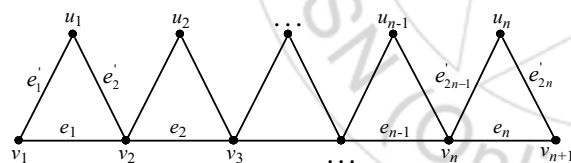
**Figure 3.2(a):** 3-ESHL of  $C_5$       **Figure 3.2(b):** 6-ESHL of  $C_6$

**Theorem 3.4**

The triangular snake  $T_n$  ( $n \geq 2$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n\}$  be the vertices and  $\{e_1, e_2, \dots, e_n, e'_1, e'_2, \dots, e'_{2n}\}$  be the vertices of  $T_n$  which are denoted as in Fig. 3.3.



**Figure 3.3:**  $T_n$  with ordinary labeling

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by

For  $1 \leq i \leq n+1$ ,

$$f(v_i) = k + i - 2$$

For  $1 \leq i \leq n$ ,

$$f(u_i) = k + 2n + 3i - 2$$

Then the induced edge labels are as follows

For  $1 \leq i \leq n$ ,

$$f^+(e_i) = 2k + 2(i-1)$$

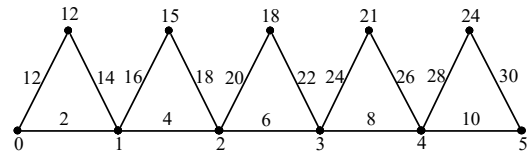
For  $1 \leq i \leq 2n$ ,

$$f^+(e'_i) = 2k + 2n + 2(i-1)$$

Therefore,  $f^+(E) = \{2k, 2k + 2, \dots, 2k + 2q - 2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph  $T_n$  is a  $k$ -even sequential harmonious graph for any  $k$ .

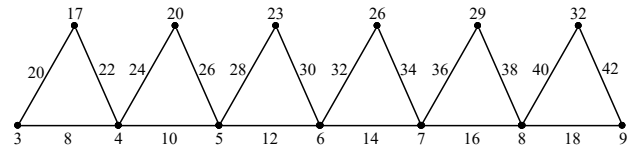
**Illustration 3.5**

(a) 1-ESHL of  $T_5$  is shown in Fig. 3.4(a).



**Figure 3.4(a):** 1-ESHL of  $T_5$

(b) 4-ESHL of  $T_6$  is shown in Fig. 3.4(b).



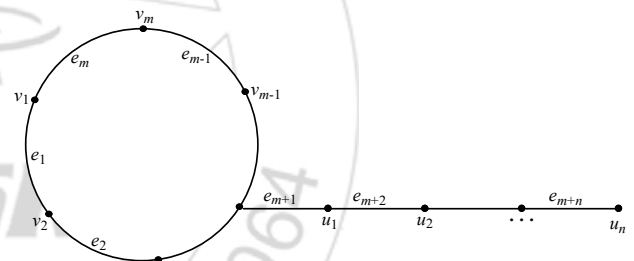
**Figure 3.4(b):** 4-ESHL of  $T_6$

**Theorem 3.6**

The graph  $C_m @ P_n$  ( $m \geq 3, n \geq 1$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{v_i, u_j; 1 \leq i \leq m, 1 \leq j \leq n\}$  be the vertices and  $\{e_i; 1 \leq i \leq m+n\}$  be the edges where  $\begin{cases} e_{m+1} = (v_i u_1), i = \frac{m+3}{2}, \text{ if } m \text{ is odd} \\ i = \frac{m+2}{2}, \text{ if } m \text{ is even} \end{cases}$  which are denoted as in Fig. 3.5.



**Figure 3.5:**  $C_m @ P_n$  with ordinary labeling

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by

$$f(v_i) = \begin{cases} k + 2i - 3 & 1 \leq i \leq \frac{m+1}{2}, \text{ if } m \text{ is odd} \\ k + 2i - 3 & 1 \leq i \leq \frac{m+2}{2}, \text{ if } m \text{ is even} \\ k + 2m - 2(i-1) & \frac{m+3}{2} \leq i \leq m, \text{ if } m \text{ is odd} \\ k + 2m - 2(i-1) & \frac{m+4}{2} \leq i \leq m, \text{ if } m \text{ is even} \end{cases}$$

For  $1 \leq j \leq n$ ,

$$f(u_j) = \begin{cases} m + k + j - 2, & \text{if } j \text{ is even} \\ m + j + k, & \text{if } j \text{ is odd} \end{cases}$$

Then the induced edge labels are:

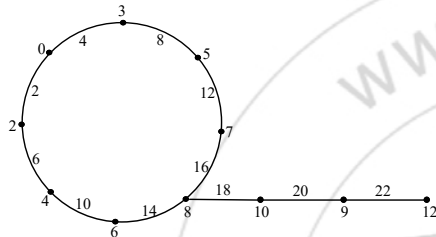
$$f^+(e_i) = \begin{cases} 2k+4(i-1) & 1 \leq i \leq \frac{m+1}{2}, \text{ if } m \text{ is odd} \\ 2k+4(i-1) & 1 \leq i \leq \frac{m}{2}, \text{ if } m \text{ is even} \\ 2k+4(m-i)+2 & \frac{m+3}{2} \leq i \leq m, \text{ if } m \text{ is odd} \\ 2k+4(m-i)+2 & \frac{m+2}{2} \leq i \leq m, \text{ if } m \text{ is even} \end{cases}$$

For  $m+1 \leq i \leq m+n$ ,  
 $f^+(e_i) = 2i+2k-2$

Therefore,  $f^+(E) = \{2k, 2k+2, \dots, 2k+2q-2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph  $C_m @ P_n$  ( $m \geq 3, n \geq 1$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

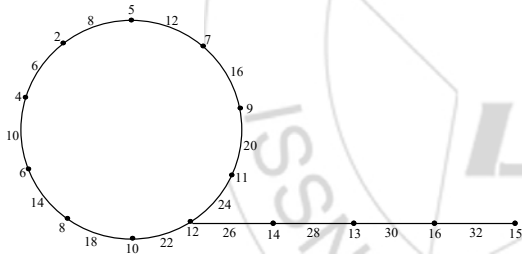
**Illustration 3.7**

(a) 1-ESHL of  $C_8 @ P_3$  is shown in Fig. 3.6(a).



**Figure 3.6(a): 1-ESHL of  $C_8 @ P_3$**

(b) 3-ESHL of  $C_{10} @ P_4$  is shown in Fig. 3.6(b).



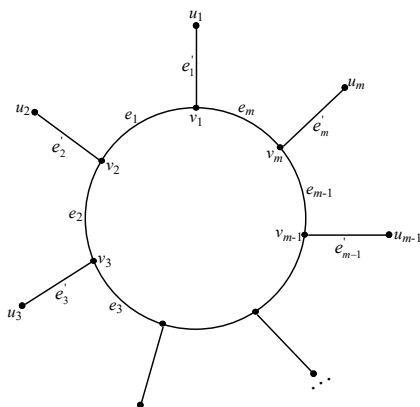
**Figure 3.6(b): 3-ESHL of  $C_{10} @ P_4$**

**Theorem 3.8**

The crown  $C_m \odot K_1$  ( $m \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{v_i, u_i, 1 \leq i \leq m\}$  be the vertices and  $\{e_i, e'_i; 1 \leq i \leq m\}$  be the edges which are denoted as in Fig. 3.7.



**Figure 3.7:  $C_m @ K_1$  with ordinary labeling**

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by

$$f(v_i) = \begin{cases} k+2i-3 & 1 \leq i \leq \frac{m+2}{2}, \text{ when } m \text{ is even} \\ k+2i-3 & 1 \leq i \leq \frac{m+1}{2}, \text{ when } m \text{ is odd} \\ k+2m-2i+2 & \frac{m+4}{2} \leq i \leq m, \text{ when } m \text{ is even} \\ k+2m-2i+2 & \frac{m+3}{2} \leq i \leq m, \text{ when } m \text{ is odd} \end{cases}$$

$f(u_i) = k+2m$

$$f(u_i) = \begin{cases} k+2m+2i-3 & 2 \leq i \leq \frac{m+2}{2}, \text{ when } m \text{ is even} \\ k+2m+2i-3 & 2 \leq i \leq \frac{m+1}{2}, \text{ when } m \text{ is odd} \\ k+4m-2i+2 & \frac{m+4}{2} \leq i \leq m, \text{ when } m \text{ is even} \\ k+4m-2i+2 & \frac{m+3}{2} \leq i \leq m, \text{ when } m \text{ is odd} \end{cases}$$

Then the induced edge labels are:

$$f^+(e_i) = \begin{cases} 2k+4(i-1) & 1 \leq i \leq \frac{m}{2}, \text{ when } m \text{ is even} \\ 2k+4(i-1) & 1 \leq i \leq \frac{m+1}{2}, \text{ when } m \text{ is odd} \\ 2k+4m-4i+2 & \frac{m+2}{2} \leq i \leq m, \text{ when } m \text{ is even} \\ 2k+4m-4i+2 & \frac{m+3}{2} \leq i \leq m, \text{ when } m \text{ is odd} \end{cases}$$

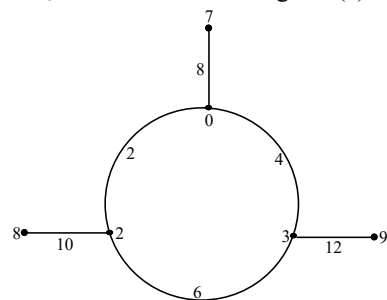
$f^+(e'_i) = 2k+2m$

$$f^+(e'_i) = \begin{cases} 2k+2m+4i-6 & 2 \leq i \leq \frac{m+2}{2}, \text{ when } m \text{ is even} \\ 2k+2m+4i-6 & 2 \leq i \leq \frac{m+1}{2}, \text{ when } m \text{ is odd} \\ 2k+6m-4i+4 & \frac{m+4}{2} \leq i \leq m, \text{ when } m \text{ is even} \\ 2k+6m-4i+4 & \frac{m+3}{2} \leq i \leq m, \text{ when } m \text{ is odd} \end{cases}$$

Therefore,  $f^+(E) = \{2k, 2k+2, \dots, 2k+2q-2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph  $C_m \odot K_1$  ( $m \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

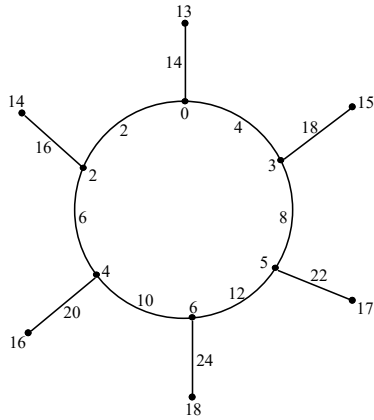
**Illustration 3.9**

(a) 1-ESHL of  $C_3 \odot K_1$  is shown in Fig. 3.8(a).



**Figure 3.8(a): 1-ESHL of  $C_3 \odot K_1$**

(b) 1-ESHL of  $C_6 \odot K_1$  is shown in Fig. 3.8(b).



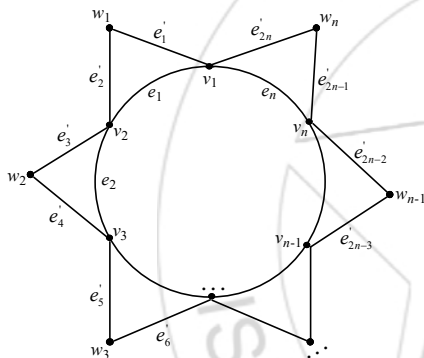
**Figure 3.8(b):** 1-ESHL of  $C_6 \odot K_1$

**Theorem 3.10**

The sunflower graph  $SF(n)$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{v_i, w_i, 1 \leq i \leq n\}$  be the vertices and  $\{e_i, 1 \leq i \leq n, e'_i, 1 \leq i \leq 2n\}$  be the edges which are denoted as in Fig. 3.9.



**Figure 3.9:** Ordinary labeling of  $SF(n)$

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by

**Case (i):  $n$  is odd**  
 For  $1 \leq i \leq \frac{n+1}{2}$ ,

$$f(v_i) = k + 2i - 3$$

For  $\frac{n+3}{2} \leq i \leq n$ ,

$$f(v_i) = k + 2n - 2i + 2$$

For  $1 \leq i \leq \frac{n+1}{2}$ ,

$$f(w_i) = k + 2n + 2i - 1$$

For  $\frac{n+3}{2} \leq i \leq n$ ,

$$f(w_i) = k + 6i - 4$$

Then the induced edge labels are:

For  $1 \leq i \leq \frac{n+1}{2}$ ,

$$f^+(e_i) = 2k + 4i - 4$$

For  $\frac{n+3}{2} \leq i \leq n$ ,

$$f^+(e_i) = 2k + 4n - 4i + 2$$

For  $1 \leq i \leq n+1$ ,

$$f^+(e'_i) = 2k + 2n + 2(i-1)$$

For  $n+2 \leq i \leq 2n$ ,

$$f^+(e'_i) = \begin{cases} 2k + 2n + 2i & i \text{ odd} \\ 2k + 2n + 2i - 4 & i \text{ even} \end{cases}$$

**Case (ii):  $n$  is even**

For  $1 \leq i \leq \frac{n+2}{2}$ ,

$$f(v_i) = k + 2i - 3$$

For  $\frac{n+4}{2} \leq i \leq n$ ,

$$f(v_i) = k + 2n - 2i + 2$$

For  $1 \leq i \leq \frac{n}{2}$ ,

$$f(w_i) = k + 2n + 2i - 1$$

For  $\frac{n+2}{2} \leq i \leq n$ ,

$$f(w_i) = k + 6i - 4$$

Then the induced edge labels are:

For  $1 \leq i \leq \frac{n}{2}$ ,

$$f^+(e_i) = 2k + 4i - 4$$

For  $\frac{n+2}{2} \leq i \leq n$ ,

$$f^+(e_i) = 2k + 4n - 4i + 2$$

For  $1 \leq i \leq n$ ,

$$f^+(e'_i) = 2k + 2n + 2(i-1)$$

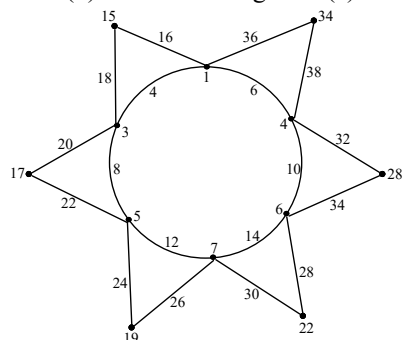
For  $n+1 \leq i \leq 2n$ ,

$$f^+(e'_i) = \begin{cases} 2k + 2n + 2i & i \text{ odd} \\ 2k + 2n + 2i - 4 & i \text{ even} \end{cases}$$

Therefore,  $f^+(E) = \{2k, 2k+2, \dots, 2k+2q-2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the sunflower graph  $SF(n)$  is a  $k$ -even sequential harmonious graph for any  $k$ .

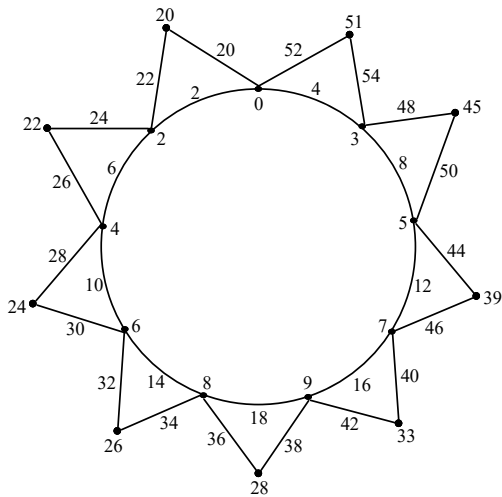
**Illustration 3.11**

(a) 2-ESHL of  $SF(6)$  is shown in Fig. 3.10(a).



**Figure 3.10(a):** 2-ESHL of  $SF(6)$

(b) 4-ESHL of  $SF(9)$  is shown in Fig. 3.10(b).



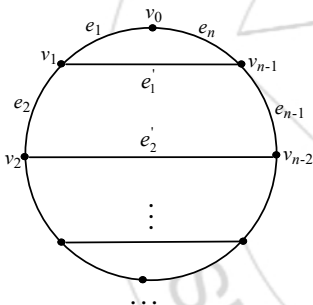
**Figure 3.10(b): 1-ESHL of  $SF(9)$**

**Theorem 3.12**

The graph  $PC_n$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$  when  $n$  is even.

**Proof**

Let  $\{v_0, v_1, \dots, v_{n-1}\}$  be the vertices and  $\{e_i, 1 \leq i \leq n, e'_i, 1 \leq i \leq \frac{n-2}{2}\}$  be the edges which are denoted as in Fig. 3.11.



**Figure 3.11:  $PC_n$  with ordinary labeling**

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by

$$f(v_i) = \begin{cases} k-1 & i=0 \\ 3i+k-2 & 1 \leq i \leq \frac{n-2}{2} \\ k+3n-3i-1 & \frac{n}{2} < i \leq n-1 \end{cases}$$

Then the induced edge labels are:

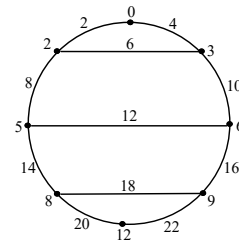
$$f^+(e_i) = \begin{cases} 6i+2k-6 & 1 \leq i \leq \frac{n}{2} \\ 2k+6n-6i+2 & \frac{n+2}{2} \leq i \leq n \end{cases}$$

$$f^+(e'_i) = 2k+6i-2 \quad 1 \leq i \leq \frac{n-2}{2}$$

Therefore,  $f^+(E) = \{2k, 2k+2, \dots, 2k+2q-2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph  $PC_n$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

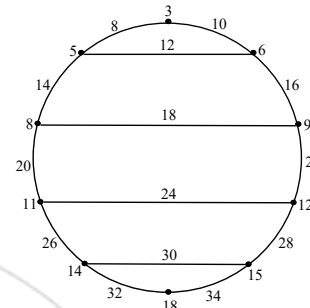
**Illustration 3.13**

(a) 1-ESHL of  $PC_8$  is shown in Fig. 3.12(a).



**Figure 3.12(a): 1-ESHL of  $PC_8$**

(b) 4-ESHL of  $PC_{10}$  is shown in Fig. 3.12(b).



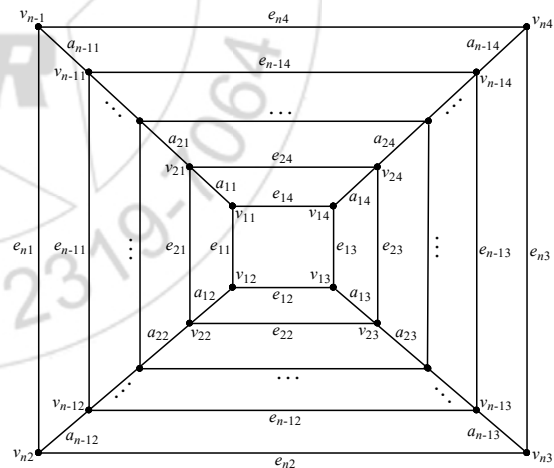
**Figure 3.12(b): 4-ESHL of  $PC_{10}$**

**Theorem 3.16**

The graph  $CH_4^n$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{v_{i1}, v_{i2}, v_{i3}, v_{i4}, 1 \leq i \leq n\}$  be the vertices and  $\{e_{i1}, e_{i2}, e_{i3}, e_{i4}, 1 \leq i \leq n, a_{i1}, a_{i2}, a_{i3}, a_{i4}, 1 \leq i \leq n-1\}$  be the edges which are denoted as in Fig. 3.15.



**Figure 3.15: Ordinary labeling of  $CH_4^n$**

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by For  $1 \leq i \leq n$ ,

$$f(v_{i1}) = \begin{cases} 8(i-1)+k-1 & i \text{ odd} \\ 8i-6+k-1 & i \text{ even} \end{cases}$$

$$f(v_{i3}) = \begin{cases} 8i-4+k-1 & i \text{ odd} \\ 8i-5+k-1 & i \text{ even} \end{cases}$$

$$f(v_{i4}) = \begin{cases} 8i-5+k-1 & i \text{ odd} \\ 8(i-1)+k-1 & i \text{ even} \end{cases}$$

$$f(v_{i2}) = k+1$$

$$f(v_{32}) = k + 16$$

$$f(v_{22}) = k + 11$$

For  $3 \leq i \leq n$ ,

$$f(v_{i2}) = \begin{cases} 8i - 4 + k - 1 & i \text{ even} \\ 8i - 6 + k - 1 & i \text{ odd} \end{cases}$$

Then the induced edge labels are:

For  $1 \leq i \leq n$ ,

$$f^+(e_{i1}) = \begin{cases} 16i - 14 + 2k - 2 & i \text{ odd} \\ 16i - 10 + 2k - 2 & i \text{ even} \end{cases}$$

$$f^+(e_{i2}) = \begin{cases} 16i - 10 + 2k - 2 & i \text{ odd} \\ 16i - 8 + 2k - 2 & i \text{ even} \end{cases}$$

$$f^+(e_{i3}) = \begin{cases} 16i - 8 + 2k - 2 & i \text{ odd} \\ 16i - 12 + 2k - 2 & i \text{ even} \end{cases}$$

$$f^+(e_{i4}) = \begin{cases} 16i - 12 + 2k - 2 & i \text{ odd} \\ 16i - 14 + 2k - 2 & i \text{ even} \end{cases}$$

For  $1 \leq i \leq n - 1$ ,

$$f^+(a_{i1}) = 16i + 2k - 8$$

$$f^+(a_{i2}) = 16i + 2k - 4$$

$$f^+(a_{i3}) = 16i + 2k - 2$$

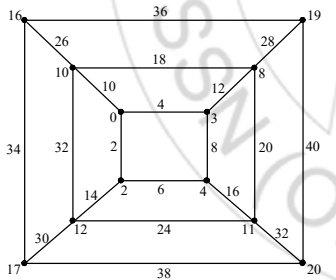
$$f^+(a_{i4}) = 16i + 2k - 6$$

Therefore,  $f^+(E) = \{2k, 2k + 2, 2k + 4, \dots, 2k + 2q - 2\}$ .

So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph  $CH_n$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

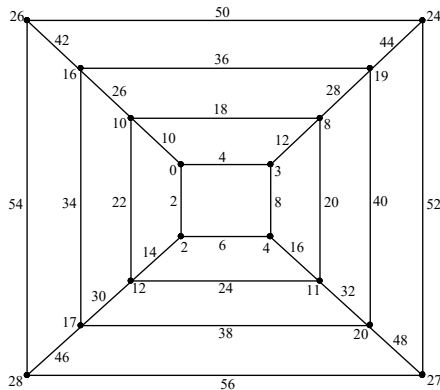
**Illustration 3.17**

(a) 1-ESHL of  $CH_4^3$  is shown in Fig. 3.16(a).



**Figure 3.16(a):** 1-ESHL of  $CH_4^3$

(b) 1-ESHL of  $CH_4^4$  is shown in Fig. 3.16(b).



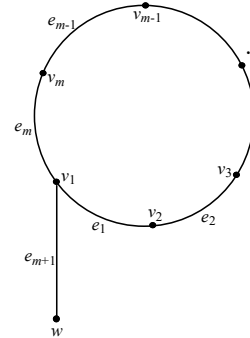
**Figure 3.16(b):** 1-ESHL of  $CH_4^4$

**Theorem 3.18**

The graph Flag  $Fl_m$  ( $m \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{v_i, w, 1 \leq i \leq m\}$  be the vertices and  $\{e_i, 1 \leq i \leq m + 1, e_{m+1} = (v_1w)\}$  be the edges which are denoted as in Fig. 3.17.



**Figure 3.17:** Flag  $Fl_m$  with ordinary labeling

First we label the vertices as follows:

Define  $f: V \rightarrow \{k - 1, k, k + 1, \dots, k + 2q - 1\}$  by

$$f(v_i) = \begin{cases} 2(i-1) + k - 1 & 1 \leq i \leq \frac{m+1}{2}, \text{ if } m \text{ is odd} \\ 2(i-1) + k - 1 & 1 \leq i \leq \frac{m+2}{2}, \text{ if } m \text{ is even} \\ \frac{m+3}{2} \leq i \leq m, \text{ if } m \text{ is odd} \\ \frac{m+4}{2} \leq i \leq m, \text{ if } m \text{ is even} \end{cases}$$

$$f(w) = 2m + k$$

Then the induced edge labels are:

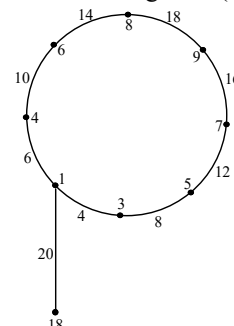
$$f^+(e_i) = \begin{cases} 4i + 2k - 4 & 1 \leq i \leq \frac{m+1}{2}, \text{ if } m \text{ is odd} \\ 4i + 2k - 4 & 1 \leq i \leq \frac{m}{2}, \text{ if } m \text{ is even} \\ 2k + 4m - 4i + 2 & \frac{m+3}{2} \leq i \leq m, \text{ if } m \text{ is odd} \\ 2k + 4m - 4i + 2 & \frac{m+2}{2} \leq i \leq m, \text{ if } m \text{ is even} \end{cases}$$

$$f^+(v_1w) = 2(m + 1) + 2k - 2$$

Therefore,  $f^+(E) = \{2k, 2k + 2, 2k + 4, \dots, 2k + 2q - 2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the graph flag  $Fl_m$  ( $m \geq 3$ ) is an  $k$ -even sequential harmonious graph for any  $k$ .

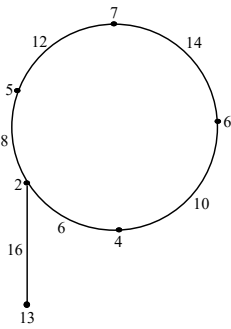
**Illustration 3.19**

(a) 2-ESHL of  $Fl_8$  is shown in Fig. 3.18(a).



**Figure 3.18(a):** 2-ESHL of  $Fl_8$

(b) 3-ESHL of  $Fl_5$  is shown in Fig. 3.18(b).



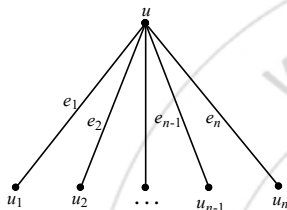
**Figure 3.18(b): 3-ESHL of  $Fl_5$**

**Theorem 3.20**

The star  $K_{1,n}$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

**Proof**

Let  $\{u_i, 1 \leq i \leq n, u\}$  be the vertices and  $\{e_i, 1 \leq i \leq n\}$  be the edges which are denoted as in Fig. 3.19.



**Figure 3.19:  $K_{1,n}$  with ordinary labeling**

First we label the vertices as follows:

Define  $f: V \rightarrow \{k-1, k, k+1, \dots, k+2q-1\}$  by

For  $1 \leq i \leq n$ ,

$$f(u_i) = k + 2i - 2$$

$$f(u) = k - 1$$

Then the induced edge labels are:

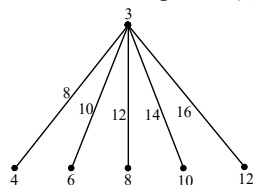
For  $1 \leq i \leq n$ ,

$$f^+(e_i) = 2k + 2(i - 1)$$

Therefore,  $f^+(E) = \{2k, 2k + 2, \dots, 2k + 2q - 2\}$ . So,  $f$  is a  $k$ -even sequential harmonious labeling and hence, the star graph  $K_{1,n}$  ( $n \geq 3$ ) is a  $k$ -even sequential harmonious graph for any  $k$ .

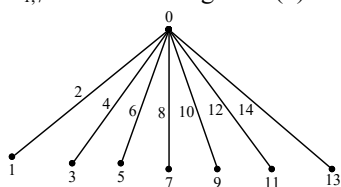
**Illustration 3.21**

(a) 4-ESHL of  $K_{1,5}$  is shown in Fig. 3.20(a).



**Figure 3.20(a): 4-ESHL of  $K_{1,5}$**

(b) 1-ESHL of  $K_{1,7}$  is shown in Fig. 3.20(b).



**Figure 3.20(b): 1-ESHL of  $K_{1,7}$**

**References**

[1] J.A. Gallian, A dynamic survey of graph labeling – The Electronic Journal of Combinatorics 15(2008), #DS6.  
 [2] B. Gayathri and V. Hemalatha, Even sequential harmonious labeling of some graphs - Presented in national Conference hold at Govt. College for Women, Pudukkottai – 28 & 29<sup>th</sup> March 2008.  
 [3] B. Gayathri and D. Muthuramakrishnan,  $k$ -even sequential harmonious labeling of some tree related graphs, International Journal of Engineering Science, Advanced Computing and Bio-technology, Vol. 3, No. 2, April – June 2012, pp. 85-92.  
 [4] B. Gayathri and D. Muthuramakrishnan, Some results on  $k$ -even sequential harmonious labeling of graphs, Elixir Applied Mathematics, 47 (2012) 9054-9057.  
 [5] D. Muthuramakrishnan, Ph.D. Thesis, March 2013,  $k$ -even sequential harmonious labeling of graphs, Bharathidasan University, Tiruchirappalli-24.  
 [6] R.L. Graham and N.J.A. Sloane, On addition bases and harmonious graphs, SIAM, J. Alg. Discrete Math., 1 (1980) 382-404.  
 [7] G.S. Singh and T.K.M. Varkey, On odd sequential and bisquential graphs, preprint.