Integral Solutions of $4w^2 - x^2 - y^2 + z^2 = t^2$

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Abstract: ***The Quadratic equation with five unknowns of the form*** $4w^2 - x^2 - y^2 + z^2 = t^2$ ***has been studied for its non-trivial distinct integral solutions. A few interesting relations among the solutions and special polygonal numbers are presented.***

Keywords: Quadratic equation with five unknowns, Integral solutions.

Mathematics Subject Classification Number: 11D09

1. Introduction

The Diophantine Equation offers an unlimited field for research because of their variety [1-3]. For an extensive review of problems, one may refer [4-15].

This communication concerns with another interesting quadratic Diophantine equation with five variables represented by $4w^2 - x^2 - y^2 + z^2 = t^2$ for determining its infinitely many non-zero distinct integral solutions. A few interesting relations among the solutions are presented.

2. Notations

$T_{10,n}$ = Decagonal number of rank n.

$T_{12,n}$ =Dodecagonal number of rank n.

$T_{14,n}$ = Tetradecagonal number of rank n.

$T_{18,n}$ = Octadecagonal number of rank n.

Gno_n = Gnomonic number of rank n.

SO_n = Stella Octangula number of rank n.

CS_n = Centered square number of rank n.

$Star_n$ = Star number of rank n.

HO_n = Hauy Octahedral number of rank n.

3. Method of Analysis

The Quadratic Diophantine equation with five unknowns under consideration is

$$4w^2 - x^2 - y^2 + z^2 = t^2 \qquad (1)$$

The substitution of the linear transformations

$$x = w + z \text{ and } y = w - z \qquad (2)$$

in (1) leads to

$$2w^2 - t^2 = z^2 \qquad (3)$$

Four different choices of solutions to (3) are presented below. Once the values of w and z are known, using (2), the corresponding values of x and y are obtained.

Pattern 1

Equation (3) can be written as,

$$w^2 - t^2 = z^2 - w^2$$

Factorizing the above equation, we have

$$(w+t)(w-t) = (z+w)(z-w)$$

$$\frac{w+t}{z+w} = \frac{z-w}{w-t} = \frac{A}{B}, B \neq 0$$

This is equivalent to the following two equations.

$$(A+B)w - At - Bz = 0$$

$$(B-A)w + Bt - Az = 0$$

Applying the method of cross multiplication, we get

$$w = w(A,B) = A^2 + B^2 \qquad (4)$$

$$t = t(A,B) = A^2 + 2AB - B^2 \qquad (5)$$

$$z = z(A,B) = B^2 + 2AB - A^2 \qquad (6)$$

Substituting (4), (5) and (6) in (2), we get

$$x = x(A,B) = 2B^2 + 2AB \qquad (7)$$

$$y = y(A,B) = 2A^2 - 2AB \qquad (8)$$

Thus (4) , (5) , (6) , (7) and (8) represent non-zero distinct integer solutions to (1) in two parameters.

Observations

1. $w(A,A) + x(A,A) = 6A^2$, is a nasty number.
2. $w(A,A) - t(A,A) + 2z(A,A) = 6A^2$, is a nasty number.
3. For all values of A and B, 2w+t+z is divisible by 2.
4. $w(A,1) - z(A,1) + t_{14,A} = t_{18,A}$
5. $w(A, A-1) = CS_A$
6. $3y(A,1) = Star_A - 1$

Pattern 2

Equation (3), can be written as

$$2w^2 - t^2 = z^2.1$$

Assuming $z = 2A^2 - B^2$ and

$$\text{write } 1 = (\sqrt{2}+1)(\sqrt{2}-1) \qquad (9)$$

Substituting (9) in (3) , using the method of factorization we get

$$(\sqrt{2}w + t)(\sqrt{2}w - t) = (\sqrt{2}A + B)^2 (\sqrt{2}A - B)^2 (\sqrt{2}+1)(\sqrt{2}-1)$$

International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2014): 5.611

New define,

$$(\sqrt{2}w+t)=(\sqrt{2}A+B)^2\left(\sqrt{2}+1\right) \qquad (10)$$

$$(\sqrt{2}w-t)=(\sqrt{2}A-B)^2\left(\sqrt{2}-1\right)$$

Equating the like terms in (10), we get

$$w=2A^2+2AB+B^2$$

$$t=2A^2+4AB+B^2$$

Thus, using the values of w and t performing a few calculations the values of x, y, z, w and t are obtained as follows:

$$\left.\begin{aligned} x&=x(A,B)=4A^2+2AB\\ y&=y(A,B)=2B^2+2AB\\ w&=w(A,B)=2A^2+2AB+B^2\\ z&=z(A,B)=2A^2-B^2\\ t&=t(A,B)=2A^2+4AB+B^2 \end{aligned}\right\} \qquad (11)$$

Thus (11) represents the non-trivial integral solutions of (1) in two parameters.

Observations

1. $x(A,-1)+y(1,A)=6A^2$, is a nasty number
2. $x(A,A)$ is a nasty number.
3. $x(1,1)+y(2,1)\equiv 0(\bmod 6)$.
4. For all values A and B, $x+y, t+w+y, t+w+x, t+w+x+y$ is divisible by 2.
5. $2x(A,1)-2t(A,1)=Gno_A^2$.
6. $nz(n,1)=So_n$.
7. $t+w+y$ is a perfect square, for all values of A and B.

Pattern 3

Consider the linear transformations

$$\left.\begin{aligned} w&=u-v\\ t&=u-2v \end{aligned}\right\} \qquad (12)$$

Substituting (12) in (3), we get

$$u^2=z^2+2v^2 \qquad (13)$$

This is in the standard form

$$x^2=Dy^2+z^2$$

Thus the corresponding solutions to (13) are

$$v=2AB$$

$$z=2A^2-B^2 \qquad (14)$$

$$u=2A^2+B^2$$

Thus, using the values of u, v, z and performing a few calculations, the values of x, y, z, w and t are obtained as follows:

$$\left.\begin{aligned} x&=x(A,B)=4A^2-2AB\\ y&=y(A,B)=2B^2-2AB\\ w&=w(A,B)=2A^2-2AB+B^2\\ t&=t(A,B)=2A^2-4AB+B^2\\ z&=z(A,B)=2A^2-B^2 \end{aligned}\right\} \qquad (15)$$

Thus (15) represents the non-trivial integral solutions of (1).

Observations

1. $x(A,1)+y(A,1)-z(1,A)=T_{12,A}$
2. $6(y+t+w)$ is a nasty number, for all values of A and B.
3. $w(n,1)+t(n,1)+z(n,1)=Star_n$.
4. $x(n+1,-n^2)+y(-n^2,n+1)-3HO_n-4T_{10,n}-8Gno_n\equiv 0(\bmod 17)$

Pattern 4

Equation (13) can be written as,

$$(u+z)(u-z)=v^2.2$$

Equating the positive and negative factors, we get

$$u+z=v^2 \qquad (16)$$

$$u-z=2 \qquad (17)$$

Solving (16) and (17), we get

$$u=\frac{v^2+2}{2} \qquad (18)$$

$$z=\frac{v^2-2}{2} \qquad (19)$$

As our interest is on finding integer solutions, choose v so that u and z are integers.

Write $v=2A+2$ (20)

Substituting (20) in (18) and (19), we get

$$\left.\begin{aligned} u&=2A^2+4A+3\\ z&=2A^2+4A+1 \end{aligned}\right\} \qquad (21)$$

Substituting (21) in (16), (17), (12) and (2) we get

$$\left.\begin{aligned} x&=x(A)=4A^2+6A+2\\ y&=y(A)=-2A\\ w&=w(A)=2A^2+2A+1\\ t&=t(A)=2A^2-1\\ z&=z(A)=2A^2+4A+1 \end{aligned}\right\} \qquad (22)$$

Thus (22) represents the non-trivial integral solution of (1) in one parameter.

Observations

1. $x(A)+z(A)+y(8A)-2=Star_A$
2. For all values of A, $z+t$ is divisible by 4.
3. $A.t(A)=SO_A$

4. Conclusion

One may search for other patterns of solution of our considered equation (1).

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