# Sum of Orthogonal Bimatrices in $C_{nxn}$

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Abstract: Let  $F \in \{R, C, H\}$ . Let  $\mathcal{U}_{n \times n}$  be the set of unitary bimatrics in  $F_{n \times n}$ , and let  $O_{n \times n}$  be the set of orthogonal bimatrices in  $F_{n \times n}$ . Suppose  $n \ge 2$ , we show that every  $A_B \in F_{n \times n}$  can be written as a sum of bimatrices in  $\mathcal{U}_{n \times n}$  and of bimatrices in  $O_{n \times n}$ . let  $A_B \in F_{n \times n}$  be given that and let  $k \ge 2$  be the least integer that is a least upper bound of the singular values of  $A_B$ . When F=C, we show that  $A_B$  can be written as a sum of k bimatrices from  $\mathcal{U}_{n\times n}$ .

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#### 1. Introduction

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of nxn complex matrices by  $\mathcal{C}_{nxn}$ . For  $A \in \mathcal{C}_{nxn}$ ,  $A^T, A^{-1}, A^{\dagger}$  and det (A) denote transpose, inverse, Moore-Penrose inverse and determinant of A respectively. If  $AA^{T} = A^{T}A = I$  then A is an orthogonal matrix, where I is the identity matrix. In this paper we study orthogonal bimatrices as a generalization of orthogonal matrices. Some of the properties of orthogonal matrices are extended to orthogonal bimatrices. Some important results of orthogonal matrices are generalized to orthogonal bimatrices.

#### 2. Basic Definitions and Results

#### Definition 1.1 [6]

A bimatrix  $A_{B}$  is defined as the union of two rectangular array of numbers  $A_1$  and  $A_2$  arranged into rows and columns. It is written as  $A_{B} = A_{1} \cup A_{2}$  with  $A_{1} \neq A_{2}$  (except zero and unit bimatrices) where,

 $a^1$ 

$$A_{1} = \begin{bmatrix} a_{11}^{1} & a_{12}^{1} & \cdots & a_{1n}^{1} \\ a_{21}^{1} & a_{22}^{1} & \cdots & a_{2n}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{1} & a_{m2}^{1} & \cdots & a_{mn}^{1} \end{bmatrix}$$
$$A_{2} = \begin{bmatrix} a_{11}^{2} & a_{12}^{2} & \cdots & a_{2n}^{2} \\ a_{21}^{2} & a_{22}^{2} & \cdots & a_{2n}^{2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{2} & a_{m2}^{2} & \cdots & a_{mm}^{2} \end{bmatrix}$$

 $a^{1}$ 

and

 $'\cup'$  is just for the notational convenience (symbol) only.

## Definition 1.2 [6]

Let  $A_{B} = A_{1} \cup A_{2}$  and  $C_{B} = C_{1} \cup C_{2}$  be any two  $m \ge n$ bimatrices. The sum  $D_B$  of the bimatrices  $A_B$  and  $C_B$  is defined as

$$D_{B} = A_{B} + C_{B} = (A_{1} \cup A_{2}) + (C_{1} \cup C_{2})$$
$$= (A_{1} + C_{1}) \cup (A_{2} + C_{2})$$

Where  $A_1 + C_1$  and  $A_2 + C_2$  are the usual addition of matrices.

#### Definition 1.3 [7]

If  $A_{\scriptscriptstyle R} = A_{\scriptscriptstyle 1} \cup A_{\scriptscriptstyle 2}$  and  $C_{\scriptscriptstyle R} = C_{\scriptscriptstyle 1} \cup C_{\scriptscriptstyle 2}$  be two bimatrices, then  $A_B$  and  $C_B$  are said to be equal (written as  $A_B = C_B$ ) if and only if  $A_1$  and  $C_1$  are identical and  $A_2$  and  $C_2$  are identical. (That is,  $A_1 = C_1$  and  $A_2 = C_2$ ).

#### Definition 1.4 [7]

Given a bimatrix  $A_{\rm B} = A_{\rm I} \cup A_{\rm 2}$  and a scalar  $\lambda$ , the product of  $\lambda$  and  $A_{B}$  written as  $\lambda A_{B}$  is defined to be

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$$\lambda A_{B} = \begin{bmatrix} \lambda a_{11}^{1} & \lambda a_{12}^{1} & \cdots & \lambda a_{1n}^{1} \\ \lambda a_{21}^{1} & \lambda a_{22}^{1} & \cdots & \lambda a_{2n}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^{1} & \lambda a_{m2}^{1} & \cdots & \lambda a_{mn}^{1} \end{bmatrix} \cup \begin{bmatrix} \lambda a_{11}^{2} & \lambda a_{12}^{2} & \cdots & \lambda a_{1n}^{2} \\ \lambda a_{21}^{2} & \lambda a_{22}^{2} & \cdots & \lambda a_{2n}^{2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^{2} & \lambda a_{m2}^{2} & \cdots & \lambda a_{mn}^{2} \end{bmatrix}$$
$$= (\lambda A_{1} \cup \lambda A_{2}).$$

That is, each element of  $A_1$  and  $A_2$  are multiplied by  $\lambda$ .

#### Remark 1.5 [7]

If  $A_B = A_1 \cup A_2$  be a bimatrix, then we call  $A_1$  and  $A_2$  as the component matrices of the bimatrix  $A_B$ .

#### Definition 1.6 [6]

If  $A_B = A_1 \cup A_2$  and  $C_B = C_1 \cup C_2$  are both *n* x *n* square bimatrices then, the bimatrix multiplication is defined as,  $A_B \times C_B = (A_1C_1) \cup (A_2C_2).$ 

## Definition 1.7 [6]

Let  $A_B^{m \times m} = A_1 \cup A_2$  be a *mxm* square bimatrix. We define  $I_B^{m \times m} = I^{m \times m} \cup I^{m \times m} = I_1^{m \times m} \cup I_2^{m \times m}$  to be the identity bimatrix.

#### Definition 1.8 [6]

Let  $A_B^{m \times m} = A_1 \cup A_2$  be a square bimatrix,  $A_B$  is a symmetric bimatrix if the component matrices  $A_1$  and  $A_2$  are symmetric matrices. i.e,  $A_1 = A_1^T$  and  $A_2 = A_2^T$ .

#### Definition 1.9 [6]

Let  $A_B^{m \times m} = A_1 \cup A_2$  be a *mxm* square bimatrix i.e,  $A_1$  and  $A_2$  are *mxm* square matrices. A skew-symmetric bimatrix is a bimatrix  $A_B$  for which  $A_B = -A_B^T$ , where  $-A_B^T = -A_1^T \cup -A_2^T$  i.e, the component matrices  $A_1$  and  $A_2$  are skew-symmetric.

#### 3. Orthogonal and Unitary Bimatrices

#### Definition 2.1 [5]

A bimatrix 
$$A_B = A_1 \cup A_2$$
 is said to be orthogonal bimatrix,  
if  $A_B A_B^T = A_B^T A_B = I_B$  (or)  
 $\left(A_1 A_1^T \cup A_2 A_2^T\right) = \left(A_1^T A_1 \cup A_2^T A_2\right) = I_1 \cup I_2.$   
(That is, the component matrices of  $A_B$  are orthogonal.)  
That is,  $A_B^T = A_B^{-1}$  (or)  $\left(A_1^T \cup A_2^T\right) = \left(A_1^{-1} \cup A_2^{-1}\right).$ 

## Remark 2.2

Let  $A_B = A_1 \cup A_2$  be a orthogonal bimatrix. If  $A_1$  and  $A_2$  are square and posses the same order then  $A_B$  is called square orthogonal bimatrix, and if  $A_1$  and  $A_2$  are of different orders then  $A_B$  is called mixed square orthogonal bimatrix.

#### Example 2.3

(1) 
$$A_{B} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{bmatrix} \cup \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

is a square orthogonal bimatrix.

(2) 
$$A_{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cup \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{bmatrix}$$

is a mixed square orthogonal bimatrix.

#### Definition 2.4 [4]

Let  $A_B = A_1 \cup A_2$  be an  $n \times n$  complex bimatrix. (A bimatrix  $A_B$  is said to be complex if it takes entries from the complex field).  $A_B$  is called a unitary bimatrix if  $A_B A_B^* = A_B^* A_B = I_B$  (or)  $\overline{A_B}^T = A_B^{-1}$ . That is,  $A_1 A_1^* \cup A_2 A_2^* = A_1^* A_1 \cup A_2^* A_2 = I_1 \cup I_2$ .

#### Example 2.5

$$A_{B} = A_{1} \cup A_{2} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{2} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$
 is a nitary bimatrix

unitary bimatrix.

In this paper, we have determined which bimatrices (if any) in  $C_{n \times n}$  can be written as a sum of unitary or orthogonal bimatrices. We let  $\mathcal{U}_{n \times n}$  and  $O_{n \times n}$  are the set of unitary and orthogonal bimatrices in the complex field. We begin with the following observation.

Let *n* be a given positive integer. Let  $G \subset F_{n \times n}$  be a group under multiplication. Then  $A_B \in F_{n \times n}$  can be written as a sum of bimatrices in G if and only if for every  $Q_B, P_B \in G$ , the bimatrix  $Q_B A_B P_B$  can be written as a sum of bimatrices in G. Notice that both  $\mathcal{U}_{n \times n}$  and  $O_{n \times n}$  are groups under multiplication.

Let  $\alpha_1, \alpha_2 \in F$  be given. Then lemma 2.6 guarantees that for each  $Q_{\scriptscriptstyle B} \in G$ , we have that  $\alpha_1 Q_1 \cup \alpha_2 Q_2$  can be written as a sum of bimatrices from G if and only if  $\alpha_1 I_1 \cup \alpha_2 I_2$  can be written as a sum of bimatrices from G.

#### Lemma 2.7

Let  $n \ge 2$  be a given integer. Let  $G \subset F_{n \times n}$  be a group under multiplication. Suppose that G contains  $K_B \equiv diag(1, -1, ..., -1)$  and the permutation bimatrices. Then every  $A_B \in F_{n \times n}$  can be written as a sum of bimatrices in G if and only if for each  $\alpha_1, \alpha_2 \in F$ ,  $\alpha_1 I_1 \cup \alpha_2 I_2$  can be written as a sum of bimatrices from G.

## 4. Sum of orthogonal bimatrices in $C_{nxn}$

The only bimatrices in the set of all orthogonal bimatrices of order 1 are ±1. Hence, not every element of  $F_{1\times 1}$  can be written as a sum of elements in the set of all orthogonal bimatrices of order 1. In fact, only the integers can be written as a sum of elements of the set of all orthogonal bimatrices of order 1.

Notice that  $\mathcal{U}_1 = \left\{ e^{i\theta_1}; \theta_1 \in R \right\} \cup \left\{ e^{i\theta_2}; \theta_2 \in R \right\}.$ 

for

each

Set

We

show

that

 $(A_1^k \cup A_2^k) = (C_1^k \cup C_2^k).$ 

$$C_{B_2} = \left(C_1^{II} \cup C_2^{II}\right) \equiv \left\{e^{i\theta_1} + e^{i\beta_1}; \theta_1, \beta_1 \in R\right\} \cup \left\{e^{i\theta_2} + e^{i\beta_2}; \theta_2, \beta_2 \in R\right\}.$$

If 
$$\theta_1, \theta_2, \beta_1, \beta_2 \in K$$
 are given, then  
 $|e^{i\theta_1} + e^{i\beta_1}| \leq 2$  and  $|e^{i\theta_2} + e^{i\beta_2}| \leq 2$ .  
Hence,  $(C_1^{II} \cup C_2^{II}) \subset (A_1^{II} \cup A_2^{II}) \equiv \{Z_1 \in C; |Z_1| \leq 2\} \cup \{Z_2 \in C; |Z_2| \leq 2\}$   
We show that  $(A_1^{II} \cup A_2^{II}) \subset (C_1^{II} \cup C_2^{II})$ . Let  
 $0 \leq r_1 \leq 2$  and  $0 \leq r_2 \leq 2$  are given.  
Set  $\beta_1 = -\theta_1$ ,  $\beta_2 = -\theta_2$  and Choose  $\theta_1$  and  $\theta_2$  so  
that,  $2 \cos \theta_1 = r_1$  and  $2\cos \theta_2 = r_2$ . Then  
 $e^{i\theta_1} + e^{-i\theta_1} = 2\cos \theta_1 = r_1$  and  $e^{i\theta_2} + e^{-i\theta_2} = 2\cos \theta_2 = r_2$ .  
If  $z_1 = r_1 e^{i\delta_1}$  and  $z_2 = r_2 e^{i\delta_2}$ , then choose  
 $\beta_1 = -\theta_1 + 2\delta_1; \beta_2 = -\theta_2 + 2\delta_2$ , and choose  
 $\theta_1, \theta_2$  so that  $2\cos(\theta_1 - \delta_1) = r_1$  and  
 $2\cos(\theta_2 - \delta_2) = r_2$ .  
Let  $k \geq 2$  be an integer.  
Set

$$C_{B_k} = \left(C_1^k \cup C_2^k\right) \equiv \left\{\sum_{j=1}^k e^{i\theta_j^1}; \theta_j^1 \in R \text{ for } j = 1, \dots, k\right\}$$

$$\cup \left\{ \sum_{j=1}^{k} e^{i\theta_{j}^{2}}; \theta_{j}^{2} \in R \text{ for } j = 1, ..., k \right\}$$

and

$$A_{B_{k}} = \left(A_{1}^{k} \cup A_{2}^{k}\right) \equiv \left\{z_{1} \in C; |z_{1}| \leq k\right\} \cup \left\{z_{2} \in C; |z_{2}| \leq k\right\}.$$
  
for each k, we have  $e^{i(\theta_{1}^{1} - \beta_{1})} + ... + e^{i(\theta_{k}^{1} - \beta_{1})} = r_{1}; e^{i(\theta_{1}^{2} - \beta_{2})} + ... + e^{i(\theta_{k}^{2} - \beta_{2})} = r_{2}$ 

First, notice that for each *k*, we  $(C_1^k \cup C_2^k) \subset (A_1^k \cup A_2^k)$ . We now show that Hence,  $(z_1 = r_1 e^{i\beta_1}; z_2 = r_2 e^{i\beta_2}) \in (C_1^k \cup C_2^k)$  if and only  $(A_1^k \cup A_2^k) \subset (C_1^k \cup C_2^k). \quad \text{If} \quad z_1 = r_1 e^{i\beta_1}; \ z_2 = r_2 e^{i\beta_2} \quad \text{if} \quad r_1, r_2 \in (C_1^k \cup C_2^k). \text{ For } (\theta_1^1, \dots, \theta_k^1; \theta_1^2, \dots, \theta_k^2) \in \mathbb{R},$ with  $r_1, r_2, \beta_1, \beta_2 \in R$  and  $r_1, r_2 \ge 0$ , then  $e^{i\theta_1^1} + e^{i\theta_2^1} + \dots + e^{i\theta_k^1} = r_i e^{i\beta_1}$  $e^{i\theta_1^2} + e^{i\theta_2^2} + \dots + e^{i\theta_k^2} = r_2 e^{i\beta_2}$ if and only if

 $f_k^1(\theta_1^1,...,\theta_k^1) \equiv e^{i\theta_1^1} + ... + e^{i\theta_k^1};$ set  $f_k^2(\theta_1^2,...,\theta_k^2) \equiv e^{i\theta_1^2} + ... + e^{i\theta_k^2}.$ 

(1)

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The case k = 2 has already been shown. Let k = 3, and suppose  $0 \le r_1 \le 3; 0 \le r_2 \le 3$  Set  $(\theta_3^1, \theta_3^2) = 0$  and set  $\theta_1^1 = \theta_2^1 = -\theta_2^1$  and  $\theta_1^2 = \theta_2^2 = -\theta_2^2$ . Then  $f_3^1(\theta_1^1, \theta_2^1, \theta_3^1) = 1 + 2Cos \ \theta_1;$  $f_3^2(\theta_1^2, \theta_2^2, \theta_3^2) = 1 + 2Cos \ \theta_2$ , and  $(\theta_1, \theta_2)$  may be  $0 \le r_1 \equiv 1 + 2Cos \ \theta_1 \le 3; \ 0 \le r_2 \equiv 1 + 2Cos \ \theta_2 \le 3.$ We use mathematical induction to show the general case. The hase cases k = 2 and k = 3 have already been shown. Assume that k > 3and suppose that  $(C_1^k \cup C_2^k) = (A_1^k \cup A_2^k).$ Consider  $f_{k+1}^{1}(\theta_{1}^{1},...,\theta_{k}^{1},\theta_{k+1}^{1}) \equiv e^{i\theta_{1}^{1}} + ... + e^{i\theta_{k}^{1}} + e^{i\theta_{k+1}^{1}};$  $f_{k+1}^{2} \left( \theta_{1}^{2}, \dots, \theta_{k}^{2}, \theta_{k+1}^{2} \right) \equiv e^{i\theta_{1}^{2}} + \dots + e^{i\theta_{k}^{2}} + e^{i\theta_{k+1}^{2}}.$ Let  $z_1 = r_1 e^{i\beta_1}$  and  $z_2 = r_2 e^{i\beta_2}$  are given with  $0 \le r_1 \le k + 1, 0 \le r_2 \le k + 1$ . We show that

 $r_1, r_2 \in (C_1^{k+1} \cup C_2^{k+1}).$ First, we show that  $(A_1^{II} \cup A_2^{II}) \subseteq (C_1^{k+1} \cup C_2^{k+1})$ . If k  $\theta_1^2 = \ldots = \theta_1^2 = \pi.$ Then

$$f_{k+1}^{1}\left(\theta_{1}^{1},...,\theta_{k}^{1},\theta_{k+1}^{1}\right) = e^{i\theta_{1}^{1}} + e^{i\theta_{2}^{1}};$$
  
$$f_{k+1}^{2}\left(\theta_{1}^{2},...,\theta_{k}^{2},\theta_{k+1}^{2}\right) = e^{i\theta_{1}^{2}} + e^{i\theta_{2}^{2}}.$$

If k is odd, choose  $\theta_{4}^{1} = ... = \theta_{k-1}^{1} = 0; \ \theta_{4}^{2} = ... = \theta_{k-1}^{2} = 0$ and  $\theta_5^1 = \ldots = \theta_L^1 = \pi;$   $\theta_5^2 = \ldots = \theta_L^2 = \pi.$ Then  $f_{k+1}^{1}(\theta_{1}^{1},...,\theta_{k}^{1},\theta_{k+1}^{1}) \equiv e^{i\theta_{1}^{1}} + e^{i\theta_{2}^{1}} + e^{i\theta_{3}^{1}};$  $f_{k+1}^{2}(\theta_{1}^{2},...,\theta_{k}^{2},\theta_{k+1}^{2}) \equiv e^{i\theta_{1}^{2}} + e^{i\theta_{2}^{2}} + e^{i\theta_{3}^{2}}.$ 

In that  $(A_1^{II} \cup A_2^{II}) \subseteq (C_1^{k+1} \cup C_2^{k+1})$  . Hence, we may assume further that  $r_1, r_2 \ge 1$ ; that is, we need to show that  $(r_1, r_2) \in (C_1^{k+1} \cup C_2^{k+1})$ for  $1 \le r_1 \le k + 1; 1 \le r_2 \le k + 1.$ 

 $\theta_{k+1}^1 = 0; \theta_{k+1}^2 = 0,$  so Choose that  $f_{k+1}^{1}(\theta_{1}^{1},...,\theta_{k}^{1},\theta_{k+1}^{1}) = f_{k}^{1}(\theta_{1}^{1},...,\theta_{k}^{1}) + 1;$ is even, choose  $\theta_3^1 = ... = \theta_{k-1}^1 = 0;$   $f_{k+1}^2(\theta_1^2, ..., \theta_k^2, \theta_{k+1}^2) = f_k^2(\theta_1^2, ..., \theta_k^2) + 1$ . Now,  $\theta_3^2 = ... = \theta_k^2, = 0$  and  $\theta_3^1 = ... = \theta_{k-1}^1 = 0;$  by our inductive burned.  $\theta_3^2 = \dots = \theta_{k-1}^2 = 0$  and  $\theta_4^1 = \dots = \theta_k^1 = \pi;$   $f_k^1 (\theta_1^1, \dots, \theta_k^1) + 1 = r_1; f_k^2 (\theta_1^2, \dots, \theta_k^2) + 1 = r_2$  has a solution since  $0 \le r_1 - 1 \le k; 0 \le r_2 - 1 \le k$ .

## Lemma 3.1

Let 
$$k \ge 2$$
 be a given integer. Let  $(A_1^k \cup A_2^k) \equiv \{z_1 \in C; |z_1| \le k\} \cup \{z_2 \in C; |z_2| \le k\}$  and let  $(C_1^k \cup C_2^k) \equiv \{\sum_{j=1}^k e^{i\theta_j^1}; \theta_j^1 \in R \text{ for } j = 1, ..., k\} \cup \{\sum_{j=1}^k e^{i\theta_j^2}; \theta_j^2 \in R \text{ for } j = 1, ..., k\}$  Then  $(A_1^k \cup A_2^k) = (C_1^k \cup C_2^k)$ .  
I. The case  $\mathcal{U}_{n \times n}$   $n \ge 2$ , Lemma 2.7 guarantees that every

Let  $\alpha_1, \alpha_2 \in C$  be given. Then there exist an integer  $k \geq 2$  and  $\theta_1^1, \theta_2^1, \dots, \theta_k^1; \theta_1^2, \theta_2^2, \dots, \theta_k^2 \in \mathbb{R}$  such that  $\alpha_1 = f_k^1(\theta_1^1, ..., \theta_k^1); \alpha_2 = f_k^2(\theta_1^2, ..., \theta_k^2).$  Now,

 $\left(A_{\!_{1}}\cup A_{\!_{2}}
ight)\!\in\! C_{_{\!n imes n}}$  can be written as a sum of matrices in  $\mathcal{U}_{n imes n}$ . Lemma 3.2

$$\left(\alpha_{1}I_{1}\cup\alpha_{2}I_{2}\right)=\left(f_{k}^{1}\left(\theta_{1}^{1},\ldots,\theta_{k}^{n}\right)\right)$$

$$= \left( e^{i\theta_{1}^{l}} I_{1} + \dots + e^{i\theta_{k}^{l}} I_{1} \right) \cup \left( e^{i\theta_{1}^{2}} I_{2} + \dots + e^{i\theta_{k}^{2}} I_{2} \right)$$

is a sum of matrices in  $\mathcal{U}_{n \times n}$ .

When n=1, every  $\alpha_1, \alpha_2 \in C$  can be written as a sum of elements of the set of all unitary bimatrices of order 1. When

Let n be a given positive integer. Then every  $(\theta_k^1)I_1 \cup (f_k^2(\theta_1^2,...,\theta_k^1)I_2) \cup A_2 \in C_{n \times n}$  can be written as a sum of matrices

## in $\mathcal{U}_{n\times n}$ . Proof

Let  $(A_1 \cup A_2) \in C_{n \times n}$  be given. We look at the number of matrices that make up the sum  $(A_1 \cup A_2)$ .

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Let  $\alpha_1, \alpha_2 \in C$  be given. If  $|\alpha_1|, |\alpha_2| \leq k$  for some positive integer k, then  $(\alpha_1, \alpha_2) \in (A_1^k \cup A_2^k)$ . Moreover,  $(\alpha_1, \alpha_2) \in (A_1^m \cup A_2^m)$  for every integer  $m \geq k$ .

For any such *m*, Lemma 3.1 guarantees that there exist  $\theta_1^1, \dots, \theta_m^1; \ \theta_1^2, \dots, \theta_m^2 \in \mathbb{R}$  such that  $\alpha_1 = e^{i\theta_1^1} + \dots + e^{i\theta_m^1}; \ \alpha_2 = e^{i\theta_1^2} + \dots + e^{i\theta_m^2}.$ 

However, if  $|\alpha_1|, |\alpha_2| < k$  then  $\alpha_1, \alpha_2 \notin (A_1^k \cup A_2^k)$ and  $\alpha_1, \alpha_2$  cannot be written as a sum of k elements of  $\mathcal{U}_1(C)$ .

Write  $(A_1 \cup A_2) = (U_1 \cup U_2)(\Sigma_1 \cup \Sigma_2)(V_1 \cup V_2)$ where  $(U_1 \cup U_2), (V_1 \cup V_2) \in C_{n \times n}$  are unitary bimatrices and

$$\begin{split} \Sigma_{1} &= diag_{B} \left(\sigma_{1}^{1},...,\sigma_{n}^{1}\right) & \text{with} \\ \sigma_{1}^{1} &\geq ... \geq \sigma_{n}^{1} \geq 0; \\ \Sigma_{2} &= diag_{B} \left(\sigma_{1}^{2},...,\sigma_{n}^{2}\right) & \text{with} \\ \sigma_{1}^{2} &\geq ... \geq \sigma_{n}^{2} \geq 0. \end{split}$$

Let k be the least integer such that  $\sigma_1^1, \sigma_1^2 \leq k$ . Suppose that  $k \geq 2$ . Then for each l, we have  $\sigma_l^1, \sigma_l^2 \in (A_1^k \cup A_2^k)$ . Moreover,  $\sigma_1^1, \sigma_1^2 \notin (A_1^{k-1} \cup A_2^{k-1})$ .

Hence,  $(A_1 \cup A_2)$  cannot be written as a sum of *k*-1 unitary bimatrices. However, for each *l*, we have  $\sigma_l^1 = e^{i\theta_{l1}^1} + \ldots + e^{i\theta_{lk}^2}; \ \sigma_l^2 = e^{i\theta_{l1}^2} + \ldots + e^{i\theta_{lk}^2},$  where each  $(\theta_{l1}^1, \ldots, \theta_{lk}^1); (\theta_{l1}^2, \ldots, \theta_{lk}^2) \in R.$ 

II. The case  $O_{n \times n}$ Let n=2. Let  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C$  be given.

Set 
$$(A_1(\alpha_1, \beta_1) \cup A_2(\alpha_2, \beta_2)) \equiv \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \cup \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix}$$
 (2)  
Choose  $\beta_1, \beta_2$  such that  $\alpha_1^2 + \beta_1^2 = 1; \alpha_2^2 + \beta_2^2 = 1$ 

For each t = 1, ..., k, set  $(U_1^t \cup U_2^t) = diag_B(e^{i\theta_{1t}^1}, ..., e^{i\theta_{nt}^1}) \cup diag_B(e^{i\theta_{1t}^2}, ..., e^{i\theta_{nt}^2})$ . Then  $(U_1^t \cup U_2^t) \in C_{n \times n}$  is unitary bimatrix and  $\sum_{t=1}^k (U_1^t \cup U_2^t) = (\Sigma_1 \cup \Sigma_2)$ . Hence,  $(A_1 \cup A_2)$ can be written as a sum of k unitary bimatrices.

Suppose that k=1. If  $\sigma_n^1 = 1$ ;  $\sigma_n^2 = 1$ , Then  $(\Sigma_1 \cup \Sigma_2) = (I_1 \cup I_2)$  and  $(A_1 \cup A_2)$  is unitary bimatrix. If  $\sigma_n^1 \neq 1$ ;  $\sigma_n^2 \neq 1$ , then for each l, we have  $\sigma_l^1, \sigma_l^2 \in (A_1^H \cup A_2^H)$ , and  $(A_1 \cup A_2)$  can be

written as a sum of two unitary bimatrices.

## Theorem 3.3

Let  $(A_1 \cup A_2) \in C_{n \times n}$  be given. Let k be the least (positive) integer so that there exist  $(U_1^1 \cup U_2^1), (U_1^2 \cup U_2^2), ..., (U_1^k \cup U_2^k) \in \mathcal{U}_{n \times n}$  satisfying  $(U_1^1 \cup U_2^1) + ... + (U_1^k \cup U_2^k) = (A_1 \cup A_2).$ 

1. If  $A_1 \cup A_2$  is unitary bimatrix, then k=1.

2. If  $A_1 \cup A_2$  is not unitary bimatrix and  $\sigma_1^1(A_1) \cup \sigma_1^2(A_2) \le 2$  then k=2. 3. If  $m \ge 2$  is an integer such that  $m < \sigma_1^1(A_1) \cup \sigma_1^2(A_2) \le m + 1$  then k = m + 1. positive integers  $m \ge k$ , For we have  $(A_1^k \cup A_2^k) \subseteq (A_1^m \cup A_2^m)$ . Hence, every  $(U_1 \cup U_2) \in \mathcal{U}_{n \times n}$  can be written as a sum of two or more elements of  $\mathcal{U}_{n \times n}$ .

It follows that every  $(A_1 \cup A_2) \in C_{n \times n}$  that can be written as a sum of *k* elements of  $\mathcal{U}_{n \times n}$  can be written as a sum of m elements of  $\mathcal{U}_{n \times n}$ .

and notice that

$$\begin{pmatrix} A_1(\pm\alpha_1,\pm\beta_1)\cup A_2(\pm\alpha_2,\pm\beta_2) \end{pmatrix} \in O_{2\times 2}$$
  
Set  $\begin{pmatrix} A_1^I\cup A_2^I \end{pmatrix} \equiv \begin{pmatrix} A_1(\alpha_1,\beta_1)\cup A_2(\alpha_2,\beta_2) \end{pmatrix}$  and

$$\left(A_1^{II} \cup A_2^{II}\right) \equiv \left(A_1\left(\alpha_1, -\beta_1\right) \cup A_2\left(\alpha_2, -\beta_2\right)\right)$$

Then

$$(A_1^I \cup A_2^I) + (A_1^{II} \cup A_2^{II}) = 2 [\alpha_1 I_1^{II} \cup \alpha_2 I_2^{II}].$$
  
Lemma 2.7 guarantees that every  $(A_1 \cup A_2) \in C_{2\times 2}$  can

be written as a sum of bimatrices from  $O_{2\times 2}$ .

We look at the case when n=3. Let  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$  be given. Set

$$\begin{bmatrix} B_{1}(\alpha_{1},\beta_{1}) \cup B_{2}(\alpha_{2},\beta_{2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_{1} & \beta_{1} \\ 0 & -\beta_{1} & \alpha_{1} \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_{2} & \beta_{2} \\ 0 & -\beta_{2} & \alpha_{2} \end{bmatrix},$$
(3)  
set  $\begin{bmatrix} C_{1}(\alpha_{1},\beta_{1}) \cup C_{2}(\alpha_{2},\beta_{2}) \end{bmatrix} = \begin{bmatrix} \alpha_{1} & 0 & \beta_{1} \\ 0 & 1 & 0 \\ -\beta_{1} & 0 & \alpha_{1} \end{bmatrix} \cup \begin{bmatrix} \alpha_{2} & 0 & \beta_{2} \\ 0 & 1 & 0 \\ -\beta_{2} & 0 & \alpha_{2} \end{bmatrix}$ (4)  
and set  $\begin{bmatrix} D_{1}(\alpha_{1},\beta_{1}) \cup D_{2}(\alpha_{2},\beta_{2}) \end{bmatrix} = \begin{bmatrix} \alpha_{1} & \beta_{1} & 0 \\ -\beta_{1} & \alpha_{1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} \alpha_{2} & \beta_{2} & 0 \\ -\beta_{2} & \alpha_{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} \alpha_{2} & \beta_{2} & 0 \\ -\beta_{2} & \alpha_{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (5)

Choose 
$$\beta_1, \beta_2$$
 so that  $\alpha_1^2 + \beta_1^2 = 1; \ \alpha_2^2 + \beta_2^2 = 1.$   
Then  $\begin{bmatrix} B_1(\pm \alpha_1, \pm \beta_1) \cup B_2(\pm \alpha_2, \pm \beta_2) \end{bmatrix}, \begin{bmatrix} C_1(\pm \alpha_1, \pm \beta_1) \cup C_2(\pm \alpha_2, \pm \beta_2) \end{bmatrix}, \text{ and } \begin{bmatrix} D_1(\pm \alpha_1, \pm \beta_1) \cup D_2(\pm \alpha_2, \pm \beta_2) \end{bmatrix}, \text{ are all elements of } O_{3\times 3}$ 

$$\begin{pmatrix} B_1^I \cup B_2^I \end{pmatrix} \equiv \begin{bmatrix} B_1(\alpha_1, \beta_1) \cup B_2(\alpha_2, \beta_2) \end{bmatrix}, \begin{pmatrix} B_1^{II} \cup B_2^{II} \end{pmatrix} \equiv \begin{bmatrix} B_1(-\alpha_1, \beta_1) \cup B_2(-\alpha_2, \beta_2) \end{bmatrix}, \begin{pmatrix} C_1^I \cup C_2^I \end{pmatrix} \equiv \begin{bmatrix} C_1(\alpha_1, \beta_1) \cup C_2(\alpha_2, \beta_2) \end{bmatrix}, \begin{pmatrix} C_1^{II} \cup C_2^{II} \end{pmatrix} \equiv \begin{bmatrix} C_1(-\alpha_1, \beta_1) \cup C_2(-\alpha_2, \beta_2) \end{bmatrix},$$

$$(D_1^I \cup D_2^I) \equiv [D_1(\alpha_1, \beta_1) \cup D_2(\alpha_2, \beta_2)],$$

And

$$\begin{pmatrix} D_1^{II} \cup D_2^{II} \end{pmatrix} \equiv \begin{bmatrix} D_1(-\alpha_1, \beta_1) \cup D_2(-\alpha_2, \beta_2) \end{bmatrix}.$$
  
Then,  

$$\begin{pmatrix} B_1^{I} \cup B_2^{I} \end{pmatrix} - \begin{pmatrix} B_1^{II} \cup B_2^{II} \end{pmatrix} + \begin{pmatrix} C_1^{I} \cup C_2^{I} \end{pmatrix} - \begin{pmatrix} C_1^{II} \cup C_2^{II} \end{pmatrix}$$
  

$$+ \begin{pmatrix} D_1^{I} \cup D_2^{I} \end{pmatrix} - \begin{pmatrix} D_1^{II} \cup D_2^{II} \end{pmatrix} = 2\begin{bmatrix} \alpha_1 I_1^{III} \cup \alpha_2 I_2^{III} \end{bmatrix}$$

Lemma 2.7 now guarantees that every  $(A_1 \cup A_2) \in C_{3\times 3}$  can be written as a sum of bimatrices in  $O_{3\times 3}$ .

Let n=2m be a positive even integer, and let  $\delta_1, \delta_2 \in C$  be given. Choose  $(A_1^I \cup A_2^I), (A_1^{II} \cup A_2^{II}) \in O_{2\times 2}$  so that  $\left[ (A_1^I \cup A_2^I) + (A_1^{II} \cup A_2^{II}) \right] = \left[ \delta_1 I_1^{II} \cup \delta_2 I_2^{II} \right].$ 

Set  $(E_1^I \cup E_2^I) = (A_1^I \cup A_2^I) \oplus ... \oplus (A_1^I \cup A_2^I) (m \text{ copies})$ and set  $(E_1^{II} \cup E_2^{II}) = (A_1^{II} \cup A_2^{II}) \oplus ... \oplus (A_1^{II} \cup A_2^{II}) (m \text{ copies}).$ 

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Then 
$$(E_1^I \cup E_2^I), (E_1^{II} \cup E_2^{II}) \in O_{2m \times 2m}$$
, and  $(A_1^I \cup A_2^I), (A_1^{II} \cup A_2^{II}) \in O_{2 \times 2}$  so that  $\left[ (E_1^I + E_1^{II}) \cup (E_2^I + E_2^{II}) \right] = (\delta_1 I_1^{2m} \cup \delta_2 I_2^{2m})$   $((A_1^I + A_1^{II}) \cup (A_2^I + A_2^{II})) = (\delta_1 I_1^{II} \cup \delta_2 I_2^{II})$ .  
Let  $n = 2m + 1 \ge 3$  be an odd integer, and let  $\delta_1, \delta_2 \in C$  be given. Choose Also, choose

$$\left(B_1^I \cup B_2^I\right), \left(B_1^H \cup B_2^H\right), \left(C_1^I \cup C_2^I\right), \left(C_1^H \cup C_2^H\right), \left(D_1^I \cup D_2^I\right), \left(D_1^H \cup D_2^H\right) \in O_{3\times 3}$$

Such that

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$$\begin{split} \left[ \begin{pmatrix} B_{1}^{I} - B_{1}^{II} \end{pmatrix} \cup \begin{pmatrix} B_{2}^{I} - B_{2}^{II} \end{pmatrix} \right] + \left[ \begin{pmatrix} C_{1}^{I} - C_{1}^{II} \end{pmatrix} \cup \begin{pmatrix} C_{2}^{I} - C_{2}^{II} \end{pmatrix} \right] + \\ \left[ \begin{pmatrix} D_{1}^{I} - D_{1}^{II} \end{pmatrix} \cup \begin{pmatrix} D_{2}^{I} - D_{2}^{II} \end{pmatrix} \right] = \begin{pmatrix} \delta_{1} I_{1}^{III} + \delta_{2} I_{2}^{III} \end{pmatrix} & \text{Let} \quad n \geq \\ \begin{pmatrix} A_{1} \cup A_{2} \end{pmatrix} \\ \text{Set} & \begin{pmatrix} A_{1} \cup A_{2}^{I} \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} A_{1}^{I} \cup A_{2}^{I} \end{pmatrix} \oplus \begin{pmatrix} B_{1}^{I} \cup B_{2}^{I} \end{pmatrix} & \text{from } O_{n \times n} \\ \begin{pmatrix} m - 1 \text{ copies of } \left( A_{1}^{I} \cup A_{2}^{I} \right) \end{bmatrix} \\ \text{Set} & \begin{pmatrix} m - 1 \text{ copies of } \left( A_{1}^{II} \cup A_{2}^{II} \right) \oplus \dots \oplus \begin{pmatrix} A_{1}^{II} \cup A_{2}^{II} \end{pmatrix} \oplus \begin{pmatrix} -B_{1}^{II} \cup -B_{1}^{II} \cup -B_{2}^{II} \end{pmatrix} \\ \begin{pmatrix} m - 1 \text{ copies of } \left( A_{1}^{II} \cup A_{2}^{II} \right) \end{bmatrix} \\ \text{Set} & \text{where } \left( Q_{1} \\ \text{Set} \\ \begin{pmatrix} E_{1}^{III} \cup E_{2}^{III} \end{pmatrix} = \begin{pmatrix} I_{1}^{2m-2} \cup I_{2}^{2m-2} \end{pmatrix} \oplus \begin{pmatrix} C_{1}^{I} \cup C_{2}^{I} \end{pmatrix}, \\ \text{Set} & \text{so that } \begin{pmatrix} A_{1} A_{1}^{T} \cup A_{2}^{III} \end{pmatrix} \\ \begin{pmatrix} E_{1}^{III} \cup E_{2}^{III} \end{pmatrix} = -\begin{pmatrix} I_{1}^{2m-2} \cup I_{2}^{2m-2} \end{pmatrix} \oplus -\begin{pmatrix} C_{1}^{II} \cup C_{2}^{II} \end{pmatrix}, \\ \text{similar.} \\ \begin{pmatrix} A_{1} \cup A_{2}^{III} \end{pmatrix} \end{pmatrix} \\ \end{pmatrix}$$

 $I \setminus I$ 

Set

$$\begin{pmatrix} E_{1}^{V} \cup E_{2}^{V} \end{pmatrix} = \begin{pmatrix} I_{1}^{2m-2} \cup I_{2}^{2m-2} \end{pmatrix} \oplus \begin{pmatrix} D_{1}^{I} \cup D_{2}^{I} \end{pmatrix},$$
  
and  
$$\begin{pmatrix} E_{1}^{VI} \cup E_{2}^{VI} \end{pmatrix} = -\begin{pmatrix} I_{1}^{2m-2} \cup I_{2}^{2m-2} \end{pmatrix} \oplus -\begin{pmatrix} D_{1}^{II} \cup D_{2}^{II} \end{pmatrix}$$

Then each  $(E_1^j \cup E_2^j) \in O_{2m+1}$ , and

$$(E_1^I \cup E_2^I) + \dots + (E_1^{VI} \cup E_2^{VI}) = (\delta_1 I_1^{2m+1} \cup \delta_2 I_2^{2m+1})$$

Hence, for every  $\alpha_1, \alpha_2 \in C$  and for every integer  $n \ge 2, (\alpha_1 I_1 \cup \alpha_2 I_2)$  can be written as a sum of bimatrices from  $O_{n \times n}$ . Lemma 3.2 guarantees that every  $(A_1 \cup A_2) \in C_{n \times n}$  can be written as a sum of bimatrices from  $O_{n \times n}$ .

Let  $n \ge 2$  be a given integer. Then every  $(A_1 \cup A_2) \in C_{n \times n}$  can be written as a sum of bimatrices from  $O_{n \times n}$ .

Proof

Suppose that  $\underline{(\boldsymbol{Q}_{1}^{\boldsymbol{A}})} \cup \boldsymbol{A}_{2} = \left[ \left( \boldsymbol{Q}_{1}^{\boldsymbol{I}} + \boldsymbol{Q}_{1}^{\boldsymbol{I}} \right) \cup \left( \boldsymbol{Q}_{2}^{\boldsymbol{I}} + \boldsymbol{Q}_{2}^{\boldsymbol{I}} \right) \right],$ where  $(Q_1^I \cup Q_2^I), (Q_1^{II} \cup Q_2^{II}) \in O_{n \times n}$ . Then that  $(A_1A_1^T \cup A_2A_2^T) = (Q_1^T A_1^T \cup Q_2^T A_2^T) (A_1 Q_1^{T} \cup A_2 Q_2^{T}),$ so that  $\left(A_1A_1^T \cup A_2A_2^T\right)$  and  $\left(A_1^TA_1 \cup A_2^TA_2\right)$  are similar. Theorem 13 of [3] ensures that  $(A_1 \cup A_2) + (Q_1S_1 \cup Q_2S_2)$ , where  $(Q_1 \cup Q_2)$ is orthogonal bimatrix and  $(S_1 \cup S_2)$  is symmetric bimatrix (or that  $(A_1\cup A_2)$  has a  $(Q_1S_1\cup Q_2S_2)$ bidecomposition).

has a  $(Q_1S_1 \cup Q_2S_2)$ that Suppose now bidecomposition. Is it true that  $(A_1 \cup A_2)$  can be written as a sum of two (complex) orthogonal bimatrices? Take the case n=1,and notice that every  $(A_1 \cup A_2) \in C_{n \times n}$ is а scalar and has а

 $(Q_1S_1 \cup Q_2S_2)$  bidecomposition.

However, only the integers can be written as a sum of orthogonal bimatrices in this case.

#### Lemma 3.5

Let an integer  $n \ge 2$  and  $0 \ne \alpha_1; 0 \ne \alpha_2 \in \Box$  be given. If  $(\alpha_1 I_1 \cup \alpha_2 I_2) = (Q_1 \cup Q_2) + (V_1 \cup V_2)$ 

is a sum of two bimatrices from  $O_{n \times n}$ , then there exists a skew-symmetric bimatrix  $(D_1 \cup D_2) \in C_{n \times n}$  such that

$$(Q_1 \cup Q_2) = \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) + (D_1 \cup D_2),$$

$$(V_1 \cup V_2) = \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) - (D_1 \cup D_2),$$
  
and

$$\left(D_1D_1^T \cup D_2D_2^T\right) = \left\lfloor \left(1 - \frac{\alpha_1^2}{4}\right)I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right)I_2\right\rfloor$$

Conversely, if there exists a skew-symmetric bimatrix  $(D_1 \cup D_2) \in C_{n \times n}$ such that

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right)I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right)I_2\right], \text{ then}$$

$$(Q_1 \cup Q_2) \equiv \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) + (D_1 \cup D_2)$$
 and

$$(V_1 \cup V_2) \equiv \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) - (D_1 \cup D_2) \text{ are in } O_{n \times n}$$
  
and  $(Q_1 \cup Q_2) + (V_1 \cup V_2) = (\alpha_1 I_1 \cup \alpha_2 I_2)$ 

#### Proof

Let an integer  $n \ge 2$  and  $0 \ne \alpha_1; 0 \ne \alpha_2 \in \Box$ be given. Suppose that  $(\alpha_1 I_1 \cup \alpha_2 I_2) \in C_{n \times n}$  can be written as a sum of two orthogonal bimatrices, say  $(\alpha_1 I_1 \cup \alpha_2 I_2) = (Q_1 \cup Q_2) + (V_1 \cup V_2),$ Write  $(Q_1 \cup Q_2) = \begin{bmatrix} a_{ij}^1 \cup a_{ij}^2 \end{bmatrix} = \begin{bmatrix} q_1^1 \dots q_n^1 \end{bmatrix} \cup \begin{bmatrix} q_1^2 \dots q_n^2 \end{bmatrix}$  $(V_1 \cup V_2) = \left\lceil b_{ij}^1 \cup b_{ij}^2 \right\rceil = \left\lceil v_1^1 \dots v_n^1 \right\rceil \cup \left\lceil v_1^2 \dots v_n^2 \right\rceil.$ Then  $b_{ii}^1 = -a_{ii}^1$  and  $b_{ij}^2 = -a_{ij}^2$  for  $i \neq j$ . Now, for each i = 1, ..., n, we have

$$\sum_{i=1}^{n} a_{ij}^{1^{2}} = q_{i}^{1^{T}} q_{i}^{1} = 1 = v_{i}^{1^{T}} v_{i}^{1} = \sum_{j=1}^{n} b_{ij}^{1^{2}} = b_{ii}^{1^{2}} + \sum_{j \neq i, j=1}^{n} a_{ij}^{1^{2}}$$

and

$$\sum_{j=1}^{n} a_{ij}^{2^{2}} = q_{i}^{2^{T}} q_{i}^{2} = 1 = v_{i}^{2^{T}} v_{i}^{2} = \sum_{j=1}^{n} b_{ij}^{2^{2}} = b_{ii}^{2^{2}} + \sum_{j \neq i, j=1}^{n} a_{ij}^{2^{2}} \left( Q_{1} \cup Q_{2} \right) \equiv \left( \frac{\alpha_{1}}{2} I_{1} \cup \frac{\alpha_{2}}{2} I_{2} \right) + \left( D_{1} \cup D_{2} \right)$$

Hence, 
$$b_{ii}^1 = \pm a_{ii}^1$$
 and  $b_{ii}^2 = \pm a_{ii}^2$ .  
Because  $(Q_1 \cup Q_2) + (V_1 \cup V_2) = (\alpha_1 I_1 \cup \alpha_2 I_2)$   
and  $\alpha_1 \neq 0; \alpha_2 \neq 0$  we have

$$b_{ii}^1 = a_{ii}^1 = \frac{\alpha_1}{2}; b_{ii}^2 = a_{ii}^2 = \frac{\alpha_2}{2}$$
 Set

$$(D_1 \cup D_2) = [d_{ij}^1] \cup [d_{ij}^2],$$
 with

$$d_{ij}^{1} = a_{ij}^{1}; d_{ij}^{2} = a_{ij}^{2}$$
 if  $i \neq j$ , and  $d_{ii}^{1} = 0; d_{ii}^{2} = 0$ , so

that 
$$(Q_1 \cup Q_2) = \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) + (D_1 \cup D_2)$$
  
and  $(V_1 \cup V_2) = \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) - (D_1 \cup D_2).$ 

Now, since  $\left( Q_1 \cup Q_2 
ight)$  and  $\left( V_1 \cup V_2 
ight)$  are orthogonal bimatrices, we have

$$\begin{pmatrix} Q_1 Q_1^T \cup Q_2 Q_2^T \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1^2}{4} I_1 \cup \frac{\alpha_2^2}{4} I_2 \end{pmatrix} + \\ \left\lfloor \frac{\alpha_1}{2} (D_1 + D_1^T) \cup \frac{\alpha_2}{2} (D_2 + D_2^T) \right\rfloor + (D_1 D_1^T \cup D_2 D_2^T) \\ = (I_1 \cup I_2)$$

$$(6)$$

And 
$$\left(V_1V_1^T \cup V_2V_2^T\right) = \left[\frac{\alpha_1^2}{4}I_1 \cup \frac{\alpha_2^2}{4}I_2\right] - \left[\frac{\alpha_1}{2}\left(D_1 + D_1^T\right) \cup \frac{\alpha_2}{2}\left(D_2 + D_2^T\right)\right] + \left(D_1D_1^T \cup D_2D_2^T\right) = \left(I_1 \cup I_2\right)$$

$$(7)$$

Subtracting equation (7) from equation (6), we get  $(D_1 \cup D_2) = -(D_1^T \cup D_2^T)$ , so that  $(D_1 \cup D_2)$ is skew-symmetric bimatrix. Moreover

 $\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left| \left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2 \right|$ 

For the converse, suppose that  $(D_1 \cup D_2) \in C_{n \times n}$  is

$$\sum_{\substack{\neq i, j=1 \\ \neq i, j=1 \\ }} \alpha_{ij} \text{ skew-symmetric bimatrix and satisfies,}} \left( D_1 D_1^T \cup D_2 D_2^T \right) = \left[ \left( 1 - \frac{\alpha_1^2}{4} \right) I_1 \cup \left( 1 - \frac{\alpha_2^2}{4} \right) I_2 \right]$$

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and set

$$\left(V_1 \cup V_2\right) \equiv \left(\frac{\alpha_1}{2}I_1 \cup \frac{\alpha_2}{2}I_2\right) - \left(D_1 \cup D_2\right)$$

Then one checks that both  $(Q_1 \cup Q_2)$  and  $(V_1 \cup V_2)$ are orthogonal bimatrices and  $[(Q_1 + V_1) \cup (Q_2 + V_2)] = (\alpha_1 I_1 \cup \alpha_2 I_2).$ 

## Remark 3.6

When 
$$\alpha_1 = 0$$
;  $\alpha_2 = 0$ , then for any orthogonal bimatrix  
 $(Q_1 \cup Q_2)$ , notice that  
 $(\alpha_1 I_1 \cup \alpha_2 I_2) = \lfloor (Q_1 + (-Q_1)) \cup (Q_2 + (-Q_2)) \rfloor$   
is a sum of two orthogonal bimatrices.

Let 
$$n=2$$
 and  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ . Set  
 $\beta_1 \equiv \sqrt{1 - \frac{\alpha_1^2}{4}}; \beta_2 \equiv \sqrt{1 - \frac{\alpha_2^2}{4}}$  (either square root).  
Then  $(D_1 \cup D_2) \equiv \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & \beta_2 \\ -\beta_2 & 0 \end{bmatrix}$  is  
skew-symmetric bimatrix and satisfies

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right)I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right)I_2\right]$$

Lemma 3.5 guarantees that  $(\alpha_1 I_1 \cup \alpha_2 I_2)$  can be written as a sum of two orthogonal bimatrices.

If n=2k and  $\alpha_1 \neq 0$ ;  $\alpha_2 \neq 0$ , set  $(E_1 \cup E_2) = (D_1 \cup D_2) \oplus ... \oplus (D_1 \cup D_2)$  (k copies) and notice that  $(E_1 \cup E_2)$  is skew-symmetric bimatrix and satisfies

$$\left(E_1E_1^T \cup E_2E_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right)I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right)I_2\right]$$

Hence, if n=2k and if  $\alpha_1$ ,  $\alpha_2$  be a scalar, then  $(\alpha_1 I_1 \cup \alpha_2 I_2)$  can be written as a sum of two orthogonal bimatrices.

## Theorem 3.7

Let *n* be a given positive integer. For each  $\alpha_1, \alpha_2 \in \Box$ and each orthogonal bimatrix  $(Q_1 \cup Q_2) \in C_{2n}, (\alpha_1 Q_1 \cup \alpha_2 Q_2)$  can be written as a sum of two orthogonal bimatrices.

#### Remark 3.8

Let an integer  $n \ge 2$  be given. If  $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$ then one checks that  $(\alpha_1 I_1 \cup \alpha_2 I_2) \in C_{n \times n}$  can be written as a sum of two orthogonal bimatrices.

### Theorem 3.9

Let  $\alpha_1, \alpha_2 \in \Box$  and let a positive integer *n* be given. Then  $(\alpha_1 I_1 \cup \alpha_2 I_2) \in C_{2n+1}$  can be written as a sum of two bimatrices from  $O_{n \times n}$  if and only if  $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$ .

## Proof

For the forward implication, let  $\alpha_1, \alpha_2 \in \Box$  and let a positive integer *n* be given. Suppose that  $(\alpha_1 I_1 \cup \alpha_2 I_2) \in C_{2n+1}$  can be written as a sum of two orthogonal bimatrices. Then  $\alpha_1 = 0$ ;  $\alpha_2 \neq 0$  (or)  $\alpha_1 \neq 0$ ;  $\alpha_2 \neq 0$ . If  $\alpha_1 = 0$ ;  $\alpha_2 = 0$ , then  $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$ . If  $\alpha_1 \neq 0$ ;  $\alpha_2 \neq 0$ , we show that  $\alpha_1 = \alpha_2 = \pm 2$ . Lemma 3.5 guarantees that there exists a skew-symmetric bimatrix  $(D_1 \cup D_2) \in C_{n \times n}$  satisfying

$$\left(D_1D_1^T \cup D_2D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right)I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right)I_2\right].$$

Now, since n is odd, the skew-symmetric bimatrix  $(D_1 \cup D_2)$  is singular. Hence,  $(D_1 D_1^T \cup D_2 D_2^T)$  is singular and  $\alpha_1 = \alpha_2 = \pm 2$ .

The backward implication can be show by direct computation.

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