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Compact Toeplitz Operators on pluriharmonic Bergman spaces

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Abstract: *On the setting of the unit ball, we characterize compact Toeplitz operators on pluriharmonic Bergman spaces $b^{1+\epsilon}$, $0<\epsilon<\infty$, in terms of the boundary vanishing conditions of the Berezin transform and certain differential quantity of the symbol. As a consequence, we characterize $\mathcal{M}$-harmonic and radial symbols of compact Toeplitz operators.*

Keywords: *Toeplitz Operators ; Berezin transform; pluriharmonic function; Bergman spaces.*

1. Introduction

Let B be the open unit ball of the complex n-space $\mathbb{C}^n$ and V denote the normalized Lebesgue volume measure on B. For $0\le\epsilon<\infty$, let $L^{1+\epsilon}=L^{1+\epsilon}(B,V)$ be the usual Lebesgue space and put

$$\|u\|_{1+\epsilon}=\left(\int_B |f|^{1+\epsilon}\,dV\right)^{\frac{1}{1+\epsilon}}$$

for $f\in L^{1+\epsilon}$. The Bergman space $A^{1+\epsilon}$ is a subspace of $L^{1+\epsilon}$ consisting of all holomorphic functions on B. A function $u\in C^2(B)$ is said to be pluriharmonic if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. So, every pluriharmonic function is just harmonic on the unit disk. The pluriharmonic Bergman space $b^{1+\epsilon}$ is the subspace of $L^{1+\epsilon}$ consisting of all pluriharmonic functions on B. It is known that $A^{1+\epsilon}$ and $b^{1+\epsilon}$ are closed subspaces of $L^{1+\epsilon}$ and hence are Banach spaces. Clearly, $A^{1+\epsilon}\subset b^{1+\epsilon}$.

We let $P: L^2\to A^2$ and $Q: L^2\to b^2$ be the Hilbert space orthogonal projections respectively. As is well known, P is the well known Bergman projection given by

$$P\varphi(z)=\int_B \frac{\varphi(w)}{(1-z.\bar{w})^{n+1}}dV(w)$$

for functions $\varphi\in L^2$. Here, $z.\bar{w}=z_1\bar{w}_1+\cdots+z_n\bar{w}_n$ denotes the Hermitian inner product on $\mathbb{C}^n$. Also, it turns out that Q is an integral operator represented by

$$Q\varphi(z)=\int_B\left(\frac{1}{(1-w.\bar{z})^{n+1}}+\frac{1}{(1-z.\bar{w})^{n+1}}-1\right)\varphi(w)dV(w)$$

for functions $\varphi\in L^2$. These integral formulas for P and Q allow us to extend the domains of P and Q to L^1. Note that Q can be rewritten as

$$Q(\varphi)=P(\varphi)+\overline{P(\bar{\varphi})}-P(\varphi)(0) \qquad (1)$$

for functions $\varphi\in L^1$.

Let $u\in L^1$. The (Bergman space) Toeplitz operator $T_u^a: A^{1+\epsilon}\to A^{1+\epsilon}$ with symbol u is the linear operators defined by

$$T_u^a f=P(uf)$$

for $f\in A^{1+\epsilon}$ with $uf\in L^1$.

Also, the (pluriharmonic Bergman space) Toeplitz operator $T_u: b^{1+\epsilon}\to b^{1+\epsilon}$ with symbol u is defined by

$$T_u f=Q(uf)$$

for functions $f\in b^{1+\epsilon}$ with $uf\in L^1$. Clearly, T_u^a and T_u are densely defined and not bounded in general.

For $z\in B$, we let k_z be the normalized holomorphic Bergman kernel given by

$$k_z(w)=\frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1-w.\bar{z})^{n+1}} \quad (w\in B).$$

Given a bounded operator T on $A^{1+\epsilon}$ or $b^{1+\epsilon}$, the Berezin transform $\tilde{T}$ is a function on B defined by

$$\tilde{T}(z)=\langle Tk_z,k_z\rangle \quad (z\in B).$$

Here and else where, we use the usual pairing

$$\langle f,g\rangle=\int_B f\bar{g}\,dV$$

whenever $f\bar{g}\in L^1$. Given $u\in L^1$, we note $\tilde{T}_u^a=\tilde{T}_u$. Thus we let $\tilde{u}$ denote the Berezin transform of T_u^a or T_u.

Note that

$$\tilde{u}(z)=\int_B u|k_z|^2\,dV \quad (z\in B).$$

In this paper, we are concerned with a characterization problem of compact Toeplitz operators on the pluriharmonic Bergman spaces $b^{1+\epsilon}$ of the ball. Recently, this problem has been studied on the Bergman spaces and harmonic Bergman spaces of the unit disk.

Axler and Zheng [1] proved that if T equals a finite sum of the form $T_{u_1}^a\dots T_{u_k}^a$ where each u_i is bounded on the unit disk, then T is compact on A^2 of the unit disk if and only if the Berezin transformon $\tilde{T}$ vanishes on the boundary of the unit disk. Later, this result was extended to the unit ball and bounded symmetric domains in [2] and [3] respectively. Recently , Miao and Zeng, [4] considered the same problem for bounded operators on $A^{1+\epsilon}$, $0 < \epsilon < \infty$, of the unit disk with certain integrable conditions and proved that the operator under consideration is compact if and only if the Berezin transfom of the operator vanishes on the boundary of the unit disk. As consequence of the result, they extended the result of Axler and Zheng [1] to all $A^{1+\epsilon}$, $0 < \epsilon < \infty$, and more general symbols in the class BT. (see the below for the definition).

The corresponding problem has been also considered for Toeplitz operators on the pluriharmonic Bergman spaces. Stroethoff [5] proved that a Toeplitz operator with bounded radial symbol is compact on b^2 of the unit disk if and only if the Berezin transform of the symbol vanishes on the boundary of the unit disk. In [6], the same was proved for positive symbols in L^1. Recently, K. Guo and D. Zheng [7] characterized compact Toeplitz operators with bounded symbol on b^2 of the unit disk in terms of the boundary vanishing condition of the Berezin transform and certain differential quantity of the symbol.

We let BT be the set of all functions $f \in L^1$ for which

$$\|f\|_{BT} = \sup_{z\in B}|\tilde{u}(z)| < \infty.$$

Note that $L^\infty \subset BT$.

In this paper, we consider the same characterizing problem of compact Toeplitz operators with symbols in BT on the unit ball. In Section 2, we first show that symbols in BT induce bounded Toeplitz operators on $A^{1+\epsilon}$ and $b^{1+\epsilon}$ respectively for $0 < \epsilon < \infty$. (see Theorem 4). In Section 3, we will extend the result of Miao and Zheng [4] to the ball (see Theorem 7). As an application, we characterize the compactness of Toeplitz operators with symbols in BT on $A^{1+\epsilon}$. In Section 4, we will use the result In Section 3, to obtain a characterization of compact Toeplitz operators with symbol in BT on $b^{1+\epsilon}$, $0 < \epsilon < \infty$, in terms of the boundary vanishing condition of the Berezin transform and certain differential quantity of the symbol (see Theorem 17). This result extends the result in [7] where ϵ is assumed to be 1 and all symbols are assumed to be in L^∞. As applications, we obtain characterizations of $\mathcal{M}$-harmonic and radial symbols in BT for which the corresponding Toeplitz operators are compact. See Corollaries 18 and 19.

2. Toeplitz Operators With Sympols In BT

Throughout this paper, we will often abbreviate inessential constants involved in inequalities by writing $A \lesssim B$ for positive quantities A and B if the ratio A/B has a positive upper bound. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Given $\epsilon \in (1,\infty)$, we let $(1+\epsilon)'$ be the conjugate exponent of $1+\epsilon$, i.e. $\frac{1}{1+\epsilon} + \frac{1}{(1+\epsilon)'}$.

For $z, w \in B$, $z \neq 0$, define

$$\varphi_z(w) = \frac{z - |z|^{-2}(w.\bar{z})z - \sqrt{1-|z|^2}\,[w - |z|^{-2}(w.\bar{z})z]}{1 - w.\bar{z}}$$

and $\varphi_0(w) = -w$. Then each φ_z, is a biholomorphic self-maps of B and $\varphi_z \circ \varphi_z$ is the identity on B. We also have

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1 - z.\bar{w}|^2}, \quad (z, w \in B) \quad (2)$$

See Section 2 of [8] for details. The pseudo hyperbolic ball $E_r(z)$ with center $z \in B$ and $r \in (0,1)$ is defined by $E_r(z) = \varphi_z(rB)$. It is well known that

$$V(E_r(z)) \approx (1-|z|^2)^{n+1} \quad (3)$$

for every $z \in B$.

Proposition 1. Let $0 \leq \epsilon < \infty$, $r \in (0,1)$ and μ be a positive Borel measure on B. Then the following quantities are all equivalent.

(a) $\sup_{0\neq f\in A^{1+\epsilon}} \frac{\int_B |f|^{1+\epsilon} d\mu}{\int_B |f|^{1+\epsilon} dV}$.

(b) $\sup_{z\in B} \int_B |k_z|^2 d\mu$.

(c) $\sup_{z\in B} \frac{\mu(E_r(z))}{V(E_r(z))}$.

(d) $\sup_{0\neq f\in b^{1+\epsilon}} \frac{\int_B |f|^{1+\epsilon} d\mu}{\int_B |f|^{1+\epsilon} dV}$.

Proof . The equivalences of (a), (b) and (c) are well known. See [9] for example. Note $A^{1+\epsilon} \subset b^{1+\epsilon}$. So, to complete the proof, we only need to show that

$$\sup_{0\neq f\in b^{1+\epsilon}} \frac{\int_B |f|^{1+\epsilon} d\mu}{\int_B |f|^{1+\epsilon} dV} \lesssim \sup_{z\in B} \frac{\mu(E_r(z))}{V(E_r(z))}.$$

Let $f \in b^{1+\epsilon}$. By Proposition 10.1, of [10] and (3), we have

$$|f(z)|^{1+\epsilon} \lesssim \int_{E_r(z)} \frac{|f(w)|}{(1-|w|^2)^{n+1}} dV(w) \lesssim \int_{E_r(z)} \frac{|f(w)|^{1+\epsilon}}{V(E_r(w))} dV(w)$$

for all $z \in B$. Note that $\chi_{E_r(z)}(w) = \chi_{E_r(w)}(z)$ for all $z, w \in B$. Here, the notation χ_F denotes the characteristic function of $F \subset B$. It follows from Fubini's theorem that

$$\int_B |f|^{1+\epsilon} d\mu \lesssim \int_B \int_B \frac{\chi_{E_r(z)}(w)|f(w)|^{1+\epsilon}}{V(E_r(w))} dV(w)\,d\mu(z)$$

$$= \int_B \int_B \frac{\chi_{E_r(w)}(z)|f(w)|^{1+\epsilon}}{V(E_r(w))} d\mu(z)\,dV(w) = \int_B \frac{\mu(E_r(w))|f(w)|^{1+\epsilon}}{V(E_r(w))} dV(w)$$

$$\leq \sup_{w\in B} \frac{\mu(E_r(w))}{V(E_r(w))} \int_B |f|^{1+\epsilon} dV$$

This completes the proof.

Given $0 < \epsilon < \infty$, it is well known that P is a bounded projection from $L^{1+\epsilon}$ onto $A^{1+\epsilon}$. We let $A_0^{1+\epsilon}$ denote the space of all functions f in $A^{1+\epsilon}$ such that $f(0) = 0$. As is well known, every pluriharmonic functions u on B has a unique decomposition $u = f + \bar{g}$ where f, g are holomorphic and $f(0) = 0$. Furthermore, if $u \in b^{1+\epsilon}$ then both $f, g \in A^{1+\epsilon}$; this is clearly a consequence of the $L^{1+\epsilon}$-boundedness of the Bergman projection P. So, we have a decomposition

$$b^{1+\epsilon} = A_0^{1+\epsilon} + \overline{A^{1+\epsilon}}.$$

Proposition 2. For $0 < \epsilon < \infty$, Q is a bounded projection from $L^{1+\epsilon}$ onto $b^{1+\epsilon}$.

Proof . First note that $|F(0)| \leq \|F\|_{1+\epsilon}$ for every $F \in A^{1+\epsilon}$ Since $Qf = Pf + \overline{P(\bar{f})} - P(f)(0)$

for every $f \in L^{1+\epsilon}$ by (1), we have by the $L^{1+\epsilon}$-boundedeness of P,

$$\|Qf\|_{1+\epsilon} = \left\| Pf + \overline{P(\bar{f})} - P(f)(0) \right\|_{1+\epsilon} \le \|Pf\|_{1+\epsilon} + \left\| P(\bar{f}) \right\|_{1+\epsilon} + |P(f)(0)| \lesssim \|Pf\|_{1+\epsilon} \lesssim \|f\|_{1+\epsilon}$$

for every $f \in L^{1+\epsilon}$. Hence Q is bounded from $L^{1+\epsilon}$ into $b^{1+\epsilon}$.

Using the fact that P is a projection from $L^{1+\epsilon}$ onto $A^{1+\epsilon}$ and the decomposition $b^{1+\epsilon} = A_0^{1+\epsilon} + \overline{A^{1+\epsilon}}$, we see Q is a projection from $L^{1+\epsilon}$ onto $b^{1+\epsilon}$. The proof is complete.

It is known that the dual of $A^{1+\epsilon}$ is $A^{(1+\epsilon)'}$ under the pairing $<,>$. Also, we have the analogous dualities for harmonic Bergman spaces.

Proposition 3. For $0 < \epsilon < \infty$, the spaces $b^{1+\epsilon}$ and $b^{(1+\epsilon)'}$ are dual to each other proposition under the pairing $<,>$.

Proof . This follows from the Hahn-Banach extension theorem and the $L^{1+\epsilon}$-boundedness of Q. This completes the proof.

The next theorem says that symbols in BT induce bounded Toeplitz operators on both $A^{1+\epsilon}$ and $b^{1+\epsilon}$ for $0 < \epsilon < \infty$.

Theorem 4. Let $u \in BT$ and $0 < \epsilon < \infty$. Then $T_u^a: A^{1+\epsilon} \to A^{1+\epsilon}$ is bounded and $\|T_u^a\| \lesssim \|u\|_{BT}$. Also, $T_u: b^{1+\epsilon} \to b^{1+\epsilon}$ is bounded and $\|T_u\| \lesssim \|u\|_{BT}$.

Proof . Let $f \in b^{1+\epsilon}$ and $g \in b^{(1+\epsilon)'}$. By HÖlder's inequality and Proposition 1,

$$|\langle T_u f, g\rangle| = |\langle uf, g\rangle| \le \int_B |\langle ufg\rangle| dV$$
$$\le \left(\int_B |f|^{1+\epsilon} |u| dV \right)^{\frac{1}{1+\epsilon}} \left(\int_B |g|^{(1+\epsilon)'} |u| dV \right)^{\frac{1}{(1+\epsilon)'}}$$
$$\lesssim \|u\|_{BT} \|f\|_{1+\epsilon} \|g\|_{(1+\epsilon)'}$$

Hence, by Proposition 3, T_u is bounded on $b^{1+\epsilon}$.

Since the dual of $A^{1+\epsilon}$ is $A^{(1+\epsilon)'}$, the si.milar argument can be applied to prove the boundedness of T_u^a on $A^{1+\epsilon}$. The proof is complete.

Corollary 5. For $\epsilon \ge 0$, Q is a bounded frgeition from $L^{1+\epsilon}$ onto $(b)^{1+\epsilon}$.

Proof . note that $|F(0)| \le \|F\|_P$ for every $F \in A^{1+\epsilon}$. Since

$$\sum_{j=1}^{r} Qf_j = \sum_{j=1}^{r} Pf_j + \sum_{j=1}^{r} \overline{P(\bar{f_j})} - \sum_{j=1}^{r} P(f_j)(0)$$

for every $f_j \in L^{1+\epsilon}$. We have by the $L^{1+\epsilon}$-boundedeness of P,

$$\sum_{j=1}^{r} \|Qf_j\|_{1+\epsilon} = \sum_{j=1}^{r} \left\| Pf_j + \overline{P(\bar{f_j})} - P(f_j)(0) \right\|_{1+\epsilon}$$
$$\le \sum_{j=1}^{r} \|Pf_j\|_{1+\epsilon} + \sum_{j=1}^{r} \left\| P(\bar{f_j}) \right\|_{1+\epsilon} + \sum_{j=1}^{r} |P(f_j)(0)| \lesssim \sum_{j=1}^{r} \|Pf_j\|_{1+\epsilon}$$
$$\lesssim \sum_{j=1}^{r} \|f_j\|_{1+\epsilon}$$

for every $f_j \in L^{1+\epsilon}$. Hence Q is bounded from $L^{1+\epsilon}$ into $b^{1+\epsilon}$.

Using the fact that P is a projection from $P: L^{1+\epsilon} \to A^{1+\epsilon}$ and the decomposition $b^{1+\epsilon} = A_0^{1+\epsilon} + \overline{A^{1+\epsilon}}$. Hence Q is $L^{1+\epsilon} \to b^{1+\epsilon}$ is a projection from [11]. The proof is complete.

(3) Compact Toeplitz Operators On $A^{1+\epsilon}$

In this section, we characterize compact Toeplitz operators with symbol in **BT** on $A^{1+\epsilon}$ in terms of the boundary vanishing property of the Berezin transform of the symbol. In fact, we generally characterize the compactness of bounded operators on $A^{1+\epsilon}$ with some integrable condition in terms of the boundary vanishing property of its Berezin transform. Our method will be based on a recent result of [4] where J.Miao and D. Zheng proved the same characterization on the unit disk.

For each point $z \in B$, let U_z be the operator defined by $U_z f = (f \circ \varphi_z) k_z$. Then, one can prove that each U_z, is bounded on $A^{1+\epsilon}$ for $\epsilon > 0$. Given a bounded operator T on $A^{1+\epsilon}, \epsilon > 0$, we define an operator T_z by $T_z = U_z T U_z$. Note that

$$\tilde{T} \circ \varphi_z = \tilde{T}_z, \quad (z \in B). \qquad (4)$$

This was proved in lemma 3 of [2] for $\epsilon = 1$. But, the same proof works for all $1 + \epsilon$.

Lemma 6. For $z \in B, c$ real , $t > -1$, we define

$$I_{c,t}(z) = \int_B \frac{(1-|w|^2)^t}{|1-z.\overline{w}|^{n+1+t+c}} dV(w), (z \in B).$$

If $c < 0$, then $I_{c,t}$ is bounded on B. If $c > 0$, then $I_{c,t}(z) \approx (1 - |z|^2)^{-c}$ as $|z| \to 1$.

Proof . See Proposition 1.4.10 of [8].

For $z \in B$, we let K_z be the holomorphic Bergman kernel given by

$$K_z(w) = \frac{1}{(1-w.\bar{z})^{n+1}}, \quad (z \in B).$$

Using Lemma 6, we see for each $0 < \epsilon < \infty$,

$$\|K_z\|_{1+\epsilon} \approx (1 - |z|^2)^{-\frac{n+1}{(1+\epsilon)'}} \qquad (5)$$

for $z \in B$.

Using the power series representation of K_z, we can write $\tilde{T}$ for a bounded operators T on $A^{1+\epsilon}$ as a power series:

$$\tilde{T}(w) = (1-|w|^2)^{n+1} \sum_{\alpha,\beta} C_\alpha C_\beta \langle Tw^\alpha, w^\beta \rangle z^\alpha \overline{w}^\beta, \qquad (w \in B) \ (6)$$

where $C_\gamma = (n + 1 + |\gamma|)!/n!\gamma!$.

Lemma 7. Let $0 < \epsilon < \infty$. Suppose $T: A^{1+\epsilon} \to A^{1+\epsilon}$ is bounded for which

$$\sup_{z \in B} \|T_z 1\|_m < \infty$$

for some $m > 1$. Then $\tilde{T}(z) \to 0$ as $|z| \to 1$ if and only if for every $t \in [1, m)$, $\|T_z 1\|_t \to 0$ as $|z| \to 1$.

Proof . First suppose that for any $t \in [1, m), \|T_z\|_t \to 0$ as $|z| \to 1$. In particular $\|T_z 1\|_1 \to 0$ as $|z| \to 1$. Hence

$$|\tilde{T}(z)| = |\langle Tk_z, k_z\rangle| = \langle U_z T U_z 1, 1\rangle \le \|T_z 1\|_1$$

Thus, we have $\tilde{T}(z) \to 0$ as $|z| \to 1$.

Now suppose $\tilde{T}(z) \to 0$ as $|z| \to 1$. Fix $t \in [1,m)$ and show, $\|T_z 1\|_t \to 0$ as $|z| \to 1$. By (5), we note that

$$\left|\langle T_z w^\alpha, w^\beta\rangle\right| = (1-|z|^2)^{n+1}\left|\langle T[w^\alpha \circ \varphi_z k_z], w^\beta \circ \varphi_z k_z\rangle\right| \leq (1-|z|^2)^{n+1}\|T\|\|K_z\|_{1+\epsilon}\|K_z\|_{(1+\epsilon)'} \lesssim \|T\|$$

for any $z \in B$ and multi-indices α, β. Hence $|\langle T_z w^\alpha, w^\beta\rangle|$ is uniformly bounded for $z \in B$ and multi-indices α, β. By (4) and (6),

$$\tilde{T}(\varphi_z(w)) = \tilde{T}_z(w) = (1-|w|^2)^{n+1}\sum_{\alpha,\beta} C_\alpha C_\beta < \langle T_z w^\alpha, w^\beta\rangle z^\alpha \overline{w}^\beta \quad ,(z,w \in B)$$

Since $|\varphi_z(w)| \to 1$ as $|z| \to 1$ for each $w \in B$, by the same argument of the proof of Lemma 14 of [4], we can show that $\langle T_z 1, w^\alpha\rangle \to 0$ as $|z| \to 1$ for every multi index α. For $w \in B$, we note that

$$(T_z 1)(w) = \langle T_z 1, K_w\rangle = \sum_\alpha C_\alpha \langle T_z 1, w^\alpha\rangle w^\alpha$$

Also, the same method as in the proof of Lemma 14 of [4] can be applied to show that $\|T_z\|_t \to 0$ as $|z| \to 1$.

The following is the main result of this section.

Theorem 8. Let $0 < \epsilon < \infty$ and $(1+\epsilon)_1 = min\{1+\epsilon, (1+\epsilon)'\}$. Suppose T is bounded on $A^{1+\epsilon}$ for which

$$\sup_{z\in B}\|T_z 1\|_m < \infty \ and \ \sup_{z\in B}\|T_z^* 1\|_m < \infty$$

for some $m > \frac{n+2}{(1+\epsilon)_1 - 1}$. Then T is compact on $A^{1+\epsilon}$ if and only if $\tilde{T}(z) \to 0$ as $|z| \to 1$.

Proof . First suppose T is compact on $A^{1+\epsilon}$. By (5), we note that

$$\tilde{T}(z) = \langle Tk_z, k_z\rangle = (1-|z|^2)^{n+1}\langle TK_z, K_z\rangle \approx \langle T\frac{K_z}{\|K_z\|_{1+\epsilon}}, \frac{K_z}{\|K_z\|_{(1+\epsilon)'}}\rangle$$

for every $z \in B$. Since $K_z/\|K_z\|_{1+\epsilon} \to 0$ weakly in $A^{1+\epsilon}$ as $|z| \to 1$, we have $\tilde{T}(z) \to 0$ as $|z| \to 1$.

Suppose $\tilde{T}(z) \to 0$ as $|z| \to 1$. By Lemma 7, we have $\|T_z\|_t \to 0$ as $|z| \to 1$ for every $t \in [1,m)$. Fix t such that $\frac{n+2}{(1+\epsilon)_1 - 1} < t < m$ in the rest of the proof. To prove the compactness of T, we first note that

$$(T^* K_w)(z) = \langle T^* K_w, K_z\rangle = \langle K_w, TK_z\rangle = \overline{(TK_z)(w)}, \quad (z,w \in B) \ (7)$$

It follows that

$$(Tf)(w) = \langle Tf, K_w\rangle = \langle f, T^* K_w\rangle = \int_B f(z)\overline{(T^* K_w)(z)}dV(z) = \int_B f(z)(TK_z)(w)dV(z) \qquad (8)$$

for every $f \in A^{1+\epsilon}$. For each $0 < r < 1$, define an operator T_r on $A^{1+\epsilon}$ by

$$T_r f(w) = \int_{rB} f(z)(TK_z)(w)dV(z), \qquad (w \in B)$$

By (5), we have

$$\int_B\left(\int_B |TK_z(w)\chi_{rB}(z)|^{1+\epsilon}dV(w)\right)^{(1+\epsilon)'-1} dV(z) \leq \int_{rB}\left(\int_B |TK_z|^{1+\epsilon}dV\right)^{(1+\epsilon)'-1} dV(z) \leq \int_{rB}\|T\|^{(1+\epsilon)'}\|K_z\|_{1+\epsilon}^{(1+\epsilon)'}dV(z) \leq \|T\|^{(1+\epsilon)'}\int_{rB}\frac{1}{(1-|z|^2)^{n+1}}dV(z) \leq \frac{\|T\|^{(1+\epsilon)'}}{(1-r^2)^{n+1}}$$

for each r. Using Exercise 7 on page 181 of [12], we see that each T_r is compact on $A^{1+\epsilon}$. Hence, to prove the compactness of T, we only need to show that $\|T - T_r\| \to 0$ as $r \to 1$. Note that

$$[(T-T_r)f](w) = \int_B f(z)T(w,z)dV(z), (w \in B, \ f \in A^{1+\epsilon})$$

where $T(w,z) = (TK_z)(w)\chi_r(z)$ and $\chi_r = \chi_{B\backslash rB}$. Let $h(z) = \frac{1}{(1-|z|^2)^\alpha}$ where $\alpha = \frac{(n+1)((1+\epsilon)_1 - 1)}{(n+2)(1+\epsilon)_1}$.

Note that

$$TK_z(w) = \frac{T_z 1(\varphi_z(w))k_z(w)}{(1-|z|^2)^{\frac{n+1}{2}}} \ , \ (z,w \in B).$$

Using (2), we see

$$\frac{|k_z(w)|}{(1-|z|^2)^{\frac{n+1}{2}}(1-|\varphi_z(w)|^2)^{\alpha(1+\epsilon)}} = \frac{h(z)^{1+\epsilon}(1-|w|^2)^{-\alpha(1+\epsilon)}}{|1-z.\overline{w}|^{n+1-2\alpha(1+\epsilon)}}, (z,w \in B)$$

Since the real Jacobian of φ_z is $|k_z|^2$ and $k_z(\varphi_z(w))k_z(w) = w$ for every $z, w \in B$, we have by a change of variables and Hölder's inequality,

$$\int_B |T(w,z)|h(w)^{1+\epsilon}dV(w) = \int_B \frac{|(TK_z)(w)\chi_r(z)|}{(1-|z|^2)^{\alpha(1+\epsilon)}}dV(w)$$

$$= \frac{\chi_{B\backslash rB}(z)}{(1-|z|^2)^{\frac{n+1}{2}}}\int_B \frac{|T_z 1(\varphi_z(w))k_z(w)|}{(1-|w|^2)^{\alpha(1+\epsilon)}}dV(w)$$

$$= \frac{\chi_{B\backslash rB}(z)}{(1-|z|^2)^{\frac{n+1}{2}}}\int_B \frac{|T_z 1(w)k_z(\varphi_z(w))||k_z(w)|^2}{(1-|\varphi_z(w)|^2)^{\alpha(1+\epsilon)}}dV(w)$$

$$= \chi_{B\backslash rB}(z)h(z)^{(1+\epsilon)}\int_B \frac{|T_z 1(w)|(1-|w|^2)^{-\alpha(1+\epsilon)}}{(1-z.\overline{w})^{n+1-2\alpha(1+\epsilon)}}dV(w)$$

$$= \chi_{B\backslash rB}(z)h(z)^{1+\epsilon}\left(\int_B |T_z 1|^t dV\right)^{\frac{1}{t}}\left(\int_B \frac{(1-|w|^2)^{-\alpha(1+\epsilon)t'}}{|1-z.\overline{w}|^{t'(n+1-2\alpha(1+\epsilon))}}dV(w)\right)^{\frac{1}{t'}}$$

On the other hand, since $\frac{n+2}{(1+\epsilon)_1 - 1} < t < m$, one can easily check that $-\alpha(1+\epsilon)t' > -1$ and $t'(n+1-2\alpha(1+\epsilon)) > n+1-\alpha(1+\epsilon)t'$

It follows from Lemma 6 that

$$\sup_{z\in B}\int_B \frac{(1-|w|^2)^{-\alpha(1+\epsilon)t'}}{|1-z.\overline{w}|^{t'(n+1-2\alpha(1+\epsilon))}}dV(w) < \infty.$$

Hence

$$\int_B |T(w,z)|h(w)^{1+\epsilon}dV(w) \lesssim h(z)^{1+\epsilon}\sup_{r<|z|<1}\|T_z 1\|_t, (z \in B) \quad (9)$$

By (7), we note $(T^* K_w)(z) = \overline{TK_z(w)}$ for all $z, w \in B$. The similar method we have done above gives

$$\int_B |T(w,z)|h(z)^{(1+\epsilon)'}dV(z) = \int_B \frac{|TK_z(w)\chi_r(z)|}{(1-|z|^2)^{\alpha(1+\epsilon)'}}dV(z) = \int_B \frac{|(T^* K_w)(z)\chi_r(z)|}{(1-|z|^2)^{\alpha(1+\epsilon)'}}dV(z) \lesssim h(w)^{(1+\epsilon)'}\sup_{w\in B}\|T_z^* 1\|_t, (w \in B).(10)$$

Now, the well known Schur's test (see in Theorem 3.2.2 of [13] for example), together with (9) and (10), implies that

$$\|T-T_r\| \lesssim \left(\sup_{r<|z|<1} \|T_z 1\|_t\right)^{\frac{1}{1+\epsilon}} \left(\sup_{w\in B}\|T_z^* 1\|_t\right)^{\frac{1}{(1+\epsilon)'}}$$

Since $1<\frac{n+2}{(1+\epsilon)_1-1}<t<m$ and $\|T_z 1\|_t \to 0$, as $|z|\to 1$ we have $\|T-T_r\|\to 0$ as $r\to 1$. So, T is compact on $A^{1+\epsilon}$. The proof is complete.

As an immediate consequence of Theorem 8, we characterize compactness of operators T on $A^{1+\epsilon}$ where T is a finite product of operators of the form $T^a_{u_1}\dots T^a_{u_k}$ where each $u_i\in BT$. Before doing this, we first have a couple of lemmas.

Lemma 9. Let $u\in BT$ and $0<\epsilon<\infty$. For each $z\in B$, $T^a_{u\circ\varphi_z}$ is bounded on $A^{1+\epsilon}$. Moreover, $\|T^a_{u\circ\varphi_z}\|\le C\|u\|_{BT}$ for some constant C independent of z.

Proof . By Theorem 4, we have

$\|T^a_{u\circ\varphi_z}\|\le C\|u\circ\varphi_z\|_{BT}$ for some constant C independent of z. Note that $\widetilde{u\circ\varphi_z}=\tilde{u}\circ\varphi_z$, for all $z\in B$. Hence

$$\|u\circ\varphi_z\|_{BT}=\sup_{w\in B}|\widetilde{u\circ\varphi_z}(w)|=\sup_{w\in B}|\tilde{u}(\varphi_z(w))|=\|u\|_{BT}$$

The proof is complete.

Lemma 10. Let $0<\epsilon<\infty$ and T be a finite sum of operators of the form $T^a_{u_1}\dots T^a_{u_k}$ where each $u_i\in BT$. Then,

$$\sup_{z\in B}\|T_z 1\|_{1+\epsilon}<\infty \quad and \quad \sup_{z\in B}\|T_z^* 1\|_{1+\epsilon}<\infty$$

for every $(1+\epsilon)\in(1,\infty)$.

Proof . Let $(1+\epsilon)\in(1,\infty)$ and $z\in B$. Without loss of generality, we may assume $T=T^a_{u_1}\dots T^a_{u_k}$. We note that U_zU_z is the identity and $U_zT^a_{u_i}U_z=T^a_{u_i\circ\varphi_z}$ for each i. It follows from Lemma 9 that

$$\|T_z 1\|_{1+\epsilon}=\left\|T^a_{u_1\circ\varphi_z}\dots T^a_{u_k\circ\varphi_z}\right\|_{1+\epsilon}\lesssim\|u_1\|_{BT}\dots\|u_k\|_{BT}$$

Since $\|\bar{u}_i\|_{BT}=\|u_i\|_{BT}$ and $T^*=T^a_{\bar{u}_k}\dots T^a_{\bar{u}_1}$, we also have

$$\|T_z^* 1\|_{1+\epsilon}=\left\|T^a_{\bar{u}_k\circ\varphi_z}\dots T^a_{\bar{u}_1\circ\varphi_z}\right\|_{1+\epsilon}\lesssim\|u_1\|_{BT}\dots\|u_k\|_{BT}$$

The proof is complete.

As consequence of Theorem 8 we have the following.

Theorem 11. Let $0<\epsilon<\infty$ and T be a finite sum of operators of the from $T^a_{u_1}\dots T^a_{u_k}$ where each $u_i\in BT$. Then T is compact on $A^{1+\epsilon}$ if and only if $\tilde{T}(z)\to 0$ as $|z|\to 1$.

Proof . This follows from Lemma 10 and Theorem 8. The proof is complete

Corollary 12. Let $0<\epsilon<\infty$ and $u\in BT$. Then T^a_u is compact on $A^{1+\epsilon}$ if and only if $\tilde{u}(z)\to 0$ as $|z|\to 0$.

4- Compact Toeplitz Operators On $b^{1+\epsilon}$

In this section we consider the same characterization problem on the pluriharmonic Bergman spaces. We will use Corollary 12 to characterize BT-symbols of compact Toeplitz operators acting on $b^{1+\epsilon}$ for $0<\epsilon<\infty$.

Before proceeding to this, we need to introduce certain Hankel operators.

Given $u\in L^1$, the little Hankel operator $h_u: A^{1+\epsilon}\to A^{1+\epsilon}$ with symbol u is definied by

$$h_u(f)=P(u\bar{f})$$

for functions $f\in A^{1+\epsilon}\cap L^\infty$. The operator h_u is unbounded in general and densely defined.

The Bloch space B is the space of all holomorphic functions f on B for which the quantities

$$\sup_{z\in B}(1-|z|^2)|\nabla f(z)|<\infty$$

where $\nabla f=\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right)$ is the complex gradient of f. The little Bloch space B_0 is the subspace of B for which the additional boundary vanishing condition

$\lim_{|z|\to 1}(1-|z|^2)|\nabla f(z)|=0$ holds.

The following lemma shows that the boundedness and compactness of the little Hankel operator can be characterized by Bloch functions.

Lemma 13. Let $u\in L^1$ and $0<\epsilon<\infty$. Then h_u is bounded on $A^{1+\epsilon}$ if and only if $P_u\in B$. Moreover, h_u is compact on $A^{1+\epsilon}$ if and only if $P_u\in B_0$.

Proof . In the case of $n=1$, it has been proved in [14] that for holomorphic u, h_u is bounded on $A^{1+\epsilon}$ if and only if $u\in B$, and h_u is compact on $A^{1+\epsilon}$ if and only if $u\in B_0$. But, this result can be easily extended to the ball. On the other hand, since $h_u=h_{Pu}$, we have the desired result. This completes the proof.

We remark in passing that given $u\in BT$ in Proposition 3.2 in [15] implies $P_u\in B$. So, by Lemma 13. h_u is bounded on $A^{1+\epsilon}$ for all $0<\epsilon<\infty$. Also, the same is true for $h_{\bar{u}}$ because $\bar{u}\in BT$.

Lemma 14. Let $u\in BT$ and $0<\epsilon<\infty$. Then the following statements hold for every $f\in A^{1+\epsilon}$ and $g\in A^{(1+\epsilon)'}$.

(a) $\langle T_u f,g\rangle=\langle T^a_u f,g\rangle$.

(b) $\langle T_u f,\bar{g}\rangle=\langle g,h_{\bar{u}}f\rangle$.

(c) $\langle T_u\bar{f},g\rangle=\langle h_u f,g\rangle$.

(d) $\langle T_u\bar{f},\bar{g}\rangle=\langle g,T^a_{\bar{u}}f\rangle$.

Proof . Fix $f\in A^{1+\epsilon}$ and $g\in A^{(1+\epsilon)'}$. We first note that $\overline{P(u\bar{f})}(0)=P(uf)(0)$. It follows that

$$\langle T_u f,g\rangle=\langle Q(uf),g\rangle=\langle P(uf),g\rangle+\langle\overline{P(\overline{uf})},g\rangle-P(uf)(0)\langle 1,g\rangle$$
$$=\langle T^a_u f,g\rangle+P(\overline{uf})(0)\bar{g}(0)-P(uf)(0)\bar{g}(0)=\langle T^a_u f,g\rangle$$

and hence we have (a). Similarly, we see

$$\langle T_u f,\bar{g}\rangle=\langle Q(uf),\bar{g}\rangle=\langle P(uf),\bar{g}\rangle+\langle\overline{P(\overline{uf})},\bar{g}\rangle-P(uf)(0)\langle 1,\bar{g}\rangle$$
$$=P(uf)(0)g(0)+\langle g,P(\overline{uf})\rangle-P(uf)(0)g(0)=\langle g,h_{\bar{u}}f\rangle$$

Hence (b) follows.

Also, the remaining two cases can be proved by similar arguments. This completes the proof.

Given $0<\epsilon<\infty$ and a pluriharmonic function $u=f+\bar{g}\in A_0^{1+\epsilon}+\overline{A^{1+\epsilon}}$, we can see $\|f\|_{1+\epsilon}+\|g\|_{1+\epsilon}\approx\|u\|_{1+\epsilon}$.

Proposition 15. Let $0<\epsilon<\infty$ and $u\in BT$. Then we have

$$\|T_u f\|_{1+\epsilon}\lesssim\|T^a_u f\|_{1+\epsilon}+\|h_{\bar{u}}f\|_{1+\epsilon}$$

and

$$\|T_u \bar{f}\|_{1+\epsilon} \lesssim \|T_{\bar{u}}^a f\|_{1+\epsilon} + \|h_u f\|_{1+\epsilon}$$
for every $f \in A^{1+\epsilon}$.

Proof . Fix $f \in A^{1+\epsilon}$. By Lemma 14, we have
$$\langle T_u f, a + \bar{b}\rangle = \langle T_u^a f, a\rangle + \langle b, h_{\bar{u}} f\rangle$$
for every $a + \bar{b} \in b^{(1+\epsilon)'}$. It follows that
$$\|T_u f\|_{1+\epsilon} = \sup_{\substack{a+\bar{b}\in b^{(1+\epsilon)'} \\ \|a+\bar{b}\|_{(1+\epsilon)'}\le 1}} |\langle T_u f, a+\bar{b}\rangle| = \sup_{\substack{a+\bar{b}\in b^{(1+\epsilon)'} \\ \|a+\bar{b}\|_{(1+\epsilon)'}\le 1}} |\langle T_u^a f, a\rangle + \langle b, h_{\bar{u}} f\rangle|$$
$$\le \sup_{\substack{a\in A(1+\epsilon)' \\ \|a\|_{(1+\epsilon)'}\le c_{1+\epsilon}}} |\langle T_u^a f, a\rangle| + \sup_{\substack{b\in A(1+\epsilon)' \\ \|b\|_{(1+\epsilon)'}\le c_{1+\epsilon}}} |\langle h_{\bar{u}} f, b\rangle|$$
$$\le C_{1+\epsilon}(\|T_u^a f\|_{1+\epsilon} + \|h_{\bar{u}} f\|_{1+\epsilon})$$

for some constant $C_{1+\epsilon}$. Hence we have $\|T_u f\|_{1+\epsilon} \lesssim \|T_u^a f\|_{1+\epsilon} + \|h_{\bar{u}} f\|_{1+\epsilon}$ for every $f \in A^{1+\epsilon}$. Using the similar argument, we also see that $\|T_u \bar{f}\|_{1+\epsilon} \lesssim \|T_{\bar{u}}^a f\|_{1+\epsilon} + \|h_u f\|_{1+\epsilon}$ for every $f \in A^{1+\epsilon}$. The proof is completes.

Proposition 16. Let $0 < \epsilon < \infty$. If a sequence $u_n = f_n + \overline{g_n} \in A_0^{1+\epsilon} + \overline{A^{1+\epsilon}}$, converges to 0 weakly in $b^{1+\epsilon}$, then f_n and g_n converge to 0 weakly in $A^{1+\epsilon}$. Also, if a sequence $h_n \in A^{1+\epsilon}$ converges to 0 weakly in $A^{1+\epsilon}$ then h_n and $\bar{h}_n$ converge to 0 weakly in $b^{1+\epsilon}$.

Proof . Let $\varphi \in A^{(1+\epsilon)'}$. Since $f_n(0) = 0$, we first have
$$\overline{g_n(0)} = u_n(0) = \langle u_n, 1\rangle$$
for each n. It follows that
$$\langle f_n, \varphi\rangle = \langle u_n - \overline{g_n}, \varphi\rangle = \langle u_n, \varphi\rangle - \bar{\varphi}(0)\langle u_n, 1\rangle$$
for each n. Since $u_n \to 0$ weakly in $b^{1+\epsilon}$, we have $\langle u_n, \varphi\rangle$ and $\langle u_n, 1\rangle$ converge to 0 as $n \to \infty$. Hence $f_n \to 0$ weakly in $b^{1+\epsilon}$. Similarly,
$$\langle g_n, \varphi\rangle = \langle \bar{u}_n - \bar{f}_n, \varphi\rangle = \langle \bar{u}_n, \varphi\rangle - \bar{f}_n(0)\bar{\varphi}(0) = \langle \bar{\varphi}, u_n\rangle \to 0$$
as $n \to \infty$. Hence $g_n \to 0$ weakly in $b^{1+\epsilon}$.

To prove the remaining part, let $a + \bar{b} \in b^{(1+\epsilon)'}$. Then
$$\langle h_n, a + \bar{b}\rangle = \langle h_n, a\rangle + h_n(0)\bar{b}(0)$$
for each n. Since $h_n \in A^{1+\epsilon}$ converges to 0 weakly in $A^{1+\epsilon}$, we have $h_n \to 0$ uniformly on every compact subsets. Note $a \in A^{(1+\epsilon)'}$. It follows that $\langle h_n, a + \bar{b}\rangle \to 0$ as $n \to \infty$. Hence $h_n \to 0$ weakly in $b^{1+\epsilon}$. Similarly, we can also see $\overline{h_n} \to 0$ weakly in $b^{1+\epsilon}$.

Lemma 17. Let $u \in BT$ and $0 < \epsilon < \infty$. Then $T_u^a, T_{\bar{u}}^a, h_u$ and $h_{\bar{u}}$ are compact on $A^{1+\epsilon}$ if and only if T_u is compact on $b^{1+\epsilon}$.

Proof . First suppose $T_u^a, T_{\bar{u}}^a, h_u$ and $h_{\bar{u}}$ are compact on $A^{1+\epsilon}$. Let $u_n = f_n + \overline{g_n} \in A_0^{1+\epsilon} + \overline{A^{1+\epsilon}}$ be a sequence converging to 0 weakly in $b^{1+\epsilon}$. By Proposition 15 we see
$$\|T_u(u_n)\|_{1+\epsilon} \lesssim \|T_u^a f_n\|_{1+\epsilon} + \|h_{\bar{u}} f_n\|_{1+\epsilon} + \|T_{\bar{u}}^a g_n\|_{1+\epsilon} + \|h_u g_n\|_{1+\epsilon}$$
for each n. Since $T_u^a, T_{\bar{u}}^a, h_u$ and $h_{\bar{u}}$ are compact on $A^{1+\epsilon}$ and $g_n, f_n \to 0$ weakly in $b^{1+\epsilon}$ by Proposition 16 we see $\|T_u u_n\|_{1+\epsilon} \to 0$ as $n \to \infty$. Hence T_u is compact on $b^{1+\epsilon}$.

Now suppose T_u is compact on $b^{1+\epsilon}$. Let f_n be a sequence converging to 0 weakly in $A^{1+\epsilon}$. By Lemma 14 we have
$$\|T_u^a f_n\|_{1+\epsilon} = \sup_{\substack{a\in A(1+\epsilon)' \\ \|a\|_{(1+\epsilon)'}\le 1}} |\langle T_u^a f_n, a\rangle| = \sup_{\substack{a\in A(1+\epsilon)' \\ \|a\|_{(1+\epsilon)'}\le 1}} |\langle T_u f_n, a\rangle| \le \sup_{\substack{a\in b^{(1+\epsilon)'} \\ \|a\|_{1+\epsilon}\le 1}} |\langle T_u f_n, a\rangle|$$
$$\lesssim \|T_u f_n\|_{1+\epsilon}$$
for each n . Since f_n converges to 0 weakly in $b^{1+\epsilon}$ by Proposition 16, we have $\|T_u^a f_n\|_{1+\epsilon} \to 0$ as $n \to \infty$. So T_u^a is compact. Also,
$$\|h_{\bar{u}} f_n\|_{1+\epsilon} = \sup_{\substack{a\in A(1+\epsilon)' \\ \|a\|_{(1+\epsilon)'}\le 1}} |\langle h_{\bar{u}} f_n, a\rangle| = \sup_{\substack{a\in A(1+\epsilon)' \\ \|a\|_{(1+\epsilon)'}\le 1}} |\langle T_u f_n, \bar{a}\rangle| \le \sup_{\substack{a\in b^{(1+\epsilon)'} \\ \|a\|_{1+\epsilon}\le 1}} |\langle T_u f, \bar{a}\rangle|$$
$$\lesssim |\langle T_u f_n\rangle|_{1+\epsilon}$$
for each n , which gives the compactness of $h_{\frac{a}{u}}$.

By the similar arguments, we show the compactness of h_u and $T_{\frac{a}{u}}$. This completes the proof.

Corollary 18. Let $\epsilon > 0$ and $u \in BT$. Then we have
$$\sum_{j=1}^{n} \|T_u f_j\|_{1+\epsilon} \lesssim \sum_{j=1}^{n} \|T_{\bar{u}}^a f_j\|_{1+\epsilon} + \sum_{j=1}^{n} \|h_{\bar{u}} f_j\|_{1+\epsilon}$$
and
$$\sum_{j=1}^{n} \|T_u \bar{f}_j\|_{1+\epsilon} \lesssim \sum_{j=1}^{n} \|T_{\bar{u}}^a f_j\|_{1+\epsilon} + \sum_{j=1}^{n} \|h_u f_j\|_{1+\epsilon}$$
for every $f_j \in A^{1+\epsilon}$.

Proof . Let $f_j \in A^{1+\epsilon}$, fixed. By Lemma 14, and [11] we have
$$\sum_{j=1}^{n} \langle T_u f_j, a + \overline{a+\epsilon}\rangle = \sum_{j=1}^{n} \langle T_u^a f_j, a\rangle + \sum_{j=1}^{n} \langle a+\epsilon, h_{\bar{u}} f_j\rangle$$
for every $a + \overline{a+\epsilon} \in (a+\epsilon)^{(1+\epsilon)'}$. It follows that
$$\sum_{j=1}^{n} \|T_u f_j\|_{1+\epsilon} = \sup_{\substack{a+\overline{a+\epsilon}\in(a+\epsilon)^{(1+\epsilon)'} \\ \|a+\overline{a+\epsilon}\|_{(1+\epsilon)'}\le 1}} \sum_{j=1}^{n} |\langle T_u f_j, a + \overline{a+\epsilon}\rangle|$$
$$= \sup_{\substack{a+\overline{a+\epsilon}\in(a+\epsilon)^{(1+\epsilon)'} \\ \|a+\overline{(a+\epsilon)}\|_{(1+\epsilon)'}\le 1}} \sum_{j=1}^{n} |\langle T_u^a f_j, a\rangle + \langle a+\epsilon, h_{\bar{u}} f_j\rangle|$$
$$\le \sup_{\substack{a\in A(1+\epsilon)' \\ \|a\|_{(1+\epsilon)'}\le c_{(1+\epsilon)}}} \sum_{j=1}^{n} |\langle T_u^a f_j, a\rangle| + \sup_{\substack{a+\epsilon\in A(1+\epsilon)' \\ \|a+\epsilon\|_{(1+\epsilon)'}\le c_{(1+\epsilon)}}} \sum_{j=1}^{n} |\langle h_{\bar{u}} f_j, a+\epsilon\rangle|$$
$$\le C_{(1+\epsilon)} \sum_{j=1}^{n} \left(\|T_u^a f_j\|_{(1+\epsilon)} + \|h_{\bar{u}} f_j\|_{(1+\epsilon)}\right)$$
for some constant $C_{(1+\epsilon)}$. Hence we have
$$\|T_u f_j\|_{(1+\epsilon)} \lesssim \|T_u^a f_j\|_{(1+\epsilon)} + \|h_{\bar{u}} f_j\|_{(1+\epsilon)}$$
for every $f_j \in A^{(1+\epsilon)}$. Using the similar argument, we see that
$$\sum_{j=1}^{n} \|T_u \bar{f}_j\|_{(1+\epsilon)} \lesssim \sum_{j=1}^{n} \|T_{\bar{u}}^a f_j\|_{(1+\epsilon)} + \sum_{j=1}^{n} \|h_u f_j\|_{(1+\epsilon)}$$
also see that for every $f_j \in A^{(1+\epsilon)}$. The proof is completes.

Now, we characterize compact Toeplitz operators with symbol in BT on the pluriharrsnonie Bergman spaces. On the unit disk, the following was proved in [7] where the case $\epsilon = 1$ and bounded symbols are assumed.

Theorem 19. Let $u \in BT$ and $0 < \epsilon < \infty$. Then T_u is compact on $b^{1+\epsilon}$ if and only if $\tilde{u}(z) \to 0$ as $|z| \to 1$ and
$$\lim_{|z|\to 1} (1 - |z|^2)\left(|\nabla U(z)| + |\nabla \bar{U}(z)|\right) = 0 \quad (11)$$
where $U = Qu$ is the pluriharmonic part of u.

Proof . First suppose T_u is compact on $b^{1+\epsilon}$. By Lemma 17, we see that $T_u^a, T_{\bar{u}}^a, h_u$ and $h_{\bar{u}}$ are compact on $A^{1+\epsilon}$. Since $T_u^a, T_{\bar{u}}^a$ are compact on $A^{1+\epsilon}$ and $\tilde{u} = \tilde{\bar{u}}$, we have by Corollary 12, $\tilde{u}(z) \to 0$ as $|z| \to 1$. Also. Since h_u and $h_{\bar{u}}$ are compact on $A^{1+\epsilon}$, we have by Lemma 13,

$$\lim_{|z|\to 1} (1-|z|^2)\,|\nabla Pu(z)| = 0$$

and

$$\lim_{|z|\to 1}(1-|z|^2)\,|\nabla P\bar{u}(z)| = 0.$$

On the other hand, since

$$U = Qu = Pu + \overline{P\bar{u}} - Pu(0)$$

by (1), we see $|\nabla U| = |\nabla Pu|$ and $|\nabla \bar{U}| = |\nabla P\bar{u}|$. Hence we have (11).

Conversely, suppose $\tilde{u}(z) \to 0$ as $|z| \to 1$ and (11) holds. Since $\tilde{u} = \tilde{\bar{u}}$, we see T_u^a and $T_{\bar{u}}^a$ are compact by Corollary 12. As we see before, (11) implies that

$$\lim_{|z|\to 1}(1-|z|^2)\,|\nabla Pu(z)| = 0$$

and

$$|\nabla P\bar{u}(z)| = 0$$

These two conditions above are in turn equivalent to the compactness of h_u and $h_{\bar{u}}$ by Lemma 13. Now, by Lemma 17 we see T_u is compact on $b^{1+\epsilon}$, as desired. The proof is complete.

As consequences of Theorem 18, we have the following corollaries. A function $u \in C^2(B)$ is called M-harmonic on B if its invariant Laplacian vanishes on B. An an application of the invariant mean value property implies $\tilde{u} = u$. See Chapter 4 of [8] for details.

Corollary 20. Let $0 < \epsilon < \infty$ and $u \in BT$ be M-harrnonic on B. Then T_u is compact on $b^{1+\epsilon}$ if and only if $u = 0$ on B.

Proof . Suppose T_u is compact on $b^{1+\epsilon}$. Since $\tilde{u} = u$, the compactness of T_u implies u vanishes on the boundary of B by Theorem 19. Now, the maximum principle (see Theorem 4.3.2 of [8]), we have $u = 0$ on B.

The converse implication is clear. The proof is complete.

Given a radial function $u \in L^1$, it is not hard to see

$$P_u = \int_B u\,dV. \qquad (12)$$

The following is an extension of Theorem 4.1 of [5] where the case $\epsilon = 1$ and bounded symbols are assumed.

Corollary 21. Let $0 < \epsilon < \infty$ and $u \in BT$ be a radial function on B. Then T_u is compact on $b^{1+\epsilon}$ if and only if $\tilde{u}(z) \to 0$ as $|z| \to 1$.

Proof . Since $u \in BT$ is radial, so is $\bar{u}$. By (12),

$$Qu = Pu + \overline{P\bar{u}} - Pu(0) = \int_B u\,dV.$$

Now, the result follows from Theorem 19. The proof is complete.

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