Integral Solutions of $4w^2 - x^2 - y^2 + z^2 = t^2$

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Abstract: The Quadratic equation with five unknowns of the form $4w^2 - x^2 - y^2 + z^2 = t^2$ has been studied for its non-trivial distinct integral solutions. A few interesting relations among the solutions and special polygonal numbers are presented.

Keywords: Quadratic equation with five unknowns, Integral solutions.

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1. Introduction

The Diophantine Equation offers an unlimited field for research because of their variety [1-3]. For an extensive review of problems, one may refer [4-15].

This communication concerns with another interesting quadratic Diophantine equation with five variables represented by $4w^2 - x^2 - y^2 + z^2 = t^2$ for determining its infinitely many non-zero distinct integral solutions. A few interesting relations among the solutions are presented.

2. Notations

- $D_{10,n} =$ Decagonal number of rank n.
- $D_{12,n} =$Dodecagonal number of rank n.
- $D_{14,n} =$ Tetradecagonal number of rank n.
- $D_{18,n} =$ Octadecagonal number of rank n.
- $Go_{n} =$ Gnomonic number of rank n.
- $Stn =$ Stella Octangula number of rank n.
- $CS_{n} =$ Centered square number of rank n.
- $St_{n} =$ Star number of rank n.
- $HO_{n} =$ Hauy Octahedral number of rank n.

3. Method of Analysis

The Quadratic Diophantine equation with five unknowns under consideration is

$$4w^2 - x^2 - y^2 + z^2 = t^2 \quad (1)$$

The substitution of the linear transformations

$$x = w + z \quad \text{and} \quad y = w - z \quad (2)$$

in (1) leads to

$$2w^2 - t^2 = z^2 \quad (3)$$

Four different choices of solutions to (3) are presented below. Once the values of w and z are known, using (2), the corresponding values of x and y are obtained.

Pattern 1

Equation (3) can be written as,

$$w^2 - t^2 = z^2 - w^2 \quad (4)$$

Factorizing the above equation, we have

$$w + t = z - w \quad (5)$$

$$\frac{z + w}{w - t} = \frac{A}{B} \quad B \neq 0 \quad (6)$$

This is equivalent to the following two equations

$$(A + B)w - At - Bz = 0 \quad (7)$$

$$(B - A)w + Bt - Az = 0 \quad (8)$$

Applying the method of cross multiplication, we get

$$w = w(A, B) = A^2 + B^2 \quad (9)$$

$$t = t(A, B) = A^2 + 2AB - B^2 \quad (10)$$

$$z = z(A, B) = B^2 + 2AB - A^2 \quad (11)$$

Substituting (4), (5) and (6) in (2), we get

$$x = x(A, B) = 2B^2 + 2AB \quad (12)$$

$$y = y(A, B) = 2A^2 - 2AB \quad (13)$$

Thus (4) , (5) , (6) , (7) and (8) represent non-zero distinct integer solutions to (1) in two parameters.

Observations

1. $w(A, A) + x(A, A) = 6A^2$, is a nasty number.
2. $w(A, A) - t(A, A) + z(A, A) = 6A^2$, is a nasty number.
3. For all values of A and B, $2w - t + z$ is divisible by 2.
4. $w(A, 1) - z(A, 1) + t_{11,1} = t_{11,1}$
5. $w(A, A - 1) = CS_{A}$
6. $3y(A, 1) = Star_{A}$

Pattern 2

Equation (3), can be written as

$$2w^2 - t^2 = z^2 \quad (14)$$

Assuming $z = 2A^2 - B^2$ and

write $1 = (\sqrt{2} + 1)(\sqrt{2} - 1) \quad (15)$

Substituting (15) in (14), using the method of factorization we get

$$(\sqrt{2}w + t)(\sqrt{2}w - t) = (\sqrt{2}A + B)^2 (\sqrt{2}A - B)^2 (\sqrt{2} + 1)(\sqrt{2} - 1) \quad (16)$$

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New define,
\[
\begin{align*}
(\sqrt{2}w + t) &= (\sqrt{2}A + B)^2 (\sqrt{2} + 1) \\
(\sqrt{2}w - t) &= (\sqrt{2}A - B)^2 (\sqrt{2} - 1)
\end{align*}
\]
Equating the like terms in (10), we get
\[
\begin{align*}
w &= 2A^2 + 2AB + B^2 \\
t &= 2A^2 + 4AB + B^2
\end{align*}
\]\nThus, using the values of \(w\) and \(t\) performing a few calculations the values of \(x\), \(y\), \(z\), \(u\), and \(v\) are obtained as follows:
\[
\begin{align*}
x &= x(A, B) = 4A^2 - 2AB \\
y &= y(A, B) = 2B^2 + 2AB \\
w &= w(A, B) = 2A^2 - 2AB + B^2 \\
t &= t(A, B) = 2A^2 - 4AB + B^2 \\
z &= z(A, B) = 2A^2 - 2B^2
\end{align*}
\]
Thus (11) represents the non-trivial integral solutions of (1).

Observations
1. \(x(A, 1) + y(A, 1) - z(1, 4) = T_{12,4}\)
2. \(5(y + t + w)\) is a nasty number, for all values of \(A\) and \(B\).
3. \(w(n, 1) + t(n, 1) + z(n, 1) = Star_n\).
4. \((x(n+1-n^2)+y(n-n^2)+z(n-1)+3H_2)+4T_{n^2}-6G_{n^2}=0 \mod(17)\)

Pattern 4

Equation (13) can be written as,
\[
(u + z)(u - z) = v^2 \cdot 2
\]
Equate the positive and negative factors, we get
\[
\begin{align*}
u + z &= v^2 \\
u - z &= 2
\end{align*}
\]
Solving (16) and (17), we get
\[
\begin{align*}
u &= \frac{v^2 + 2}{2} \\
z &= \frac{v^2 - 2}{2}
\end{align*}
\]
As our interest is on finding integer solutions, choose \(v\) so that \(u\) and \(z\) are integers.

Write \(v = 2A + 2\) \hspace{1cm} (20)
Substituting (20) in (18) and (19), we get
\[
\begin{align*}
u &= 2A^2 + 4A + 3 \\
z &= 2A^2 + 4A + 1
\end{align*}
\]
Substituting (21) in (16), (17), (12) and (2) we get
\[
\begin{align*}
x &= x(A) = 4A^2 + 6A + 2 \\
y &= y(A) = -2A \\
w &= w(A) = 2A^2 + 2A + 1 \\
t &= t(A) = 2A^2 - 1 \\
z &= z(A) = 2A^2 + 4A + 1
\end{align*}
\]
Thus (22) represents the non-trivial integral solution of (1) in one parameter.

Observations
1. \(x(A) + z(A) + y(8A) - 2 = Star_A\)
2. For all values of \(A\), \(z + t\) is divisible by 4.
3. \(A.(t(A)) = SO_A\)
4. Conclusion

One may search for other patterns of solution of our considered equation (1).

References


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Dr. G. Janaki received the Ph.D., M.Sc., and M.Phil., degree in mathematics from Bharathidasan University, Trichy, South India. She completed her Ph.D. degree National College, Bharathidasan University. She has published many papers in international and national level journals. Her research area is “Number Theory”.

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