International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2015): 6.391

A General Class of Polynominals and a Modified Sequence of Functions Associated with Fractional Integral Operators

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Abstract: *In this paper, we studied generalized fractional calculus operators of Saigo type. In the paper, we have proved two important theorems related to the general class of polynomials and a modified sequence of functions by application of generalized fractional calculus. Some particular special cases have been also discussed.*

Keywords: Fractional integral operator; general class of polynomials; modified sequence of functions

AMS Subject Classification 2010 : 26A33, 45P05

1. Introduction

Let $\nu, \mu, \gamma, \delta \in C$ (the field of complex number) with $\mathrm{Re}(\alpha) > 0$, and $\sigma \in R^{+}$ defined by [4], then

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta} f(x) = \frac{\sigma x^{\mu}}{\Gamma(\alpha)} \int_0^x (x^{\sigma} - t^{\sigma})^{\alpha-1} {}_2F_1\left(\gamma, \delta; \alpha; 1 - \frac{t^{\sigma}}{x^{\sigma}}\right) t^{\nu} f(t)\,dt \tag{1.1}$$

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta} f(x) = \frac{\sigma x^{\mu}}{\Gamma(\alpha)} \int_x^{\infty} (t^{\sigma} - x^{\sigma})^{\alpha-1} {}_2F_1\left(\gamma, \delta; \alpha; 1 - \frac{t^{\sigma}}{x^{\sigma}}\right) t^{\nu} f(t)\,dt \tag{1.2}$$

If we set $\sigma=1, \nu=0, \mu=-\alpha-\beta, \gamma=\alpha+\beta, \delta=-\eta$ in (1.1), we get

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t)\,dt \tag{1.3}$$

which is the Saigo FIO of Reimann – Liouville type. Again, if we set $\sigma=1, \gamma=\alpha+\beta, \nu=-\alpha-\beta, \mu=0, \delta=-\eta$ in (1.2), we get

$$J_{x,\infty}^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} F\left(\alpha+\beta; -\eta; \alpha; 1 - \frac{t}{x}\right) t^{-\alpha-\beta} f(t)\,dt, \tag{1.4}$$

which is the Saigo FIO ([2], [3]) of Weyl type.

The modified sequence of function with c = 0 is given by

$$S_{\eta}^{\alpha',\beta,0}[x, r, q, A, B, m, \ell] = \sum_{\ell,p,u,\nu} \phi(\ell, p, u, \nu)\, x^{w}, \tag{1.5}$$

where

$$\sum_{\ell,p,u,\nu} = \sum_{\nu=0}^{n} \sum_{u=0}^{\nu} \sum_{p=0}^{n} \sum_{\ell=0}^{p}$$

$$\phi(\ell,p,u,\nu) = B^{qn-p} L^{n} \frac{(-1)^{p} (-\nu)_u (-p)_{\ell} (\alpha')_p}{u!\, \nu!\, \ell!\, p!} \frac{(-\alpha'-q_n)_{\ell}}{(1-\alpha'-p)_{\ell}} \cdot \left(\frac{\ell + m + ru}{L}\right)_n A^{p} B^{\nu} \tag{1.6}$$

and $w = Ln + p + r\nu$ (p, ν = 0, 1, ..., n)
and the general class of polynomial is defined by

$$S_N^M[X] = \sum_{i=0}^{(N/M)} \frac{(-N)_{Mi}}{i!} A_{N,i}\, x^{i}; N = 0, 1, 2, \ldots\ldots \tag{1.7}$$

2. Results used for Proving Main Theorems

Lemma 2.1. If $k > \max\{0, \sigma(\gamma+\delta-\alpha) - \nu - 1\}$ then

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta} x^{K} = \frac{\Gamma\{\sigma^{-1}(\nu+K+1)\}\, \Gamma\{\sigma^{-1}(\nu+K+1)+\alpha-\gamma-\delta\}}{\Gamma\{\sigma^{-1}(\nu+K+1)+\alpha-\gamma\}\, \Gamma\{\sigma^{-1}(\nu+K+1)+\alpha-\delta\}} \cdot x^{\alpha\sigma+\mu+\nu+K-\sigma+1} \tag{2.1}$$

Proof. By application of (1.1), we can write

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta} x^{K} = \frac{\sigma x^{\mu}}{\Gamma(\alpha)} \int_0^x (x^{\sigma} - t^{\sigma})^{\alpha-1} {}_2F_1\left(\gamma, \delta; \alpha; 1 - \frac{t^{\sigma}}{x^{\sigma}}\right) t^{\nu} t^{K}\,dt \tag{2.2}$$

Putting $t = x(1-y)^{1/\sigma}$ and making dx/dt = 0, equation (2.2) reduces to

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta} x^{K} = \frac{x^{\alpha\sigma+\mu}}{\Gamma(\alpha)} \int_0^1 y^{\alpha-1} {}_2F_1(\gamma, \delta; \alpha; y)\, t^{\nu+K-\sigma+1}\,dy \tag{2.3}$$

Since

$t = x(1-y)^{1/\sigma} \Rightarrow t^{\nu+K-\sigma+1} = x^{\nu+K-\sigma+1}(1-y)^{(\nu+K-\sigma+1)/\sigma}$,

therefore

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta} x^{K} = \frac{x^{\alpha\sigma+\mu+\nu+K-\sigma+1}}{\Gamma(\alpha)} \int_0^1 y^{\alpha-1}(1-y)^{\{(\nu+K+1)/\sigma\}-1} {}_2F_1(\gamma,\delta;\alpha;y)\,dy$$

$$= \frac{x^{\alpha\sigma+\mu+\nu+K-\sigma+1}}{\Gamma(\alpha)} \int_0^1 y^{\alpha-1}(1-y)^{\{((\nu+K+1)/\sigma)+\alpha\}-\alpha-1} {}_2F_1(\gamma,\delta;\alpha;y)\,dy \tag{2.4}$$

By application of the formula (2.5) :

$$\int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1} {}_2F_1(a,b;\lambda;t)\,dt = \frac{\Gamma(\lambda)\Gamma(c-\lambda)}{\Gamma(c)} \cdot {}_2F_1(a,b;c;1)$$

where $\mathrm{Re}\, c > 0$, $\mathrm{Re}\, \lambda > 0$; (2.5)

the equation (2.4) reduces to

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{x^{\alpha\sigma+\mu+\nu+K-\sigma+1}}{\Gamma(\alpha)} \cdot \Gamma(\alpha)\, \frac{\Gamma\{(\nu+K+1)/\sigma\}}{\Gamma\{(\nu+K+1+\alpha\sigma)/\sigma\}}$$

$$\cdot\, {}_2F_1\left(\gamma,\delta;(\nu+K+1+\alpha\sigma)/\sigma;1\right) \tag{2.6}$$

Again on making use of the formula (2.7) :

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \text{for } \operatorname{Re}(c-a-b) > 0, \tag{2.7}$$

we get,

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = x^{\alpha\sigma+\mu+\nu+K-\sigma+1} \frac{\Gamma\left(\frac{\nu+K+1}{\sigma}\right)\Gamma\left(\frac{\nu+K+1}{\sigma}+\alpha-\gamma-\delta\right)}{\Gamma\left(\frac{\nu+K+1}{\sigma}+\alpha-\gamma\right)\Gamma\left(\frac{\nu+K+1}{\sigma}+\alpha-\delta\right)} \tag{2.8}$$

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{\Gamma\left(\sigma^{-1}(\nu+K+1)\right)\Gamma\left(\sigma^{-1}(\nu+K+1)+\alpha-\gamma-\delta\right)}{\Gamma\left(\sigma^{-1}(\nu+K+1)+\alpha-\gamma\right)\Gamma\left(\sigma^{-1}(\nu+K+1)+\alpha-\delta\right)}$$

$$x^{\alpha\sigma+\mu+\nu+K-\sigma+1} \tag{2.9}$$

provided that $K > \max\{0, \sigma(\gamma+\delta-\alpha)\} - \nu - 1$.

Special Case 2.2. If we put $\sigma=1, \mu=-\alpha-\beta, \delta=-\eta, \nu=0, \gamma=\alpha+\beta$, in Lemma (2.1),

we get

$$I_{0,\nu}^{\alpha,\beta}\, x^K = \frac{\Gamma(K+1)\Gamma(-\beta+\eta+K+1)}{\Gamma(-\beta+K+1)\Gamma(\alpha+\eta+K+1)}\, x^{K-\beta}, \tag{2.10}$$

which is the formula due to [4, Lemma 1].

Lemma 2.3. If $K > \min\{(1-\alpha)\sigma-\nu-1, \sigma(1-\gamma-\delta)-\nu-1\}$ then

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{\Gamma\left(1-\alpha-\sigma^{-1}(\nu+K+1)\right)\Gamma\left(1-\gamma-\delta-\sigma^{-1}(\nu+K+1)\right)}{\Gamma\left(1-\gamma-\sigma^{-1}(\nu+K+1)\right)\Gamma\left(1-\delta-\sigma^{-1}(\nu+K+1)\right)}\, x^{\mu-\sigma+\alpha\sigma+\nu+K+1} \tag{2.11}$$

Proof. By application of (1.2), we can write

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{\sigma x^{\mu}}{\Gamma(\alpha)} \int_x^{\infty} t^{\alpha\sigma-\sigma}\left(1-\frac{x^{\sigma}}{t^{\sigma}}\right)^{\alpha-1} {}_2F_1\left(\gamma,\delta;\alpha;1-\frac{x^{\sigma}}{t^{\sigma}}\right) t^{\nu} t^{K}\, dt$$

$$= \frac{\sigma x^{\mu}}{\Gamma(\alpha)} \int_x^{\infty} t^{\alpha\sigma-\sigma+\nu+K}\left(1-\frac{x^{\sigma}}{t^{\sigma}}\right)^{\alpha-1} {}_2F_1\left(\gamma,\delta;\alpha;1-\frac{x^{\sigma}}{t^{\sigma}}\right) dt \tag{2.12}$$

Putting $x = t(1-y)^{1/\sigma}$, and making $dx/dt = 0$, we get

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{x^{\mu-\sigma+\alpha\sigma+\nu+K+1}}{\Gamma(\alpha)}$$

$$\int_0^1 y^{\alpha-1}(1-y)^{-(\alpha\sigma+\nu+K+1)/\sigma}\, {}_2F_1(\gamma,\delta;,\alpha;y)\, dy$$

$$= \frac{x^{\mu-\sigma+\alpha\sigma+\nu+K+1}}{\Gamma(\alpha)} \int_0^1 y^{\alpha-1}(1-y)^{1-\{(\nu+K+1)/\sigma\}-\alpha-1}$$

$${}_2F_1(\gamma,\delta;,\alpha;y)\, dy \tag{2.13}$$

On using the formula (2.5), we get

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{x^{\mu-\sigma+\alpha\sigma+\nu+K+1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma\left(1-\frac{\nu+K+1}{\sigma}-\alpha\right)}{\Gamma\left(1-\frac{\nu+K+1}{\sigma}\right)}$$

$$F\left(\gamma,\delta;1-\frac{\nu+K+1}{\sigma};1\right) \tag{2.14}$$

Therefore, applying (2.7), we get

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\, x^K = \frac{\Gamma\left(1-\alpha-\sigma^{-1}(\nu+K+1)\right)\Gamma\left(1-\gamma-\delta-\sigma^{-1}(\nu+K+1)\right)}{\Gamma\left(1-\gamma-\sigma^{-1}(\nu+K+1)\right)\Gamma\left(1-\delta-\sigma^{-1}(\nu+K+1)\right)}$$

$$x^{\mu-\sigma+\alpha\sigma+\nu+K+1} \tag{2.15}$$

provided that

$$K > \min\{\sigma(1-\alpha)-\nu-1, \sigma(1-\gamma-\delta)-\nu-1\}.$$

If we put $\sigma=1, \gamma=\alpha+\beta, \nu=-\alpha-\beta, \mu=0, \delta=-\eta$ in (2.15), we get

$$J_{x,\infty}^{\alpha,\beta,\eta}\, x^K = \frac{\Gamma(\beta-K)\Gamma(\eta-K)}{\Gamma(-K)\Gamma(\alpha+\beta+\eta-K)}\, x^{K-\beta}$$

which is the formula due to [4, Lemma 2].

Lemma 2.4. Let $K > \max\{0, \sigma(\gamma+\delta-\alpha)\} - \nu - 1$ where σ is non-zero positive integer then

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\} = \frac{\sigma^{\nu+K+3/2} x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{\sigma-1/2}\lambda^{\nu+K+1}}$$

$$\sum_{h=1}^{\sigma} \frac{\prod_{\substack{j=1\\ j\neq h}}^{\sigma}\Gamma(a_h-a_j)\prod_{j=1}^{2}\Gamma(b_j-a_h+1)}{\prod_{j=\alpha+1}^{\sigma+2}\Gamma(1+a_j-a_h)}$$

$$\left(\frac{\sigma}{\lambda x}\right)^{\sigma(a_h-1)} {}_2F_{\sigma+1}\left(\begin{matrix}1+b_1-a_h, 1+b_2-a_h\\ 1+a_1-a_h,\ldots,1+a_{h-1}-a_h, 1-a_{h+1}-a_h,\ldots,1+a_{\sigma+2}-a_h;\left(\frac{-\lambda x}{\sigma}\right)^{\sigma}\end{matrix}\right); \tag{2.17}$$

where

$$a_j = \frac{-\nu-K+j+1}{\sigma};\ j=1,\ldots,\sigma,$$

$a_{\sigma+1}=\alpha-\gamma$, $a_{\sigma+2}=\alpha-\delta$,
$b1 = 0$, $b2 = \alpha-\gamma-\delta$.

Proof.

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\}=\frac{\sigma x^{\mu}}{\Gamma(\alpha)}$$

$$\int_0^x (x^\sigma-t^\sigma)^{\alpha-1}\ {}_2F_1\left(\gamma,\delta;\alpha;1-\frac{t^\sigma}{x^\sigma}\right)t^\nu t^K e^{-\lambda t}\,dt \tag{2.18}$$

We make use of the following formula [1, p.536]

$$\int_0^{w^{K/L}} t^{\mu}(w-t^{L/K})^{c-1}\times{}_2F_1\left(a,b;c;1-\frac{t^{L/K}}{w}\right)e^{-\lambda t}\,dt$$

$$=\frac{K^{1-c}L^{\mu+1/2}w^{c-1}\Gamma(c)}{(2\pi)^{(L-1)/2}\lambda^{\mu+1}}$$

$$\times G_{2K+L,2K}^{2K,L}\left\{\frac{1}{w^K}\left(\frac{L}{\lambda}\right)^L\middle|\begin{matrix}\Delta(L,-\mu),\Delta(K,c-a),\Delta(K,c-b)\\ \Delta(K,0),\Delta(K,c-a-b)\end{matrix}\right\} \tag{2.19}$$

where $Re(\mu)>-1, w, Re(c)>0, Re(c-a-b+K\mu/L)>-K/L$.

Thus, put
$w = x^\sigma, L/K=\sigma$, i.e. $L=\sigma$, $K=1$, $a=\gamma$, $b=\delta$, $c=\alpha$, $\mu=\nu+K$

we get

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\}=\frac{\sigma x^{\mu}}{\Gamma(\alpha)}$$

$$\int_0^x (x^\sigma-t^\sigma)^{\alpha-1}\ {}_2F_1\left(\gamma,\delta;\alpha;1-\frac{t^\sigma}{x^\sigma}\right)t^\nu t^K e^{-\lambda t}\,dt$$

$$=\frac{\sigma x^{\mu}}{\Gamma(\alpha)}\,\frac{\sigma^{\nu+K+1/2}x^{\alpha\sigma-\sigma}\Gamma(\alpha)}{(2\pi)^{(\sigma-1)/2}\lambda^{\nu+K+1}}$$

$$G_{2+\sigma,2}^{2,\sigma}\left\{\left(\frac{\sigma}{\lambda x}\right)^{\sigma}\middle|\begin{matrix}\Delta(\sigma,-\nu-K),\Delta(1,\alpha-\gamma),\Delta(1,\alpha-\delta)\\ \Delta(1,0),\Delta(1,\alpha-\gamma-\beta)\end{matrix}\right\}; \tag{2.20}$$

where

$Re(\nu+K)>-1$, $Re(\alpha)>0$, $Re(\alpha-\gamma-\delta+(\nu+K)/\sigma)>-1/\sigma$
i.e. $K>-1-\nu$, $K>\sigma(\gamma+\delta-\alpha)-\nu-1$
i.e. $K>\max|0,\sigma(\gamma+\delta-\alpha)-\nu-1|$
Let

$$\Delta(t,a_j)=\frac{a_j+0}{t},\frac{a_j+1}{t},\ldots,\frac{a_j+t-1}{t};$$

Then

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\}=\frac{\sigma^{\nu+K+3/2}x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}\lambda^{\nu+K+1}}$$

$$.\,G_{2+\sigma,2}^{2,\sigma}\left\{\left(\frac{\sigma}{\lambda x}\right)^{\sigma}\middle|\begin{matrix}\frac{-\nu-K}{\sigma},\frac{-\nu-K+1}{\sigma},\ldots,\frac{-\nu-K+\sigma-1}{\sigma},\alpha-\gamma,\alpha-\delta\\ 0,\alpha-\gamma-\delta\end{matrix}\right\} \tag{2.21}$$

Now applying the following formula [1, p.4, 1.1.10] given below :

$$G_{p,q}^{m,n}\left[Z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]=\sum_{h=1}^{n}\frac{\prod_{\substack{j=1\\h\neq j}}^{n}\Gamma(a_h-a_j)\prod_{j=1}^{m}\Gamma(b_j-a_h+1)}{\prod_{j=n+1}^{\sigma+2}\Gamma(1+a_j-a_h)\prod_{j=m+1}^{q}\Gamma(a_h-b_j)}$$

$$.\,Z^{a_h-1}\,{}_qF_{p-1}\left(\begin{matrix}1+b_1-a_h,\ldots,1+b_q-a_h\\ 1+a_1-a_h,\ldots,1+a_p-a_h;\frac{(-1)^{q-m-n}}{Z}\end{matrix}\right); \tag{2.22}$$

where
$p>q$, and $|Z|\geq 1$,
$\sigma+2\geq 2$, and $\sigma\geq 1$;
we get

$$I_{+,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\}=\frac{\sigma^{\nu+K+3/2}x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}\lambda^{\nu+K+1}}$$

$$\sum_{h=1}^{\sigma}\frac{\prod_{\substack{j=1\\h\neq j}}^{\sigma}\Gamma(a_h-a_j)\prod_{j=1}^{2}\Gamma(b_j-a_h+1)}{\prod_{j=\sigma+1}^{\sigma+2}\Gamma(1+a_j-a_h)}$$

$$\left(\frac{\sigma}{\lambda x}\right)^{\sigma(a_h-1)}{}_2F_{\sigma+1}\left(\begin{matrix}1+b_1-a_h,1+b_2-a_h\\ 1+a_1-a_h,\ldots,1+a_{h-1}-a_h,1+a_{h+1}-a_h,\ldots,1+a_{\sigma+2}-a_h;\left(\frac{-\lambda x}{\sigma}\right)^{\sigma}\end{matrix}\right); \tag{2.23}$$

where

$$a_j=\frac{-\nu-K+j-1}{\sigma},\ j=1,\ldots,\sigma$$

$a_{\sigma+1}=\alpha-\gamma$, $b_1=0$,
$a_{\sigma+2}=\alpha-\delta$, $b_2=\alpha-\gamma-\delta$.

Special case 2.5. If we put $\sigma=1$, $\nu=0$, $\mu=-\alpha-\beta$, $\gamma=\alpha+\beta$, $\delta=-\eta$ in Lemma 2.4, we get

$$I_{0,x}^{\alpha,\beta,\eta}\{x^K e^{-\lambda x}\}=\frac{x^{-\beta-1}\Gamma(K+1)\Gamma(-\beta+\eta+K+1)}{\lambda^{K+1}\Gamma(1-\beta+K)\Gamma(1+\alpha+\eta+K)}(\lambda x)^{K+1}$$

$$\times\,{}_2F_2\left(\begin{matrix}1+K,1+\eta-\beta+K\\ 1,\ldots,1+\alpha+\eta+K\,;-\lambda x\end{matrix}\right) \tag{2.24}$$

which is the formula due to [4, Lemma 4].

Lemma 2.6. If $\lambda>0$, $\alpha>0$, $x>0$ and σ is non-zero positive integer then,

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\} = \frac{\sigma^{\nu+K+3/2} x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}\lambda^{\mu+K+1}}$$

$$\sum_{h=1}^{\sigma+2} \frac{\prod_{\substack{j=1\\ j\neq h}}^{\sigma+2}\Gamma(a_h-a_j)}{\prod_{j=1}^{2}\Gamma(a_h-b_j)} \times \left(\left(\frac{\sigma}{\lambda x}\right)^{\sigma}\right)^{a_h-1}$$

$${}_2F_{1+\sigma}\left(\begin{matrix}1+b_1-a_h, a+b_2-a_h\\ 1+a_1-a_h,\ldots,1+a_{h-1}-a_h,1+a_{h+1}-a_h,\ldots,1+a_{\sigma+2}-a_h;\end{matrix}\left(\frac{-\lambda x}{\sigma}\right)^{\sigma}\right); \quad (2.25)$$

where

$$a_j = \frac{-\nu-K+j-1}{\sigma}; \qquad j=1,\ldots,\sigma$$

$$a_{\sigma+1}=\alpha \ , \ a_{\sigma+2}=\gamma+\delta, \ b_1=\gamma, \ b_2=\delta.$$

Proof.

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\} = \frac{\sigma x^{\mu}}{\Gamma(\alpha)}$$

$$\int_x^{\infty}(t^{\sigma}-x^{\sigma})^{\alpha-1}\,{}_2F_1\left(\gamma,\delta;\alpha;1-\frac{x^{\sigma}}{t^{\sigma}}\right)t^{\nu+K}e^{-\lambda t}\,dt \quad (2.26)$$

Now we make of the following formula [1] :

$$\int_{w^{K/L}}^{\infty} t^{\mu}(t^{L/K}-w)^{c-1} \times {}_2F_1\left(a,b;c;1-\frac{w}{t^{L/K}}\right)e^{-\lambda t}dt$$

$$=\frac{K^{1-c}L^{\mu+1/2}w^{c-1}\Gamma(c)}{(2\pi)^{(L-1)/2}\lambda^{\mu+1}}$$

$$\times G_{2K+L,2K}^{0,2K+L}\left\{\frac{1}{w^K}\left(\frac{L}{\lambda}\right)^L\middle|\begin{matrix}\Delta(L,-\mu),\Delta(K,c),\Delta(K,a+b)\\ \Delta(K,a),\Delta(K,b)\end{matrix}\right\} \quad (2.27)$$

where $w, \operatorname{Re}(c)>0, \lambda>0, \alpha>0, x>0.$

Put $\mu=\nu+K, L/K=\sigma/1, a=\gamma, b=\delta, c=\alpha, w=x^{\sigma}$ in (2.27) and then applying (2.27) in (2.26), we get

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\} = \frac{\sigma x^{\mu}}{\Gamma(\alpha)}$$

$$\int_x^{\infty} t^{\nu+K}(t^{\sigma}-x^{\sigma})^{\alpha-1}\,{}_2F_1\left(\gamma,\delta;\alpha;1-\frac{x^{\sigma}}{t^{\sigma}}\right)e^{-\lambda t}\,dt$$

$$=\frac{\sigma x^{\mu}}{\Gamma(\alpha)}\frac{\sigma^{\nu+K+1/2}x^{\alpha\sigma-\sigma}\Gamma(\alpha)}{(2\pi)^{(\sigma-1)/2}\lambda^{\nu+K+1}}$$

$$G_{2+\sigma,2}^{0,2+\sigma}\left\{\frac{1}{x^{\sigma}}\left(\frac{\sigma}{\lambda}\right)^{\sigma}\middle|\begin{matrix}\Delta(\sigma,-\nu-K),\Delta(1,\alpha),\Delta(1,\gamma+\delta)\\ \Delta(1,\gamma),\Delta(1,\delta)\end{matrix}\right\} \quad (2.28)$$

Since

$$\Delta(\sigma,-\nu-K)=\frac{-\nu-K}{\sigma},\ldots,\frac{-\nu-K+\sigma-1}{\sigma},$$

$$\Delta(1,\alpha)=\alpha \ , \quad \Delta(1,\gamma)=\gamma$$

$$\Delta(1,\gamma+\delta)=\gamma+\delta, \ \Delta(1,\delta)=\delta;$$

Therefore

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\} = \frac{\sigma^{\nu+K+3/2}x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}\lambda^{\nu+K+1}}$$

$$G_{2+\sigma,2}^{0,2+\sigma}\left\{\left(\frac{\sigma}{\lambda x}\right)^{\sigma}\middle|\begin{matrix}\frac{-\nu-K}{\sigma},\frac{-\nu-K+1}{\sigma},\ldots,\frac{-\nu-K+\sigma-1}{\sigma},\alpha,\gamma+\delta\\ \gamma,\delta\end{matrix}\right\} \quad (2.29)$$

Using formula 1.1.10 of Mathai and Saxena [1] given by equation (2.22), the (2.29) reduces to

$$J_{-,\nu}^{\alpha,\sigma,\mu,\gamma,\delta}\{x^K e^{-\lambda x}\} = \frac{\sigma^{\nu+K+3/2}x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}\lambda^{\nu+K+1}}$$

$$\sum_{h=1}^{2+\sigma}\frac{\prod_{\substack{j=1\\ j\neq h}}^{2+\sigma}\Gamma(a_h-a_j)}{\prod_{j=1}^{2}\Gamma(a_h-b_j)}\times\left(\left(\frac{\sigma}{\lambda x}\right)^{\sigma}\right)^{a_h-1}$$

$$\times {}_2F_{1+\sigma}\left(\begin{matrix}1+b_1-a_h, a+b_2-a_h\\ 1+a_1-a_h,\ldots,1+a_{h-1}-a_h,1+a_{h+1}-a_h,\ldots,1+a_{\sigma+2}-a_h;\end{matrix}\left(\frac{-\lambda x}{\sigma}\right)^{\sigma}\right); \quad (2.30)$$

where

$$a_j=\frac{-\nu-K+j-1}{\sigma}, \ j=1,2,\ldots\ldots,\sigma$$

$$a_{\sigma+1}=\alpha \ , \ a_{\sigma+2}=\gamma+\delta, \ b_1=\gamma, \ b_2=\delta.$$

Special Case 2.7.

Now, if we put $\sigma=1, \gamma=\alpha+\beta, \nu=-\alpha-\beta, \mu=0, \delta=-\eta$ in Lemma (2.6) then we get

$$J_{x,\infty}^{\alpha,\beta,\eta}\{x^K e^{-\lambda x}\} = \frac{x^{\alpha-1}}{\lambda^{-\alpha-\beta+K+1}}$$

$$\left\{\frac{\Gamma(a_1-a_2)\Gamma(a_1-a_3)}{\Gamma(a_1-b_1)\Gamma(a_1-b_2)}\left(\frac{1}{\lambda x}\right)^{a_1-1}\right.$$

$$ {}_2F_2\left(\begin{matrix}1+b_1-a_1\ ,1+b_2-a_1\\ 1+a_2-a_1\ ,1+a_3-a_1\end{matrix};-\lambda x\right) + \frac{\Gamma(a_2-a_1)\Gamma(a_2-a_3)}{\Gamma(a_2-b_1)\Gamma(a_2-b_2)}(\lambda x)^{-a_2+1} $$

$$ {}_2F_2\left(\begin{matrix}1+b_1-a_2\ ,1+b_2-a_2\\ 1+a_1-a_2\ ,1+a_3-a_2\end{matrix};-\lambda x\right) + \frac{\Gamma(a_3-a_1)\Gamma(a_3-a_2)}{\Gamma(a_3-b_1)\Gamma(a_3-b_2)}(\lambda x)^{-a_3+1} $$

$$ {}_2F_2\left(\begin{matrix}1+b_1-a_3,1+b_2-a_3\\ 1+a_1-a_3,1+a_2-a_3\end{matrix};-\lambda x\right)\Bigg\} \tag{2.31} $$

where

$$ a_1=-\nu-K=\alpha+\beta-K $$
$$ a_2=\alpha,\qquad a_3=\gamma+\delta=\alpha+\beta-\eta, $$
$$ b_1=\alpha+\beta,\qquad b_2=-\eta. $$

Equation (2.31) can also be written as

$$ J^{\alpha,\beta,\eta}_{x,\infty}\{x^K e^{-\lambda x}\} = \frac{\Gamma(\beta-K)\Gamma(\eta-K)}{\Gamma(-K)\Gamma(\alpha+\beta+\eta-K)} x^{-\beta+K} $$

$$ {}_2F_2\left(\begin{matrix}K+1,-\alpha-\beta+K+1-\eta\\ -\beta+K+1,-\eta+K+1\end{matrix};-\lambda x\right) + \frac{\Gamma(-\beta+K)\Gamma(-\eta-\beta)}{\Gamma(-\beta)\Gamma(\alpha+\eta)}\lambda^{\beta-K} $$

$$ {}_2F_2\left(\begin{matrix}\beta+1,-\alpha-\eta+1\\ 1+\beta-K,1+\beta-\eta\end{matrix};-\lambda x\right) + \frac{\Gamma(\beta-\eta)\Gamma(-\eta+K)}{\Gamma(a+\beta)\Gamma(-\eta)}\lambda^{\eta-K}x^{-\beta+\eta} $$

$$ {}_2F_2\left(\begin{matrix}1+\eta,-\alpha-\beta+1\\ 1+K+\eta,1-\beta+\eta\end{matrix};-\lambda x\right) \tag{2.32} $$

which is the formula due to [4, formula 4].

Theorem 2.8. If $K>\max\{0,\sigma(\gamma+\delta-\alpha)\}-\nu-1$ where σ is non-zero positive integer then

$$ I^{\alpha,\sigma,\mu,\gamma,\delta}_{+,\nu}\{x^K e^{-\lambda x}\, S^M_N(x) S^{\alpha',\beta,0}_n(x,r,q,A,B,m,\ell)\} $$

$$ =\frac{x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}}\sum_{i=0}^{(N/M)}\sum_{\ell,p,u,\nu}\sum_{h=1}^{\sigma}\frac{(-N)_{Mi}}{i!}A_{N,i}\phi(\ell,p,u,\nu) $$

$$ \frac{\sigma^{\nu+K+i+w+3/2}}{\lambda^{\nu+K+1+w+i}}\frac{\prod_{\substack{j=1\\ j\neq h}}^{\sigma}\Gamma(a_h-a_j)\prod_{j=1}^{2}\Gamma(b_j-a_h+1)}{\prod_{j=\sigma+1}^{\sigma+2}\Gamma(1+a_j-a_h)}\left(\frac{\sigma}{\lambda x}\right)^{\sigma(a_h-1)} $$

$$ \times {}_2F_{\sigma+1}\left(\begin{matrix}1+b_1-a_h\ ,1+b_2-a_h\\ 1+a_1-a_h,\ldots,1+a_{h-1}-a_h,1+a_{h+1}-a_h,\ldots,1+a_{\sigma+2}-a_h\end{matrix};\left(\frac{-\lambda x}{\sigma}\right)^{\sigma}\right); \tag{2.33} $$

where

$$ a_j=\frac{-\nu-K-i-w+j-1}{\sigma},\qquad j=1,\ \ldots\ ,\sigma $$

$$ a_{\sigma+1}=\alpha-\gamma\ ,\quad a_{\sigma+2}=\alpha-\delta, b_1=0, b_2=\alpha-\gamma-\delta $$

$$ \sum_{\ell,p,u,\nu} = \sum_{\nu=0}^{n}\sum_{u=0}^{\nu}\sum_{p=0}^{n}\sum_{\ell=0}^{p}, $$

$$ \phi(\ell,p,u,v)=B^{qn-p}L^n\frac{(-1)^p(-v)_u(-p)_\ell(\alpha')_p}{u!\,\nu!\,\ell!\,p!}\frac{(-\alpha'-qn)_\ell}{(1-\alpha'-p)_\ell}\left(\frac{\ell+m+ru}{L}\right)_n A^pB^\nu $$

and

$$ w = Ln+p+r\nu \qquad (p,\ \nu = 0, 1, \ldots, n) $$

provided that

$K>\max\{0,\sigma(\gamma+\delta-\alpha)\}-\nu-1$ where σ is non-zero positive integer.

Proof. The proof of the theorem follows directly from Lemma (2.4).

Theorem 2.9. Let $\alpha>0,\quad \lambda>0,\quad x>0$ then

$$ J^{\alpha,\sigma,\mu,\gamma,\delta}_{-,\nu}\{x^K e^{-\lambda x}\, S^M_N(x) S^{\alpha',\beta,0}_n(x,r,q,A,B,m,\ell)\} $$

$$ =\frac{x^{\alpha\sigma+\mu-\sigma}}{(2\pi)^{(\sigma-1)/2}}\sum_{i=0}^{(N/M)}\sum_{\ell,p,u,\nu}\sum_{h=1}^{\sigma+2}\frac{(-N)_{Mi}}{i!}A_{N,i}\phi(\ell,p,u,\nu) $$

$$\frac{\sigma^{\nu+K+i+w+3/2}}{\lambda^{\nu+K+1+w+i}} \frac{\prod_{\substack{j=1\\ j\neq h}}^{\sigma+2} \Gamma(a_h - a_j)}{\prod_{j=1}^{2} \Gamma(a_j - b_j)} \left(\frac{\sigma}{\lambda x}\right)^{\sigma(a_h-1)}$$

$$\times {}_2F_{\sigma+1}\left(\begin{matrix} 1+b_1-a_h, 1+b_2-a_h \\ 1+a_1-a_h,\ldots,1+a_{h-1}-a_h, 1+a_{h+1}-a_h,\ldots,1+a_{\sigma+2}-a_h; \left(\frac{-\lambda x}{\sigma}\right)^{\sigma} \end{matrix}\right);$$

(2.34)

where

$$a_j = \frac{-\nu-K-i-w+j-1}{\sigma}, \qquad j=1, \ldots, \sigma$$

$$a_{\sigma+1} = \alpha, \quad a_{\sigma+2} = \gamma+\delta, b_1 = \gamma, b_2 = \delta$$

$$\sum_{\ell,p,u,\nu} = \sum_{\nu=0}^{n} \sum_{u=0}^{\nu} \sum_{p=0}^{n} \sum_{\ell=0}^{p}$$

$$\phi(\ell,p,u,\nu) = B^{qn-p} L^n \frac{(-1)^p (-\nu)_u (-p)_\ell (\alpha')_p}{u!\, \nu!\, \ell!\, p!} \frac{(-\alpha'-qn)_\ell}{(1-\alpha'-p)_\ell} \left(\frac{\ell+m+ru}{L}\right)_n A^p B^\nu$$

and

$w = Ln + p + r\nu$, $(p, \nu = 0, 1, 2, \ldots, n)$.

Proof. The proof of the theorem follows from Lemma (2.6).

Corollary 2.10. (A deduction from theorem 2.8.)

If $\sigma = 1, \ \mu = 0, \ \nu = 0, \ \gamma = 0, \ \delta = -\eta$

and $p = n = q = m = B = 0, \quad A = \ell = r = 1$

then $S_n^{\alpha',\beta,0}(x,r,q,A,B,m,\ell) \to 1$ therefore $K > -1$

and we get

$$I_{+,\nu}^{\alpha,1,-\eta} \left\{x^K e^{-\lambda x} S_N^M\right\} = x^{-a_1+1} \sum_{i=0}^{[N/M]} \frac{(-N)_{Mi}}{i!} \frac{A_{n,i}}{\lambda^{K+i+1}}$$

$$\frac{\Gamma(b_1-a_1+1)\,\Gamma(b_2-a_1+1)}{\Gamma(1+a_2-a_1)\,\Gamma(1+a_3-a_1)} (\lambda x)^{-a_1+1} {}_2F_2\left(\begin{matrix} 1+b_1-a_1, 1+b_2-a_1 \\ 1+a_2-a_1, 1+a_3-a_1 \end{matrix}; -\lambda x\right);$$

(2.35)

where

$$a_1 = -K-i, \quad a_2 = \alpha-\gamma, \quad a_3 = \alpha-\delta$$

$$b_1 = 0, \quad b_2 = \alpha-\gamma-\delta.$$

Therefore,

$$R_{0,x}^{\alpha} \left\{x^K e^{-\lambda x} S_N^M\right\} = x^{\alpha-1} \sum_{i=0}^{[N/M]} \frac{(-N)_{Mi}}{i!} \frac{A_{n,i}}{\lambda^{K+i+1}}$$

$$\frac{\Gamma(K+i+1)\,\Gamma(\alpha+\eta+K+i+1)}{\Gamma(1+\alpha+K+i)\,\Gamma(\alpha+\eta+K+i+1)} (\lambda x)^{K+i+1}$$

$$\times {}_2F_2\left(\begin{matrix} 1+K+i, 1+\alpha+\eta+K+i \\ 1+\alpha+K+i, 1+\alpha+\eta+K+i \end{matrix}; -\lambda x\right)$$

$$= x^{\alpha+K} \sum_{i=0}^{[N/M]} \frac{(-N)_{Mi}}{i!} A_{N,i} \frac{\Gamma(K+i+1)}{\Gamma(1+\alpha+K+i)}$$

$${}_1F_1(1+K+i; 1+\alpha+K+i; -\lambda x)\, x^i$$

(2.36)

By making use of ${}_1F_1(a;c;z) = e^z \ {}_1F_1(c-a;c;-z)$,

we get

$$R_{0,x}^{\alpha} \{x^K e^{-\lambda x} S_N^M\} = x^{\alpha+K} e^{-\lambda x} \sum_{i=0}^{[N/M]} \frac{(-N)_{Mi}}{i!}$$

$$A_{N,i} \frac{\Gamma(K+i+1)}{\Gamma(1+\alpha+K+i)} \times {}_1F_1(\alpha; \alpha+K+i+1; \lambda x)\, x^i$$

(2.37)

Now by making use of

$$L_\nu^\beta(Z) = \frac{\Gamma(\nu+\beta+1)}{\Gamma(\beta+1)} {}_1F_1(-\nu; \beta+1; Z);$$

we get

$$R_{0,x}^{\alpha} \{x^K e^{-\lambda x} S_N^M\} = x^{\alpha+K} \Gamma(1-\alpha) e^{-\lambda x}$$

$$\sum_{i=0}^{[N/M]} \frac{(-N)_{Mi}}{i!} A_{N,i} L_{-\alpha}^{\alpha+K+i}(\lambda x);$$

(2.38)

which is the Corollary obtained by [4].

3. Acknowledgement

Thanks to Prof. P.L. Sethi, J.N.V. University, Jodhpur for valuable suggestions to improve the paper.

4. Thanks to Referee

A lot of thanks to Referee and Reviewers.

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