Sum of Orthogonal Bimatrices in $C_{n \times n}$

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Abstract: Let $F \in \{R, C, H\}$. Let $U_{non}$ be the set of unitary bimatrices in $F_{non}$, and let $O_{non}$ be the set of orthogonal bimatrices in $F_{non}$. Suppose $n \geq 2$, we show that every $A_b \in F_{non}$ can be written as a sum of bimatrices in $U_{non}$ and of bimatrices in $O_{non}$. Let $A_b \in F_{non}$ be given that and let $k \geq 2$ be the least integer that is a least upper bound of the singular values of $A_b$. When $F=C$, we show that $A_b$ can be written as a sum of $k$ bimatrices from $U_{non}$.

Keywords: Orthogonal matrix, bimatrix, orthogonal bimatrix, unitary bimatrix, sum of orthogonal bimatrices, sum of unitary bimatrices.

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1. Introduction

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of $n \times n$ complex matrices by $C_{n \times n}$.

For $A \in C_{n \times n}$, $A^T, A^{-1}, A^1$ and det $(A)$ denote transpose, inverse, Moore-Penrose inverse and determinant of $A$ respectively. If $AA^T = A^T A = I$ then $A$ is an orthogonal matrix, where $I$ is the identity matrix. In this paper we study orthogonal bimatrices as a generalization of orthogonal matrices. Some of the properties of orthogonal matrices are extended to orthogonal bimatrices. Some important results of orthogonal matrices are generalized to orthogonal bimatrices.

2. Basic Definitions and Results

Definition 1.1 [6]

A bimatrix $A_{b}$ is defined as the union of two rectangular array of numbers $A_1$ and $A_2$ arranged into rows and columns.

It is written as $A_{b} = A_{o} \cup A_{2}$ with $A_{1} \neq A_{2}$ (except zero and unit bimatrices) where,

$$
A_{1} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

and

$$
A_{2} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

$\cup$' is just for the notational convenience (symbol) only.

Definition 1.2 [6]

Let $A_{b} = A_{1} \cup A_{2}$ and $C_{b} = C_{1} \cup C_{2}$ be any two $m \times n$ bimatrices. The sum $D_{b}$ of the bimatrices $A_{b}$ and $C_{b}$ is defined as

$$
D_{b} = A_{b} + C_{b} = (A_{1} \cup A_{2}) + (C_{1} \cup C_{2}) = (A_{1} + C_{1}) \cup (A_{2} + C_{2})
$$

Where $A_{1} + C_{1}$ and $A_{2} + C_{2}$ are the usual addition of matrices.

Definition 1.3 [7]

If $A_{b} = A_{1} \cup A_{2}$ and $C_{b} = C_{1} \cup C_{2}$ be two bimatrices, then $A_{b}$ and $C_{b}$ are said to be equal (written as $A_{b} = C_{b}$) if and only if $A_{1}$ and $C_{1}$ are identical and $A_{2}$ and $C_{2}$ are identical. (That is, $A_{1} = C_{1}$ and $A_{2} = C_{2}$).

Definition 1.4 [7]

Given a bimatrix $A_{b} = A_{1} \cup A_{2}$ and a scalar $\lambda$, the product of $\lambda$ and $A_{b}$ written as $\lambda A_{b}$ is defined to be
\[ \lambda A_B = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix} \cup \begin{bmatrix} \lambda a_{11}^2 & \lambda a_{12}^2 & \cdots & \lambda a_{1n}^2 \\ \lambda a_{21}^2 & \lambda a_{22}^2 & \cdots & \lambda a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^2 & \lambda a_{m2}^2 & \cdots & \lambda a_{mn}^2 \end{bmatrix} = (\lambda A_1 \cup \lambda A_2). \]

That is, each element of \( A_1 \) and \( A_2 \) are multiplied by \( \lambda \).

**Remark 1.5** [7]

If \( A_B = A_1 \cup A_2 \) be a bimatrix, then we call \( A_1 \) and \( A_2 \) as the component matrices of the bimatrix \( A_B \).

**Definition 1.6** [6]

If \( A_B = A_1 \cup A_2 \) and \( C_B = C_1 \cup C_2 \) are both \( n \times n \) square bimatrices then, the bimatrix multiplication is defined as, \( A_B \times C_B = (A_1 C_1) \cup (A_2 C_2) \).

**Definition 1.7** [6]

Let \( A_{B_{m\times n}} = A_1 \cup A_2 \) be a \( m \times n \) square bimatrix. We define \( I_{m\times n}^B = I_{m\times n}^1 \cup I_{m\times n}^2 \) to be the identity bimatrix.

**Definition 1.8** [6]

Let \( A_{B_{m\times n}} = A_1 \cup A_2 \) be a square bimatrix, \( A_B \) is a symmetric bimatrix if the component matrices \( A_1 \) and \( A_2 \) are symmetric matrices. i.e, \( A_1 = A_1^T \) and \( A_2 = A_2^T \).

**Definition 1.9** [6]

Let \( A_{B_{m\times n}} = A_1 \cup A_2 \) be a \( m \times n \) square bimatrix i.e, \( A_1 \) and \( A_2 \) are \( m \times n \) square matrices. A skew-symmetric bimatrix is a bimatrix \( A_B \) for which \( A_B = -A_B^T \), where \( -A_B^T = -A_1^T \cup -A_2^T \) i.e, the component matrices \( A_1 \) and \( A_2 \) are skew-symmetric.

**3. Orthogonal and Unitary Bimatrices**

**Definition 2.1** [5]

A bimatrix \( A_B = A_1 \cup A_2 \) is said to be orthogonal bimatrix, if
\[ A_B A_B^T = A_B^T A_B = I_B \quad \text{(or)} \quad \left( A_1 A_1^T \cup A_2 A_2^T \right) = \left( A_1^T A_1 \cup A_2^T A_2 \right) = I_1 \cup I_2. \]

(That is, the component matrices of \( A_B \) are orthogonal.) That is, \( A_B^T = A_B^{-1} \) (or) \( A_1^T \cup A_2^T = A_1^{-1} \cup A_2^{-1} \).

**Remark 2.2**

Let \( A_B = A_1 \cup A_2 \) be a orthogonal bimatrix. If \( A_1 \) and \( A_2 \) are square and posses the same order then \( A_B \) is called square orthogonal bimatrix, and if \( A_1 \) and \( A_2 \) are of different orders then \( A_B \) is called mixed square orthogonal bimatrix.

**Example 2.3**

\[
A_B = \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ 2 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 2 \end{bmatrix}
\]

is a square orthogonal bimatrix.

\[
A_B = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \cup \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

is a mixed square orthogonal bimatrix.

**Definition 2.4** [4]

Let \( A_B = A_1 \cup A_2 \) be an \( n \times n \) complex bimatrix. (A bimatrix \( A_B \) is said to be complex if it takes entries from the complex field). \( A_B \) is called a unitary bimatrix if \( A_B A_B^* = A_B^* A_B = I_B \) (or) \( A_B^T = A_B^{-1} \).

That is, \( A_1 A_1^* \cup A_2 A_2^* = A_1^* A_1 \cup A_2^* A_2 = I_1 \cup I_2 \).

**Example 2.5**

\[
A_B = A_1 \cup A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{\sqrt{2}} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}
\]

is a unitary bimatrix.

In this paper, we have determined which bimatrices (if any) in \( C_{m\times n} \) can be written as a sum of unitary or orthogonal bimatrices. We let \( U_{m\times n} \) and \( O_{m\times n} \) are the set of unitary and orthogonal bimatrices in the complex field. We begin with the following observation.

**Lemma 2.6**
Let \( n \) be a given positive integer. Let \( G \subseteq F_{non} \) be a group under multiplication. Then \( A_b \in F_{non} \) can be written as a sum of bimatrices in \( G \) if and only if for every \( Q_b, P_b \in G \), the bimatrix \( \Phi_b \) can be written as a sum of bimatrices in \( G \). Notice that both \( u_{non} \) and \( o_{non} \) are groups under multiplication.

Let \( \alpha_1, \alpha_2 \in F \) be given. Then lemma 2.6 guarantees that for each \( Q_b \in G \), we have that \( \alpha_1 Q_b \cup \alpha_2 Q_2 \) can be written as a sum of bimatrices from \( G \) if and only if \( \alpha_1 I_1 \cup \alpha_2 I_2 \) can be written as a sum of bimatrices from \( G \).

**Lemma 2.7**

Let \( n \geq 2 \) be a given integer. Let \( G \subseteq F_{non} \) be a group under multiplication. Suppose that \( G \) contains \( K_B \equiv \text{diag} (-1, ..., -1) \) and the permutation bimatrices. Then every \( A_b \in F_{non} \) can be written as a sum of bimatrices in \( G \) if and only if for each \( \alpha_1, \alpha_2 \in F \), \( \alpha_1 I_1 \cup \alpha_2 I_2 \) can be written as a sum of bimatrices from \( G \).

4. **Sum of orthogonal bimatrices in** \( C_{nano} \)

The only bimatrices in the set of all orthogonal bimatrices of order 1 are \( \pm 1 \). Hence, not every element of \( F_{2x1} \) can be written as a sum of elements in the set of all orthogonal bimatrices of order 1. In fact, only the integers can be written as a sum of elements of the set of all orthogonal bimatrices of order 1.

Notice that \( \bar{U}_i = \{ e^{i\theta}; \theta \in R \} \cup \{ e^{i\beta}; \beta \in R \} \).

Set \( \bar{C}_{B_2} = \left( \begin{array}{cc} C_{1}^{II} & C_{2}^{II} \end{array} \right) \equiv \left\{ e^{i\theta} + e^{i\beta}; \theta, \beta \in R \right\} \cup \left\{ e^{i\theta} + e^{i\beta}; \theta, \beta \in R \right\} \).

We show that for each \( k \), we have \( A_{B_2} = \left( \begin{array}{cc} \bar{C}_{B_2} \end{array} \right) \equiv \left\{ z_1 \in C_1; |z_1| \leq k \right\} \cup \left\{ z_2 \in C_2; |z_2| \leq k \right\} \).

First, notice that for each \( k \), we have \( \bar{C}_{B_2} \). We now show that \( \bar{C}_{B_2} \). If \( z_1 = r e^{i\theta}, z_2 = r e^{i\beta} \) with \( r_1, r_2, \theta, \beta \in R \) and \( r_1, r_2 \geq 0 \), then

\[
\begin{align*}
\left| e^{i\theta} + e^{i\beta} + ... + e^{i\theta} \right| &= r_1; \\
\left| e^{i\theta} + e^{i\beta} + ... + e^{i\beta} \right| &= r_2.
\end{align*}
\]

If \( \theta, \beta \in R \) are given, then

\[
\left| e^{i\theta} + e^{i\beta} \right| \leq 2 \text{ and } \left| e^{i\theta} + e^{i\beta} \right| \leq 2.
\]

Hence,

\[
\left( C_{1}^{II} \cup C_{2}^{II} \right) \equiv \left( A_{B_2} \right)
\]

If \( z_1 = r e^{i\theta}, z_2 = r e^{i\beta} \), then choose \( \theta = -\theta_1, \beta = -\beta_2 \) such that \( 2 \cos \theta_1 = r_1 \) and \( 2 \cos \theta_2 = r_2 \). Then

\[
e^{i\theta} + e^{-i\beta} = 2 \cos \theta_1 = r_1 \quad \text{and} \quad e^{i\theta} + e^{-i\beta} = 2 \cos \theta_2 = r_2.
\]

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Set $A_k$ may be $\{ \emptyset \}$. Then can be written as a sum of matrices $A_1 \cup A_k$. We use mathematical induction to show the general case. The base cases $k = 2$ and $k = 3$ have already been shown. Assume that $k > 3$ and suppose that $(C^1_k \cup C^2_k) = (A^1_k \cup A^k_k)$. Consider $f_{k+1}^1 \left( \theta_1^1, ..., \theta_k^1, \theta_{k+1}^1 \right) = e^{i\theta_1^1} + ... + e^{i\theta_k^1} + e^{i\theta_{k+1}^1}$; $f_{k+1}^2 \left( \theta_1^2, ..., \theta_k^2, \theta_{k+1}^2 \right) = e^{i\theta_1^2} + ... + e^{i\theta_k^2} + e^{i\theta_{k+1}^2}$. Let $z_1 = r_1^1 e^{i\theta_1^1}$ and $z_2 = r_2^2 e^{i\theta_2^2}$ are given with $0 \leq r_1 \leq k + 1, 0 \leq r_2 \leq k + 1$. We show that $r_1, r_2 \in (C^1_{k+1} \cup C^2_{k+1})$. 

First, we show that $(A^1_{k+1} \cup A^2_{k+1}) \subseteq (C^1_{k+1} \cup C^2_{k+1})$. If $k$ is even, choose $\theta_1^1 = ... = \theta_{k+1}^1 = 0$; $\theta_3^1 = ... = \theta_{k-1}^1 = \pi$; $\theta_4^1 = ... = \theta_k^1 = \pi$. Then $f_{k+1}^1 \left( \theta_1^1, ..., \theta_k^1, \theta_{k+1}^1 \right) = f_k^1 \left( \theta_1^1, ..., \theta_k^1 \right) + 1$; $f_{k+1}^2 \left( \theta_1^2, ..., \theta_k^2, \theta_{k+1}^2 \right) = f_k^2 \left( \theta_1^2, ..., \theta_k^2 \right) + 1$. 

Lemma 3.1

Let $k \geq 2$ be a given integer. Let $(A^1_k \cup A^2_k) = \{ z \in C; z \leq k \} \cup \{ z \in C; z \leq k \}$. Then $(A^1_k \cup A^2_k) = (C^1_k \cup C^2_k)$. 

I. The case $U_{n \geq n}$

Let $\alpha_1, \alpha_2 \in C$ be given. Then there exist an integer $k \geq 2$ and $\theta_1^k, \theta_2^k, ..., \theta_k^k \in R$ such that

$$\alpha_1 = f_k^1 \left( \theta_1^k, ..., \theta_k^k \right); \alpha_2 = f_k^2 \left( \theta_1^k, ..., \theta_k^k \right).$$

Now, notice that

$$(\alpha_1 I_1 \cup \alpha_2 I_2) = f_k^1 \left( \theta_1^k, ..., \theta_k^k \right) I_1 \cup f_k^2 \left( \theta_1^k, ..., \theta_k^k \right) I_2 = e^{i\theta_1^k} I_1 + ... + e^{i\theta_k^k} I_1 \cup e^{i\theta_1^k} I_2 + ... + e^{i\theta_k^k} I_2$$

is a sum of matrices in $U_{n \geq n}$. 

When $n = 1$, every $\alpha_1, \alpha_2 \in C$ can be written as a sum of elements of the set of all unitary unimodular matrices of order 1. When $n \geq 2$, Lemma 2.7 guarantees that every 

$$\left( A_1 \cup A_2 \right) \in C_{n \geq n}$$

can be written as a sum of matrices in $U_{n \geq n}$.

Lemma 3.2

Let $n$ be a given positive integer. Then every 

$$\left( U_{n \geq n} \right) A_2 \in C_{n \geq n}$$

can be written as a sum of matrices in $U_{n \geq n}$.

Proof

Let $\left( A_1 \cup A_2 \right) \in C_{n \geq n}$ be given. We look at the number of matrices that make up the sum $\left( A_1 \cup A_2 \right)$. 

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Let $\alpha_1, \alpha_2 \in C$ be given. If $|\alpha_1|, |\alpha_2| \leq k$ for some positive integer $k$, then $(\alpha_1, \alpha_2) \in (A^k \cup A^k)$. Moreover, $(\alpha_1, \alpha_2) \in (A^m \cup A^m)$ for every integer $m \geq k$.

For any such $m$, Lemma 3.1 guarantees that there exist $\theta_1^1, \ldots, \theta_m^1; \theta_1^2, \ldots, \theta_m^2 \in R$ such that

$$\alpha_1 = e^{i\theta_1^1} + \ldots + e^{i\theta_m^1}; \quad \alpha_2 = e^{i\theta_1^2} + \ldots + e^{i\theta_m^2}.$$ 

However, if $|\alpha_1|, |\alpha_2| < k$ then $\alpha_1, \alpha_2 \not\in (A^k \cup A^k)$ and $\alpha_1, \alpha_2$ cannot be written as a sum of $k$ elements of $U(C)$.

Write $(A_1 \cup A_2) = (U_1 \cup U_2)(\Sigma_1 \cup \Sigma_2)(V_1 \cup V_2)$ where $(U_1 \cup U_2), \quad (V_1 \cup V_2) \in C_{n \times n}$ are unitary bimatrices and

$$\Sigma_1 = \text{diag}_B \left( \sigma_1^1, \ldots, \sigma_n^1 \right) \quad \text{with} \quad \sigma_1^1 \geq \ldots \geq \sigma_n^1 \geq 0;$$

$$\Sigma_2 = \text{diag}_B \left( \sigma_1^2, \ldots, \sigma_n^2 \right) \quad \text{with} \quad \sigma_1^2 \geq \ldots \geq \sigma_n^2 \geq 0.$$ 

Let $k$ be the least integer such that $\sigma_1^1, \sigma_1^2 \leq k$. Suppose that $k \geq 2$. Then for each $l$, we have

$$\sigma_l^1, \sigma_l^2 \in (A^k \cup A^k).$$ 

Moreover,

$$\sigma_1^1, \sigma_1^2 \not\in (A_1 \cup A_2).$$ 

Hence, $(A_1 \cup A_2)$ cannot be written as a sum of $k$ unitary bimatrices. However, for each $l$, we have

$$\sigma_l^1 = e^{i\theta_1^l} + \ldots + e^{i\theta_m^l}; \quad \sigma_l^2 = e^{i\theta_1^l} + \ldots + e^{i\theta_m^l},$$ 

where each $(\theta_1^l, \ldots, \theta_m^l); \quad (\theta_1^l, \ldots, \theta_m^l) \in R$.

II. The case $O_{n \times n}$

Let $n=2$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C$ be given.

Set $(A_1(\alpha_1, \beta_1) \cup A_2(\alpha_2, \beta_2)) = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \cup \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix}$ (2)

Choose $\beta_1, \beta_2$ such that $\alpha_1^2 + \beta_1^2 = 1; \alpha_2^2 + \beta_2^2 = 1$.
and notice that
\[
(A_1(\pm \alpha_1, \pm \beta_1) \cup A_2(\pm \alpha_2, \pm \beta_2)) \in O_{2 \times 2}
\]
Set \(A'_1 \cup A'_2 := (A_1(\alpha_1, \beta_1) \cup A_2(\alpha_2, \beta_2))\) and \(A''_1 \cup A''_2 := (A_1(\alpha_1, -\beta_1) \cup A_2(\alpha_2, -\beta_2))\)

Then \( (A'_1 \cup A'_2 \) \( + A''_2 \) \( ) + (A''_1 \cup A''_2 \) \( ) = 2(\alpha_1 I''_1 \cup \alpha_2 I''_2) \].

Lemma 2.7 guarantees that every \((A_1 \cup A_2) \in C_{2 \times 2}\) can be written as a sum of bimatrices from \(O_{2 \times 2}\).

We look at the case when \(n=3\). Let \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in F\) be given.

Set
\[
[B_1(\alpha_1, \beta_1) \cup B_2(\alpha_2, \beta_2)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 \\ 0 & -\beta_1 & \alpha_1 \end{bmatrix} \quad \text{and} \quad [C_1(\alpha_1, \beta_1) \cup C_2(\alpha_2, \beta_2)] = \begin{bmatrix} \alpha_1 & 0 & \beta_1 \\ 0 & 1 & 0 \\ -\beta_1 & 0 & \alpha_1 \end{bmatrix}
\]

and set \([D_1(\alpha_1, \beta_1) \cup D_2(\alpha_2, \beta_2)] = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ -\beta_1 & \alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Choose \(\beta_1, \beta_2\) so that \(\alpha_1^2 + \beta_1^2 = 1; \alpha_2^2 + \beta_2^2 = 1\). Then
\[
[B_1(\pm \alpha_1, \pm \beta_1) \cup B_2(\pm \alpha_2, \pm \beta_2)], \quad \text{and} \quad C_1(\pm \alpha_1, \pm \beta_1) \cup C_2(\pm \alpha_2, \pm \beta_2)
\]

and \([D_1(\pm \alpha_1, \pm \beta_1) \cup D_2(\pm \alpha_2, \pm \beta_2)], \quad \text{are all} \quad B_1(\pm \alpha_1, \pm \beta_1) \cup B_2(\pm \alpha_2, \pm \beta_2)
\]

elements of \(O_{3 \times 3}\) \(\alpha_1 I''_1 \cup \alpha_2 I''_2\) \].

Set \(\begin{bmatrix} B'_1 \cup B'_2 \bigr] := B_1(\alpha_1, \beta_1) \cup B_2(\alpha_2, \beta_2), \]

\(\begin{bmatrix} B''_1 \cup B''_2 \bigr] := B_1(-\alpha_1, \beta_1) \cup B_2(-\alpha_2, \beta_2), \]

\(\begin{bmatrix} C'_1 \cup C'_2 \bigr] := C_1(\alpha_1, \beta_1) \cup C_2(\alpha_2, \beta_2), \]

\(\begin{bmatrix} C''_1 \cup C''_2 \bigr] := C_1(-\alpha_1, \beta_1) \cup C_2(-\alpha_2, \beta_2), \]

\(\begin{bmatrix} D'_1 \cup D'_2 \bigr] := D_1(\alpha_1, \beta_1) \cup D_2(\alpha_2, \beta_2), \]

\(\begin{bmatrix} D''_1 \cup D''_2 \bigr] := D_1(-\alpha_1, \beta_1) \cup D_2(-\alpha_2, \beta_2) \]

Set \((E'_1 \cup E'_2) = (A'_1 \cup A'_2) \oplus \ldots \oplus (A'_1 \cup A'_2)(m \text{ copies}) \)

and \((E''_1 \cup E''_2) = (A''_1 \cup A''_2) \oplus \ldots \oplus (A''_1 \cup A''_2)(m \text{ copies}) \).
Lemma 3.2 guarantees that every \( A \in C \) has a \( b \)-decomposition. Let \( n \geq 2 \) be a given integer. Then every \( \alpha \in C \) can be written as a sum of bimatrices from \( O_{n \times n} \). Hence, for every \( \alpha \in C \) and for every integer \( n \geq 2 \), \((\alpha I_1 \cup \alpha I_2)\) can be written as a sum of bimatrices from \( O_{n \times n} \). Lemma 3.2 guarantees that every \( (A \cup A) \in C_{n \times n} \) can be written as a sum of bimatrices from \( O_{n \times n} \).

Theorem 3.4

Let \( n \geq 2 \) be a given integer. Then every \( (A \cup A) \in C_{n \times n} \) can be written as a sum of bimatrices from \( O_{n \times n} \).

Proof

Suppose \( (A \cup A) \in C_{n \times n} \) and set

\[
\begin{align*}
\left( E_1 \cup E_2 \right) &= (I_1 + I_2) \\
\left( E_3 \cup E_4 \right) &= A \\
\left( E_5 \cup E_6 \right) &= B \\
\left( E_7 \cup E_8 \right) &= C
\end{align*}
\]

Then each \( (E_1 \cup E_2) \in O_{2m+1} \), and

\[
\begin{align*}
\left( E_1 \cup E_2 \right) + \ldots + \left( E_7 \cup E_8 \right) &= \left( \delta I_1^{2m+1} \cup \delta I_2^{2m+1} \right)
\end{align*}
\]

Hence, for every \( \alpha_1, \alpha_2 \in C \) and for every integer \( n \geq 2 \), \((\alpha_1 I_1 \cup \alpha_2 I_2)\) can be written as a sum of bimatrices from \( O_{n \times n} \).
is a sum of two bimatrices from $O_{n,m}$, then there exists a skew-symmetric bimatrix $(D_1 \cup D_2) \in C_{n,m}$ such that

$$(Q_1 \cup Q_2) = \left( \frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2 \right) + (D_1 \cup D_2),$$

$$(V_1 \cup V_2) = \left( \frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2 \right) - (D_1 \cup D_2),$$

and

$$\left( D_1 D_1^T \cup D_2 D_2^T \right) = \left[ \left( 1 - \frac{\alpha_1^2}{4} \right) I_1 \cup \left( 1 - \frac{\alpha_2^2}{4} \right) I_2 \right].$$

Conversely, if there exists a skew-symmetric bimatrix $(D_1 \cup D_2) \in C_{n,m}$ such that

$$(D_1 D_1^T \cup D_2 D_2^T) = \left[ \left( 1 - \frac{\alpha_1^2}{4} \right) I_1 \cup \left( 1 - \frac{\alpha_2^2}{4} \right) I_2 \right],$$

then

$$(Q_1 \cup Q_2) = \left( \frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2 \right) + (D_1 \cup D_2),$$

and

$$(V_1 \cup V_2) = \left( \frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2 \right) - (D_1 \cup D_2),$$

and $(Q_1 \cup Q_2) + (V_1 \cup V_2) = (\alpha_1 I_1 \cup \alpha_2 I_2)$.

**Proof**

Let an integer $n \geq 2$ and $0 \neq \alpha_1; 0 \neq \alpha_2 \in \mathbb{R}$ be given. Suppose that $(\alpha_1 I_1 \cup \alpha_2 I_2) \in C_{n,m}$ can be written as a sum of two orthogonal bimatrices, say

$$(\alpha_1 I_1 \cup \alpha_2 I_2) = (Q_1 \cup Q_2) + (V_1 \cup V_2).$$

Write

$$(Q_1 \cup Q_2) = \left[ a_{ij}^1 \cup a_{ij}^2 \right] = \left[ q_1^1 \ldots q_n^1 \right] \cup \left[ q_1^2 \ldots q_n^2 \right]$$

and

$$(V_1 \cup V_2) = \left[ b_{ij}^1 \cup b_{ij}^2 \right] = \left[ v_1^1 \ldots v_n^1 \right] \cup \left[ v_1^2 \ldots v_n^2 \right].$$

Then $b_{ij}^1 = -a_{ij}^1$ and $b_{ij}^2 = -a_{ij}^2$ for $i \neq j$.

Now, for each $i = 1, \ldots, n$, we have

$$\sum_{j=1}^{n} a_{ij}^1 = q_i^1, q_i^1 = 1 = v_i^1, v_i^1 = \sum_{j=1}^{n} b_{ij}^1 = b_{ij}^1 + \sum_{j \neq i, j=1}^{n} a_{ij}^1,$$

and

$$\sum_{j=1}^{n} a_{ij}^2 = q_i^2, q_i^2 = 1 = v_i^2, v_i^2 = \sum_{j=1}^{n} b_{ij}^2 = b_{ij}^2 + \sum_{j \neq i, j=1}^{n} a_{ij}^2.$$
and let a positive integer $n$ be given. Suppose that $(\alpha_1 I_1 + \alpha_2 I_2) \in C_{n \times n}$ can be written as a sum of two orthogonal bimatrices.

**Remark 3.8**

Let an integer $n \geq 2$ be given. If $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$ then one checks that $(\alpha_1 I_1 + \alpha_2 I_2) \in C_{n \times n}$ can be written as a sum of two orthogonal bimatrices.

**Theorem 3.9**

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and let a positive integer $n$ be given. Then $(\alpha_1 I_1 + \alpha_2 I_2) \in C_{n \times n}$ can be written as a sum of two orthogonal bimatrices $O_{n \times n}$ if and only if $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$.

**Proof**

For the forward implication, let $\alpha_1, \alpha_2 \in \mathbb{R}$ and let a positive integer $n$ be given. Suppose that $(\alpha_1 I_1 + \alpha_2 I_2) \in C_{n \times n}$ can be written as a sum of two orthogonal bimatrices. Then $\alpha_1 = 0; \alpha_2 \neq 0$ (or) $\alpha_1 \neq 0; \alpha_2 = 0$. If $\alpha_1 = 0; \alpha_2 = 0$, then $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$. If $\alpha_1 \neq 0; \alpha_2 \neq 0$, we show that $\alpha_1 = \alpha_2 = \pm 2$. Lemma 3.5 guarantees that there exists a skew-symmetric bimatrix $(D_1 \cup D_2) \in C_{n \times n}$ satisfying

$$(D_1 D_1^T \cup D_2 D_2^T) = \left[ \begin{array}{ccc} 1 - \alpha_1^2 & 0 \\ 0 & 1 - \alpha_2^2 \\ \end{array} \right] I_1 \cup \left[ \begin{array}{ccc} 1 - \alpha_1^2 & 0 \\ 0 & 1 - \alpha_2^2 \\ \end{array} \right] I_2.$$

Now, since $n$ is odd, the skew-symmetric bimatrix $(D_1 \cup D_2)$ is singular. Hence, $(D_1 D_1^T \cup D_2 D_2^T)$ is singular and $\alpha_1 = \alpha_2 = \pm 2$.

The backward implication can be shown by direct computation.

**References**


