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New Results on k-Fibonacci Numbers and Powers of its Golden Ratio Expressed as Continued Fraction

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Abstract: ***This paper uses the tools of two very important branches of Number Theory – Continued Fractions and theory of Fibonacci numbers. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, others by preserving the recurrence relation.The relationship between the golden ratio and continued fractions is commonly known about throughout the mathematical world. The convergents of the continued fraction are the ratios of consecutive Fibonacci numbers. The continued fractions for the powers of the golden ratio also exhibit an interesting relationship with the Fibonacci numbers. The ratios of anyk-Fibonacci sequence $\{F_{k,n}\}$is expressed by means of continued fraction. We find simple closed form continued fraction expansions for ϕ_k^r, for any positive integer r.***

Keywords: Continued fraction, Fibonacci sequence, *k*-Fibonacci, Golden ratio.

1. Introduction

The Fibonacci sequence $\{F_n\}_{n\geq 1}$ is a series of numbers that begins with$F_1 = 1, F_2 = 1$ and each next term is the sum of the previous two terms.The number of properties of these sequences were studied by many researchers. S. Falcon [7],[8],[9] defined *k*–Fibonacci numbers by the recurrence equation$F_{k,n} = k\,F_{k,n-1} + F_{k,n-2};\ n \geq 2$with initial condition $F_{k,0} = 0, F_{k,1} = 1$.As a particular case,

- if$k = 1$,we obtain the classical Fibonacci sequence $0, 1, 2, 3, 5, 8, \ldots$
- if $k = 2$, we getthe Pell sequence of numbers $0, 1, 2, 5, 12, 29, 70, \ldots$.

Continued fractions offer a useful means of expressing numbers and functions. In the early ages, 300 BC – 200 AD, mathematicians used other algorithms and methods to express numbers and to express solutions of Diophantine equations. Many of these algorithms were studied and modeled in the development of the continued fractions. Throughout the eighteenth century, use of continued fractions was limited to the area of mathematics. Since the beginning of the twentieth century, continued fractions have become more common in various other areas [2], [3]. It encodes much useful information about the algebraic structure of a number and frequently arises in approximation theory and dynamical systems. Van der Poorten [4] wrote that the elementary nature and simplicity of the theory of continued fractions is mostly buried in the literature. Our work is an outgrowth of [1], [3], [5]. We refer the readers to see these papers for some basic information on continued fractions, and to the book [6],[10] for more details.

Every real number αcan be represented in the form:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

We call such a representation a continued fraction. We can simplify this representation by restricting thea_i,called the partial quotients, to be positive integers. To construct the continued fraction, one must simply let:

$$a_0 = [\alpha]$$
$$a_1 = \frac{1}{\alpha - [\alpha]}$$
$$\vdots$$

In order to simplify the notation, the usual convention will be used:$\alpha = [a_0, a_1, a_2, \ldots]$.

Clearly, α is rational if and only if its continued fraction is finite, and a beautiful theorem of Lagrange asserts that α is a quadratic irrational if and only if the continued fraction expansion is periodic.

The continued fraction expansion consisting of the number 1 repeated indefinitely represents the '*golden mean*'. This satisfies the quadratic equation $x^2 = x + 1$.The convergents of this continued fraction are obtained as the ratio of the successive terms of the Fibonacci sequence.Continued fractions turn out to be convenient representations of numbers since they provide the best approximation to a given number.

2. Properties of *k*-Fibonacci sequence

In this section we determine some interesting identities related with sequence $\{F_{k,n}\}$and use these identities to express the ratio $\frac{F_{k,n}}{F_{k,n-r}}$as finite continued fraction expansion.

Theorem 2.1:$F_{k,n} = \dfrac{\left(\frac{k+\sqrt{k^2+4}}{2}\right)^n - \left(\frac{k-\sqrt{k^2+4}}{2}\right)^n}{\sqrt{k^2+4}} = \dfrac{\alpha^n_{F_{k,n}} - \beta^n_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$.

Proof: We prove the result by the principle of mathematical induction. For $n = 1$, we have

$F_{k,1} = \dfrac{\left(\frac{k+\sqrt{k^2+4}}{2}\right) - \left(\frac{k-\sqrt{k^2+4}}{2}\right)}{\sqrt{k^2+4}} = 1$, which proves the result for $n = 1$.

Now, assume that result holds for all positive integers up to some positive integer m. Thus both $F_{k,m} = \frac{\alpha^m_{F_{k,n}} - \beta^m_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$ and $F_{k,m-1} = \frac{\alpha^{m-1}_{F_{k,n}} - \beta^{m-1}_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$ holds.

This gives, $F_{k,m+1} = kF_{k,m} + F_{k,m-1}$

$$= k\left(\frac{\alpha^m_{F_{k,n}} - \beta^m_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right) + \left(\frac{\alpha^{m-1}_{F_{k,n}} - \beta^{m-1}_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right)$$

$$= \frac{k\left(\alpha^m_{F_{k,n}} - \beta^m_{F_{k,n}}\right) + \left(\alpha^{m-1}_{F_{k,n}} - \beta^{m-1}_{F_{k,n}}\right)}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$= \frac{\alpha^{m-1}_{F_{k,n}}\left(k\alpha_{F_{k,n}} + 1\right) - \beta^{m-1}_{F_{k,n}}\left(k\beta_{F_{k,n}} + 1\right)}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}.$$

Now, $k\alpha_{F_{k,n}} + 1 = k\left(\frac{k+\sqrt{k^2+4}}{2}\right) + 1 = \frac{k^2 + k\sqrt{k^2+4} + 2}{2}$

$$= \frac{k^2 + 2k\sqrt{k^2+4} + k^2 + 4}{4}$$

$$= \left(\frac{k+\sqrt{k^2+4}}{2}\right)^2 = \left(\alpha_{F_{k,n}}\right)^2.$$

Similarly it can be shown that $k\beta_{F_{k,n}} + 1 = \left(\beta_{F_{k,n}}\right)^2$. Thus,

$$F_{k,m+1} = \frac{\alpha^{m-1}_{F_{k,n}}\left(k\alpha_{F_{k,n}} + 1\right) - \beta^{m-1}_{F_{k,n}}\left(k\beta_{F_{k,n}} + 1\right)}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$= \frac{\left(\alpha_{F_{k,n}}\right)^{m-1}\left(\alpha_{F_{k,n}}\right)^2 - \left(\beta_{F_{k,n}}\right)^{m-1}\left(\beta_{F_{k,n}}\right)^2}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$= \frac{\left(\alpha_{F_{k,n}}\right)^{m+1} - \left(\beta_{F_{k,n}}\right)^{m+1}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}.$$

$$\therefore F_{k,m+1} = \frac{\left(\frac{k+\sqrt{k^2+4}}{2}\right)^{m+1} - \left(\frac{k-\sqrt{k^2+4}}{2}\right)^{m+1}}{\sqrt{k^2+4}} = \frac{\alpha^{m+1}_{F_{k,n}} - \beta^{m+1}_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}.$$

This proves the result for $n = m + 1$ and thus for all positive integers$n$.

We now obtain the limiting ratio of two consecutive k −Fibonacci numbers.

Lemma 2.2: $\phi_{F_{k,n}} = \lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = \frac{k+\sqrt{k^2+4}}{2}$.

Proof: We note that the sequence$\{x_n\}_{n=1}^{\infty} = \left\{\frac{F_{k,n}}{F_{k,n-1}}\right\}_{n=1}^{\infty}$ is convergent. Let this sequence converge to some real number$x$.Now $\frac{F_{k,n+1}}{F_{k,n}} = k + \frac{F_{k,n-1}}{F_{k,n}}$.

Then $\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = k + \lim_{n\to\infty} \frac{F_{k,n-2}}{F_{k,n-1}} = k + \frac{1}{\lim_{n\to\infty} \frac{F_{k,n-1}}{F_{k,n-2}}}$.

This gives $x = k + \frac{1}{x} \Rightarrow x^2 - kx - 1 = 0$. Solving for x yields $x = \frac{k \sqrt{k^2+4}}{2}$. Considering only the positive root, we get the required result.

We denote $\alpha_{F_{k,n}} = \frac{k+\sqrt{k^2+4}}{2}$and $\beta_{F_{k,n}} = \frac{k-\sqrt{k^2+4}}{2}$.

We now show how extended Binet formula for $F_{k,n}$ is useful to derive the above value of $\phi_{F_{k,n}}$.

Lemma 2.3: [*Alternate proof*]

$\phi_{F_{k,n}} = \lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = \frac{k+\sqrt{k^2+4}}{2}$.

Proof: Using extended Binet formula for $F_{k,n}$, we get

$$\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = \lim_{n\to\infty} \frac{\frac{\alpha^n_{F_{k,n}} - \beta^n_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}}{\frac{\alpha^{n-1}_{F_{k,n}} - \beta^{n-1}_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}}$$

$$= \lim_{n\to\infty} \frac{\alpha^n_{F_{k,n}} - \beta^n_{F_{k,n}}}{\alpha^{n-1}_{F_{k,n}} - \beta^{n-1}_{F_{k,n}}} = \lim_{n\to\infty} \frac{1 - \left(\frac{\beta_{F_{k,n}}}{\alpha_{F_{k,n}}}\right)^n}{\frac{1}{\alpha_{F_{k,n}}} - \frac{1}{\beta_{F_{k,n}}}\left(\frac{\beta_{F_{k,n}}}{\alpha_{F_{k,n}}}\right)^n}.$$

Now we note that$\left|\beta_{F_{k,n}}\right| < \alpha_{F_{k,n}}$. Also $\lim_{n\to\infty}\left(\frac{\beta_{F_{k,n}}}{\alpha_{F_{k,n}}}\right)^n \to 0$ (when n is sufficiently large).

Thus, $\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = \frac{k+\sqrt{k^2+4}}{2}$, as required.

Remark:

1) $\alpha_{F_{k,n}} - \beta_{F_{k,n}} = \left(\frac{k+\sqrt{k^2+4}}{2}\right) - \left(\frac{k-\sqrt{k^2+4}}{2}\right) = \sqrt{k^2+4}$.

2) $\alpha_{F_{k,n}}\beta_{F_{k,n}} = \left(\frac{k+\sqrt{k^2+4}}{2}\right)\left(\frac{k-\sqrt{k^2+4}}{2}\right) = -1$.

Catalan's Identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan, a Belgian mathematician who worked for the Belgian Academy of Science in the field of Number Theory. Here we obtain analogues result.

Lemma 2.4: [*Extended Catalan Identity*]

$$F_{k,n-r}\,F_{k,n+r} - F^2_{k,n} = (-1)^{n+1-r}F^2_{k,r}.$$

Proof: By using theorem 2.1 and $\alpha_{F_{k,n}}\beta_{F_{k,n}} = -1$, we get

LHS

$$= \frac{\left(\alpha^{n-r}_{F_{k,n}} - \beta^{n-r}_{F_{k,n}}\right)\left(\alpha^{n+r}_{F_{k,n}} - \beta^{n+r}_{F_{k,n}}\right) - \alpha^{2n}_{F_{k,n}} + 2\alpha^n_{F_{k,n}}\beta^n_{F_{k,n}} - \beta^{2n}_{F_{k,n}}}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}$$

$$= \frac{\left(-\alpha^{n+r}_{F_{k,n}}\beta^{n-r}_{F_{k,n}} - \alpha^{n-r}_{F_{k,n}}\beta^{n+r}_{F_{k,n}} + 2\alpha^n_{F_{k,n}}\beta^n_{F_{k,n}}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}$$

$$= \frac{\left(-\alpha^{n+2r-r}_{F_{k,n}}\beta^{n-r}_{F_{k,n}} - \alpha^{n-r}_{F_{k,n}}\beta^{n+2r-r}_{F_{k,n}} + 2\alpha^n_{F_{k,n}}\beta^n_{F_{k,n}}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}$$

$$= \frac{\left[\begin{array}{c}-\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r}\alpha^{2r}_{F_{k,n}} - \left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r}\beta^{2r}_{F_{k,n}} \\ +2\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^n\end{array}\right]}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}$$

$$= \frac{\left[-(-1)^{n-r}\alpha^{2r}_{F_{k,n}} - (-1)^{n-r}\beta^{2r}_{F_{k,n}} + 2(-1)^n\right]}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}$$

$$= \frac{\left[(-1)^{n-r+1}\alpha^{2r}_{F_{k,n}} + (-1)^{n-r+1}\beta^{2r}_{F_{k,n}} + 2(-1)^n\right]}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}$$

$$= \frac{(-1)^{n-r+1}}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}\left[\alpha^{2r}_{F_{k,n}} + \beta^{2r}_{F_{k,n}} + 2(-1)^{r-1}\right]$$

$$= \frac{(-1)^{n-r+1}}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}\left[\alpha^{2r}_{F_{k,n}} + \beta^{2r}_{F_{k,n}} - 2(-1)^r\right]$$

$$= \frac{(-1)^{n-r+1}}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)^2}\left[\alpha^{2r}_{F_{k,n}} + \beta^{2r}_{F_{k,n}} - 2\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^r\right]$$

$$= (-1)^{n-r+1}\left(\frac{\alpha^r_{F_{k,n}} - \beta^r_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right)^2 = (-1)^{n-r+1}F^2_{k,r}.$$

The following is analogues to one of the oldest identities involving the Fibonacci numbers, which was discovered in 1680 by Jean-Dominique Cassini, a French Astronomer.

Lemma 2.5: [*Extended Cassini identity*]

$$F_{k,n-1}\, F_{k,n+1} - F_{k,n}^2 = (-1)^n.$$

Proof: The proof follows immediately by taking $r = 1$in Catalan's identity.

We next obtain extended d' Ocagne'sIdentity.

Lemma 2.6: $F_{k,n} = F_{k,n-r}F_{k,r+1} + F_{k,n-r-1}F_{k,r}$.

Proof: We write the extended Binet's formula from theorem 2.1 in the form$F_{k,n} = C_1^*\alpha_{F_{k,n}}^n + C_2^*\beta_{F_{k,n}}^n$,

Where$C_1^* = \frac{1}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}} = \frac{1}{\sqrt{k^2+4}}, C_2^* = \frac{-1}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}} = \frac{-1}{\sqrt{k^2+4}}$.

We now express two consecutive terms of this sequence by using matrices as $\begin{bmatrix} F_{k,n-r} \\ F_{k,n-r-1} \end{bmatrix} = \begin{bmatrix} \alpha_{F_{k,n}}^{n-r} & \beta_{F_{k,n}}^{n-r} \\ \alpha_{F_{k,n}}^{n-r-1} & \beta_{F_{k,n}}^{n-r-1} \end{bmatrix} \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix}$. Then

$$\begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} = \begin{bmatrix} \alpha_{F_{k,n}}^{n-r} & \beta_{F_{k,n}}^{n-r} \\ \alpha_{F_{k,n}}^{n-r-1} & \beta_{F_{k,n}}^{n-r-1} \end{bmatrix}^{-1} \begin{bmatrix} F_{k,n-r} \\ F_{k,n-r-1} \end{bmatrix}$$

$$= \frac{1}{\alpha_{F_{k,n}}^{n-r}\beta_{F_{k,n}}^{n-r-1} - \alpha_{F_{k,n}}^{n-r-1}\beta_{F_{k,n}}^{n-r}} \begin{bmatrix} \beta_{F_{k,n}}^{n-r-1} & -\beta_{F_{k,n}}^{n-r} \\ -\alpha_{F_{k,n}}^{n-r-1} & \alpha_{F_{k,n}}^{n-r} \end{bmatrix} \begin{bmatrix} F_{k,n-r} \\ F_{k,n-r-1} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} F_{k,n-r}\beta_{F_{k,n}}^{n-r-1} - F_{k,n-r-1}\beta_{F_{k,n}}^{n-r} \\ -F_{k,n-r}\alpha_{F_{k,n}}^{n-r-1} + F_{k,n-r-1}\alpha_{F_{k,n}}^{n-r} \end{bmatrix}}{\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}.$$

This gives

$$C_1^* = \frac{F_{k,n-r}\beta_{F_{k,n}}^{n-r-1} - F_{k,n-r-1}\beta_{F_{k,n}}^{n-r}}{\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)} \text{and}$$

$$C_2^* = \frac{-F_{k,n-r}\alpha_{F_{k,n}}^{n-r-1} + F_{k,n-r-1}\alpha_{F_{k,n}}^{n-r}}{\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}.$$

Thus $C_1^* = \frac{F_{k,n-r} - F_{k,n-r-1}\beta_{F_{k,n}}}{\alpha_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$, $C_2^* = \frac{-F_{k,n-r} + F_{k,n-r-1}\alpha_{F_{k,n}}}{\beta_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$.

Substituting values of $C_1^*$and C_2^*, we get

$$F_{k,n} = C_1^*\alpha_{F_{k,n}}^n + C_2^*\beta_{F_{k,n}}^n$$

$$= \frac{\left(F_{k,n-r} - F_{k,n-r-1}\beta_{F_{k,n}}\right)\alpha_{F_{k,n}}^n}{\alpha_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)} + \frac{\left(-F_{k,n-r} + F_{k,n-r-1}\alpha_{F_{k,n}}\right)\beta_{F_{k,n}}^n}{\beta_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

$$= \frac{\alpha_{F_{k,n}}^{r+1}\left(F_{k,n-r} - F_{k,n-r-1}\beta_{F_{k,n}}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)} + \frac{\beta_{F_{k,n}}^{r+1}\left(-F_{k,n-r} + F_{k,n-r-1}\alpha_{F_{k,n}}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

$$= \frac{F_{k,n-r}\left(\alpha_{F_{k,n}}^{r+1} - \beta_{F_{k,n}}^{r+1}\right) - \left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)F_{k,n-r-1}\left(\alpha_{F_{k,n}}^{r} - \beta_{F_{k,n}}^{r}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

$$= F_{k,n-r}\left(\frac{\alpha_{F_{k,n}}^{r+1} - \beta_{F_{k,n}}^{r+1}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right) - \left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)F_{k,n-r-1}\left(\frac{\alpha_{F_{k,n}}^{r} - \beta_{F_{k,n}}^{r}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right).$$

Now since$\alpha_{F_{k,n}}\beta_{F_{k,n}} = -1$, by using extended Binet formula for k −Fibonacci numbers, we get

$$F_{k,n} = F_{k,n-r}F_{k,r+1} + F_{k,n-r-1}F_{k,r}.$$

3. Continued fractions of powers of the *k*-Fibonacci sequences

We now obtain the continued fraction expansion of ϕ_k^r .

Theorem 3:

$$\frac{F_{k,n}}{F_{k,n-r}} = \begin{cases} \left[F_{k,r+1} + F_{k,r-1}, \frac{F_{k,n-r}}{F_{k,n-2r}}\right] & ;\ \text{if } r \text{ is odd} \\ \left[F_{k,r+1} + F_{k,r-1} - 1, 1, \frac{F_{k,n-r}}{F_{k,n-2r}} - 1\right] & ;\ \text{if } r \text{ is even,} \end{cases}$$

which yields

$$\phi_k^r = \begin{cases} \left[F_{k,r+1} + F_{k,r-1}, \overline{F_{k,r+1} + F_{k,r-1}}\right] & ;\text{if } r \text{ is odd} \\ \left[F_{k,r+1} + F_{k,r-1} - 1, \overline{1, F_{k,r+1} + F_{k,r-1} - 2}\right] & ;\text{if } r \text{ is even.} \end{cases}$$

Proof: We have,

$$\frac{F_{k,n}}{F_{k,n-r}} = \frac{F_{k,r+1}F_{k,n-r} + F_{k,r}F_{k,n-r-1}}{F_{k,n-r}}$$

$$= F_{k,r+1} + \frac{F_{k,r}F_{k,n-r-1}}{F_{k,n-r}}$$

$$= F_{k,r+1} + \frac{k\, F_{k,r-1}F_{k,n-r-1} + F_{k,r-2}F_{k,n-r-1}}{F_{k,n-r}}$$

$$= F_{k,r+1} + \frac{k\, F_{k,r-1}F_{k,n-r-1} + F_{k,r-1}F_{k,n-r-2} + F_{k,r-2}F_{k,n-r-1} - F_{k,r-1}F_{k,n-r-2}}{F_{k,n-r}}$$

$$= F_{k,r+1} + F_{k,r-1}\frac{F_{k,n-r}}{F_{k,n-r}} + \frac{F_{k,r-2}F_{k,n-r-1} - F_{k,r-1}F_{k,n-r-2}}{F_{k,n-r}}$$

$$= F_{k,r+1} + F_{k,r-1} + \frac{F_{k,r-2}F_{k,n-r-1} - F_{k,r-1}F_{k,n-r-2}}{F_{k,n-r}}.$$

Now by Lemma 2.6, we have

$$F_{k,n-r-1} = F_{k,r}F_{k,n-r-r} + F_{k,r-1}F_{k,n-r-r-1} \text{and}$$

$$F_{k,n-r-2} = F_{k,r-1}F_{k,n-r-r-1} + F_{k,r-2}F_{k,n-r-r-2}.$$

This gives

$$\frac{F_{k,n}}{F_{k,n-r}} = F_{k,r+1} + F_{k,r-1} + \frac{F_{k,r-2}\left(F_{k,r}F_{k,n-2r} + F_{k,r-1}F_{k,n-2r-1}\right) - F_{k,r-1}\left(F_{k,r-1}F_{k,n-2r} + F_{k,r-2}F_{k,n-2r-1}\right)}{F_{k,n-r}} = F_{k,r+1} + F_{k,r-1}$$

$$+\frac{F_{k,r-2}(F_{k,r}F_{k,n-2r})-F_{k,r-1}(F_{k,r-1}F_{k,n-2r})}{F_{k,n-r}}$$

$$=F_{k,r+1}+F_{k,r-1}+\frac{F_{k,n-2r}[F_{k,r-2}F_{k,r}-(F_{k,r-1})^2]}{F_{k,n-r}}.$$

Once again using Lemma 1, we get

$$\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}+\frac{F_{k,n-2r}(-1)^{r-1}}{F_{k,n-r}}.$$

This gives $\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}+\frac{(-1)^{r-1}}{F_{k,n-r}/F_{k,n-2r}}$.

Now if r is odd, then we get

$$\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}+\frac{1}{F_{k,n-r}/F_{k,n-2r}}.$$

Thus

$$\phi_k^r=\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}+\frac{1}{\lim_{n\to\infty}\frac{F_{k,n-r}}{F_{k,n-2r}}}$$

$$=F_{k,r+1}+F_{k,r-1}+\frac{1}{\phi_k^r}$$

$$=F_{k,r+1}+F_{k,r-1}+\frac{1}{F_{k,r+1}+F_{k,r-1}+\frac{1}{F_{k,r+1}+F_{k,r-1}+\ddots}}.$$

Also if r is even, then we get

$$\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}+\frac{-1}{F_{k,n-r}/F_{k,n-2r}}.$$

In this case we manipulate further. We write it as

$$\frac{F_{k,n}}{F_{k,n-r}}=(F_{k,r+1}+F_{k,r-1}-1)+1-\frac{1}{F_{k,n-r}/F_{k,n-2r}}.$$

Now,

$$1-\frac{1}{\frac{F_{k,n-r}}{F_{k,n-2r}}}=\frac{F_{k,n-r}-F_{k,n-2r}}{F_{k,n-r}}$$

$$=\frac{1}{F_{k,n-r}/(F_{k,n-r}-F_{k,n-2r})}$$

$$=\frac{1}{((F_{k,n-r}-F_{k,n-2r})+F_{k,n-2r})/(F_{k,n-r}-F_{k,n-2r})}$$

$$=\frac{1}{1+(F_{k,n-2r}/(F_{k,n-r}-F_{k,n-2r}))}$$

$$=\frac{1}{1+(1/((F_{k,n-r}-F_{k,n-2r})/F_{k,n-2r}))}$$

$$=\frac{1}{1+(1/((F_{k,n-r}/F_{k,n-2r})-1))}.$$

Thus $\frac{F_{k,n}}{F_{k,n-r}}=(F_{k,r+1}+F_{k,r-1}-1)$

$$+\frac{1}{1+(1/((F_{k,n-r}/F_{k,n-2r})-1))}.$$

Taking the limit as $n\to\infty$, we get

$$\phi_k^r=\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}-1+\frac{1}{1+\left(\frac{1}{(\phi_k^r-1)}\right)}.$$

$$\therefore\phi_k^r=F_{k,r+1}+F_{k,r-1}-1$$

$$+\frac{1}{1+\frac{1}{F_{k,r+1}+F_{k,r-1}-2+\frac{1}{1+\frac{1}{F_{k,r+1}+F_{k,r-1}-2+\ddots}}}}.$$

Finally the continued fraction of ϕ_k^r follows easily.

4. Conclusion

- The techniques developed in this paper have allowed us to determine closed form expressions for the continued fraction expansion of some special quadratic numbers.
- We have been able to prove the structure of the continued fraction of a sizeable class of numbers. Although it was pretty clear at the outset that was a nice structure to this class, we have successfully proven it, and can now use these results to possibly derive similar results for other classes.

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