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Sum of Orthogonal Bimatrices in C_{nxn}

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Abstract: Let $F \in \{R, C, H\}$. Let $\mathcal{U}_{n\times n}$ be the set of unitary bimatrics in $F_{n\times n}$, and let $O_{n\times n}$ be the set of orthogonal bimatrices in $F_{n\times n}$. Suppose $n \geq 2$. we show that every $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in $\mathcal{U}_{n\times n}$ and of bimatrices in $O_{n\times n}$. let $A_B \in F_{n\times n}$ be given that and let $k \geq 2$ be the least integer that is a least upper bound of the singular values of A_B. When $F=C$, we show that A_B can be written as a sum of k bimatrices from $\mathcal{U}_{n\times n}$.

Keywords: Orthogonal matrix, bimatrix, orthogonal bimatrix, unitary bimatrix, sum of orthogonal bimatrices, sum of unitary bimatrices.

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1. Introduction

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of *nxn* complex matrices by $\mathcal{C}_{nxn}$. For $A \in \mathcal{C}_{nxn}$, $A^T, A^{-1}, A^{\dagger}$ and $\det(A)$ denote transpose, inverse, Moore-Penrose inverse and determinant of A respectively. If $AA^T = A^TA = I$ then A is an orthogonal matrix, where I is the identity matrix. In this paper we study orthogonal bimatrices as a generalization of orthogonal matrices. Some of the properties of orthogonal matrices are extended to orthogonal bimatrices. Some important results of orthogonal matrices are generalized to orthogonal bimatrices.

2. Basic Definitions and Results

Definition 1.1 [6]

A bimatrix A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$ with $A_1 \neq A_2$ (except zero and unit bimatrices) where,

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix}$$ and

$$A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

'$\cup$' is just for the notational convenience (symbol) only.

Definition 1.2 [6]

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be any two *m* x *n* bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as

$$D_B = A_B + C_B = (A_1 \cup A_2) + (C_1 \cup C_2) = (A_1 + C_1) \cup (A_2 + C_2)$$

Where $A_1 + C_1$ and $A_2 + C_2$ are the usual addition of matrices.

Definition 1.3 [7]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices, then A_B and C_B are said to be equal (written as $A_B = C_B$) if and only if A_1 and C_1 are identical and A_2 and C_2 are identical. (That is, $A_1 = C_1$ and $A_2 = C_2$).

Definition 1.4 [7]

Given a bimatrix $A_B = A_1 \cup A_2$ and a scalar λ, the product of λ and A_B written as λA_B is defined to be

$$\lambda A_B = \begin{bmatrix} \lambda a_{11}^1 & \lambda a_{12}^1 & \cdots & \lambda a_{1n}^1 \\ \lambda a_{21}^1 & \lambda a_{22}^1 & \cdots & \lambda a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^1 & \lambda a_{m2}^1 & \cdots & \lambda a_{mn}^1 \end{bmatrix} \cup \begin{bmatrix} \lambda a_{11}^2 & \lambda a_{12}^2 & \cdots & \lambda a_{1n}^2 \\ \lambda a_{21}^2 & \lambda a_{22}^2 & \cdots & \lambda a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^2 & \lambda a_{m2}^2 & \cdots & \lambda a_{mn}^2 \end{bmatrix}$$
$$= \left(\lambda A_1 \cup \lambda A_2\right).$$

That is, each element of A_1 and A_2 are multiplied by λ.

Remark 1.5 [7]

If $A_B = A_1 \cup A_2$ be a bimatrix, then we call A_1 and A_2 as the component matrices of the bimatrix A_B.

Definition 1.6 [6]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ are both *n* x *n* square bimatrices then, the bimatrix multiplication is defined as, $A_B \times C_B = \left(A_1C_1\right) \cup \left(A_2C_2\right)$.

Definition 1.7 [6]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *mxm* square bimatrix. We define $I_B^{m\times m} = I^{m\times m} \cup I^{m\times m} = I_1^{m\times m} \cup I_2^{m\times m}$ to be the identity bimatrix.

Definition 1.8 [6]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a square bimatrix, A_B is a symmetric bimatrix if the component matrices A_1 and A_2 are symmetric matrices. i.e, $A_1 = A_1^T$ and $A_2 = A_2^T$.

Definition 1.9 [6]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *mxm* square bimatrix i.e, A_1 and A_2 are *mxm* square matrices. A skew-symmetric bimatrix is a bimatrix A_B for which $A_B = -A_B^T$, where $-A_B^T = -A_1^T \cup -A_2^T$ i.e, the component matrices A_1 and A_2 are skew-symmetric.

3. Orthogonal and Unitary Bimatrices

Definition 2.1 [5]

A bimatrix $A_B = A_1 \cup A_2$ is said to be orthogonal bimatrix, if $A_B A_B^T = A_B^T A_B = I_B$ (or) $\left(A_1A_1^T \cup A_2A_2^T\right) = \left(A_1^TA_1 \cup A_2^TA_2\right) = I_1 \cup I_2$.

(That is, the component matrices of A_B are orthogonal.)

That is, $A_B^T = A_B^{-1}$ (or) $\left(A_1^T \cup A_2^T\right) = \left(A_1^{-1} \cup A_2^{-1}\right)$.

Remark 2.2

Let $A_B = A_1 \cup A_2$ be a orthogonal bimatrix. If A_1 and A_2 are square and posses the same order then A_B is called square orthogonal bimatrix, and if A_1 and A_2 are of different orders then A_B is called mixed square orthogonal bimatrix.

Example 2.3

(1) $$A_B = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{bmatrix} \cup \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

is a square orthogonal bimatrix.

(2) $$A_B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cup \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{bmatrix}$$

is a mixed square orthogonal bimatrix.

Definition 2.4 [4]

Let $A_B = A_1 \cup A_2$ be an $n \times n$ complex bimatrix. (A bimatrix A_B is said to be complex if it takes entries from the complex field). A_B is called a unitary bimatrix if $A_B A_B^* = A_B^* A_B = I_B$ (or) $\bar{A}_B^{\,T} = A_B^{-1}$.

That is, $A_1A_1^* \cup A_2A_2^* = A_1^*A_1 \cup A_2^*A_2 = I_1 \cup I_2$.

Example 2.5

$$A_B = A_1 \cup A_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$ is a unitary bimatrix.

In this paper, we have determined which bimatrices (if any) in $C_{n\times n}$ can be written as a sum of unitary or orthogonal bimatrices. We let $\mathcal{U}_{n\times n}$ and $O_{n\times n}$ are the set of unitary and orthogonal bimatrices in the complex field. We begin with the following observation.

Lemma 2.6

Let n be a given positive integer. Let $G \subset F_{n\times n}$ be a group under multiplication. Then $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in G if and only if for every $Q_B, P_B \in G$, the bimatrix $Q_B A_B P_B$ can be written as a sum of bimatrices in G. Notice that both $\mathcal{U}_{n\times n}$ and $O_{n\times n}$ are groups under multiplication.

Let $\alpha_1, \alpha_2 \in F$ be given. Then lemma 2.6 guarantees that for each $Q_B \in G$, we have that $\alpha_1 Q_1 \cup \alpha_2 Q_2$ can be written as a sum of bimatrices from G if and only if $\alpha_1 I_1 \cup \alpha_2 I_2$ can be written as a sum of bimatrices from G.

Lemma 2.7

Let $n \geq 2$ be a given integer. Let $G \subset F_{n\times n}$ be a group under multiplication. Suppose that G contains $K_B \equiv diag(1,-1,\ldots,-1)$ and the permutation bimatrices. Then every $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in G if and only if for each $\alpha_1, \alpha_2 \in F$, $\alpha_1 I_1 \cup \alpha_2 I_2$ can be written as a sum of bimatrices from G.

4. Sum of orthogonal bimatrices in $C_{n\times n}$

The only bimatrices in the set of all orthogonal bimatrices of order 1 are ±1. Hence, not every element of $F_{1\times 1}$ can be written as a sum of elements in the set of all orthogonal bimatrices of order 1. In fact, only the integers can be written as a sum of elements of the set of all orthogonal bimatrices of order 1.

Notice that $\mathcal{U}_1 = \{e^{i\theta_1}; \theta_1 \in R\} \cup \{e^{i\theta_2}; \theta_2 \in R\}$.

Set

$$C_{B_2} = (C_1^{II} \cup C_2^{II}) \equiv \{e^{i\theta_1} + e^{i\beta_1}; \theta_1, \beta_1 \in R\} \cup \{e^{i\theta_2} + e^{i\beta_2}; \theta_2, \beta_2 \in R\}.$$

If $\theta_1, \theta_2, \beta_1, \beta_2 \in R$ are given, then $|e^{i\theta_1} + e^{i\beta_1}| \leq 2$ and $|e^{i\theta_2} + e^{i\beta_2}| \leq 2$.

Hence, $(C_1^{II} \cup C_2^{II}) \subset (A_1^{II} \cup A_2^{II}) \equiv \{Z_1 \in C; |Z_1| \leq 2\} \cup \{Z_2 \in C; |Z_2| \leq 2\}$

We show that $(A_1^{II} \cup A_2^{II}) \subset (C_1^{II} \cup C_2^{II})$. Let $0 \leq r_1 \leq 2$ and $0 \leq r_2 \leq 2$ are given.

Set $\beta_1 = -\theta_1$, $\beta_2 = -\theta_2$ and Choose θ_1 and θ_2 so that, $2\,Cos\ \theta_1 = r_1$ and $2Cos\ \theta_2 = r_2$. Then $e^{i\theta_1} + e^{-i\theta_1} = 2Cos\,\theta_1 = r_1$ and $e^{i\theta_2} + e^{-i\theta_2} = 2Cos\ \theta_2 = r_2$.

If $z_1 = r_1 e^{i\delta_1}$ and $z_2 = r_2 e^{i\delta_2}$, then choose $\beta_1 = -\theta_1 + 2\delta_1; \beta_2 = -\theta_2 + 2\delta_2$, and choose $\theta_1,\ \theta_2$ so that $2Cos(\theta_1 - \delta_1) = r_1$ and $2Cos(\theta_2 - \delta_2) = r_2$.

Let $k \geq 2$ be an integer.

Set

$$C_{B_k} = (C_1^k \cup C_2^k) \equiv \left\{\sum_{j=1}^{k} e^{i\theta_j^1}; \theta_j^1 \in R \ for\ j = 1,\ldots,k\right\}$$

$$\cup \left\{\sum_{j=1}^{k} e^{i\theta_j^2}; \theta_j^2 \in R\ for\ j = 1,\ldots,k\right\}$$

and set

$$A_{B_k} = (A_1^k \cup A_2^k) \equiv \{z_1 \in C; |z_1| \leq k\} \cup \{z_2 \in C; |z_2| \leq k\}.$$

We show that for each k, we have $(A_1^k \cup A_2^k) = (C_1^k \cup C_2^k)$.

First, notice that for each k, we have $(C_1^k \cup C_2^k) \subset (A_1^k \cup A_2^k)$. We now show that $(A_1^k \cup A_2^k) \subset (C_1^k \cup C_2^k)$. If $z_1 = r_1 e^{i\beta_1}$; $z_2 = r_2 e^{i\beta_2}$ with $r_1, r_2, \beta_1, \beta_2 \in R$ and $r_1, r_2 \geq 0$, then $e^{i\theta_1^1} + e^{i\theta_2^1} + \ldots + e^{i\theta_k^1} = r_1 e^{i\beta_1}$; $e^{i\theta_1^2} + e^{i\theta_2^2} + \ldots + e^{i\theta_k^2} = r_2 e^{i\beta_2}$ if and only if

$$e^{i(\theta_1^1 - \beta_1)} + \ldots + e^{i(\theta_k^1 - \beta_1)} = r_1; \quad e^{i(\theta_1^2 - \beta_2)} + \ldots + e^{i(\theta_k^2 - \beta_2)} = r_2 \tag{1}$$

Hence, $(z_1 = r_1 e^{i\beta_1}; z_2 = r_2 e^{i\beta_2}) \in (C_1^k \cup C_2^k)$ if and only if $r_1, r_2 \in (C_1^k \cup C_2^k)$. For $(\theta_1^1, \ldots, \theta_k^1; \theta_1^2, \ldots, \theta_k^2) \in R$, set $f_k^1(\theta_1^1, \ldots, \theta_k^1) \equiv e^{i\theta_1^1} + \ldots + e^{i\theta_k^1}$; $f_k^2(\theta_1^2, \ldots, \theta_k^2) \equiv e^{i\theta_1^2} + \ldots + e^{i\theta_k^2}$.

The case $k = 2$ has already been shown. Let $k = 3$, and suppose $0 \le r_1 \le 3;\ 0 \le r_2 \le 3$ Set $\left(\theta_3^1, \theta_3^2\right) = 0$ and set $\theta_1^1 = \theta^1 = -\theta_2^1$ and $\theta_1^2 = \theta^2 = -\theta_2^2$. Then
$f_3^1\left(\theta_1^1, \theta_2^1, \theta_3^1\right) = 1 + 2Cos\ \theta_1$;
$f_3^2\left(\theta_1^2, \theta_2^2, \theta_3^2\right) = 1 + 2Cos\ \theta_2$, and $\left(\theta_1,\ \theta_2\right)$ may be chosen so that $0 \le r_1 \equiv 1 + 2Cos\ \theta_1 \le 3;\ 0 \le r_2 \equiv 1 + 2Cos\ \theta_2 \le 3$.

We use mathematical induction to show the general case. The base cases $k = 2$ and $k = 3$ have already been shown. Assume that $k > 3$ and suppose that $\left(C_1^k \cup C_2^k\right) = \left(A_1^k \cup A_2^k\right)$.

Consider $f_{k+1}^1\left(\theta_1^1, \ldots, \theta_k^1, \theta_{k+1}^1\right) \equiv e^{i\theta_1^1} + \ldots + e^{i\theta_k^1} + e^{i\theta_{k+1}^1}$;
$f_{k+1}^2\left(\theta_1^2, \ldots, \theta_k^2, \theta_{k+1}^2\right) \equiv e^{i\theta_1^2} + \ldots + e^{i\theta_k^2} + e^{i\theta_{k+1}^2}$. Let $z_1 = r_1 e^{i\beta_1}$ and $z_2 = r_2 e^{i\beta_2}$ are given with $0 \le r_1 \le k+1, 0 \le r_2 \le k+1$. We show that $r_1, r_2 \in \left(C_1^{k+1} \cup C_2^{k+1}\right)$.

First, we show that $\left(A_1^{II} \cup A_2^{II}\right) \subseteq \left(C_1^{k+1} \cup C_2^{k+1}\right)$.If k is even, choose $\theta_3^1 = \ldots = \theta_{k-1}^1 = 0$; $\theta_3^2 = \ldots = \theta_{k-1}^2 = 0$ and $\theta_4^1 = \ldots = \theta_k^1 = \pi$; $\theta_4^2 = \ldots = \theta_k^2 = \pi$. Then

$f_{k+1}^1\left(\theta_1^1, \ldots, \theta_k^1, \theta_{k+1}^1\right) = e^{i\theta_1^1} + e^{i\theta_2^1}$;
$f_{k+1}^2\left(\theta_1^2, \ldots, \theta_k^2, \theta_{k+1}^2\right) = e^{i\theta_1^2} + e^{i\theta_2^2}$.

If k is odd, choose $\theta_4^1 = \ldots = \theta_{k-1}^1 = 0$; $\theta_4^2 = \ldots = \theta_{k-1}^2 = 0$ and $\theta_5^1 = \ldots = \theta_k^1 = \pi$; $\theta_5^2 = \ldots = \theta_k^2 = \pi$. Then
$f_{k+1}^1\left(\theta_1^1, \ldots, \theta_k^1, \theta_{k+1}^1\right) \equiv e^{i\theta_1^1} + e^{i\theta_2^1} + e^{i\theta_3^1}$;
$f_{k+1}^2\left(\theta_1^2, \ldots, \theta_k^2, \theta_{k+1}^2\right) \equiv e^{i\theta_1^2} + e^{i\theta_2^2} + e^{i\theta_3^2}$.

In both cases, notice that $\left(A_1^{II} \cup A_2^{II}\right) \subseteq \left(C_1^{k+1} \cup C_2^{k+1}\right)$. Hence, we may assume further that $r_1, r_2 \ge 1$; that is, we need to show that $\left(r_1, r_2\right) \in \left(C_1^{k+1} \cup C_2^{k+1}\right)$ for $1 \le r_1 \le k+1; 1 \le r_2 \le k+1$.

Choose $\theta_{k+1}^1 = 0; \theta_{k+1}^2 = 0$, so that
$f_{k+1}^1\left(\theta_1^1, \ldots, \theta_k^1, \theta_{k+1}^1\right) = f_k^1\left(\theta_1^1, \ldots, \theta_k^1\right) + 1$;
$f_{k+1}^2\left(\theta_1^2, \ldots, \theta_k^2, \theta_{k+1}^2\right) = f_k^2\left(\theta_1^2, \ldots, \theta_k^2\right) + 1$. Now, by our inductive hypothesis, the equation $f_k^1\left(\theta_1^1, \ldots, \theta_k^1\right) + 1 = r_1$; $f_k^2\left(\theta_1^2, \ldots, \theta_k^2\right) + 1 = r_2$ has a solution since $0 \le r_1 - 1 \le k; 0 \le r_2 - 1 \le k$.

Lemma 3.1

Let $k \ge 2$ be a given integer. Let $\left(A_1^k \cup A_2^k\right) \equiv \left\{z_1 \in C; \left|z_1\right| \le k\right\} \cup \left\{z_2 \in C; \left|z_2\right| \le k\right\}$ and let $\left(C_1^k \cup C_2^k\right) \equiv \left\{\sum_{j=1}^{k} e^{i\theta_j^1}; \theta_j^1 \in R \text{ for } j = 1, \ldots, k\right\} \cup \left\{\sum_{j=1}^{k} e^{i\theta_j^2}; \theta_j^2 \in R \text{ for } j = 1, \ldots, k\right\}$ Then $\left(A_1^k \cup A_2^k\right) = \left(C_1^k \cup C_2^k\right)$.

I. The case $\mathcal{U}_{n\times n}$

Let $\alpha_1, \alpha_2 \in C$ be given. Then there exist an integer $k \ge 2$ and $\theta_1^1, \theta_2^1 \ldots, \theta_k^1;\ \theta_1^2, \theta_2^2 \ldots, \theta_k^2 \in R$ such that $\alpha_1 = f_k^1\left(\theta_1^1, \ldots, \theta_k^1\right); \alpha_2 = f_k^2\left(\theta_1^2, \ldots, \theta_k^2\right)$. Now, notice that

$$\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) = \left(f_k^1\left(\theta_1^1, \ldots, \theta_k^1\right) I_1\right) \cup \left(f_k^2\left(\theta_1^2, \ldots, \theta_k^2\right) I_2\right)$$

$$= \left(e^{i\theta_1^1} I_1 + \ldots + e^{i\theta_k^1} I_1\right) \cup \left(e^{i\theta_1^2} I_2 + \ldots + e^{i\theta_k^2} I_2\right)$$

is a sum of matrices in $\mathcal{U}_{n\times n}$.

When n=1, every $\alpha_1, \alpha_2 \in C$ can be written as a sum of elements of the set of all unitary bimatrices of order 1. When $n \ge 2$, Lemma 2.7 guarantees that every $\left(A_1 \cup A_2\right) \in C_{n\times n}$ can be written as a sum of matrices in $\mathcal{U}_{n\times n}$.

Lemma 3.2

Let n be a given positive integer. Then every $\left(A_1 \cup A_2\right) \in C_{n\times n}$ can be written as a sum of matrices in $\mathcal{U}_{n\times n}$.

Proof

Let $\left(A_1 \cup A_2\right) \in C_{n\times n}$ be given. We look at the number of matrices that make up the sum $\left(A_1 \cup A_2\right)$.

Let $\alpha_1,\ \alpha_2 \in C$ be given. If $|\alpha_1|,|\alpha_2| \le k$ for some positive integer k, then $(\alpha_1,\alpha_2) \in (A_1^k \cup A_2^k)$. Moreover, $(\alpha_1,\alpha_2) \in (A_1^m \cup A_2^m)$ for every integer $m \ge k$.
For any such m, Lemma 3.1 guarantees that there exist $\theta_1^1,\ldots,\theta_m^1;\ \theta_1^2,\ldots,\theta_m^2 \in R$ such that $\alpha_1 = e^{i\theta_1^1} + \ldots + e^{i\theta_m^1};\ \alpha_2 = e^{i\theta_1^2} + \ldots + e^{i\theta_m^2}$.

However, if $|\alpha_1|,|\alpha_2| < k$ then $\alpha_1,\alpha_2 \notin (A_1^k \cup A_2^k)$ and α_1,α_2 cannot be written as a sum of k elements of $\mathcal{U}_1(C)$.

Write $(A_1 \cup A_2) = (U_1 \cup U_2)(\Sigma_1 \cup \Sigma_2)(V_1 \cup V_2)$ where $(U_1 \cup U_2)$, $(V_1 \cup V_2) \in C_{n\times n}$ are unitary bimatrices and
$\Sigma_1 = diag_B\ (\sigma_1^1,\ldots,\sigma_n^1)$ with $\sigma_1^1 \ge \ldots \ge \sigma_n^1 \ge 0$;
$\Sigma_2 = diag_B\ (\sigma_1^2,\ldots,\sigma_n^2)$ with $\sigma_1^2 \ge \ldots \ge \sigma_n^2 \ge 0$.

Let k be the least integer such that $\sigma_1^1,\sigma_1^2 \le k$. Suppose that $k \ge 2$. Then for each l, we have $\sigma_l^1,\sigma_l^2 \in (A_1^k \cup A_2^k)$. Moreover, $\sigma_1^1,\sigma_1^2 \notin (A_1^{k-1} \cup A_2^{k-1})$.

Hence, $(A_1 \cup A_2)$ cannot be written as a sum of k-1 unitary bimatrices. However, for each l, we have $\sigma_l^1 = e^{i\theta_{l1}^1} + \ldots + e^{i\theta_{lk}^2};\ \sigma_l^2 = e^{i\theta_{l1}^2} + \ldots + e^{i\theta_{lk}^2}$, where each $(\theta_{l1}^1,\ldots,\theta_{lk}^1);(\theta_{l1}^2,\ldots,\theta_{lk}^2) \in R$.

For each $t = 1,\ldots,k,$ set $(U_1^t \cup U_2^t) = diag_B(e^{i\theta_{1t}^1},\ldots,e^{i\theta_{nt}^1}) \cup diag_B(e^{i\theta_{1t}^2},\ldots,e^{i\theta_{nt}^2})$.
Then $(U_1^t \cup U_2^t) \in C_{n\times n}$ is unitary bimatrix and $\sum_{t=1}^{k}(U_1^t \cup U_2^t) = (\Sigma_1 \cup \Sigma_2)$. Hence, $(A_1 \cup A_2)$ can be written as a sum of k unitary bimatrices.

Suppose that k=1. If $\sigma_n^1 = 1; \sigma_n^2 = 1$, Then $(\Sigma_1 \cup \Sigma_2) = (I_1 \cup I_2)$ and $(A_1 \cup A_2)$ is unitary bimatrix. If $\sigma_n^1 \ne 1;\ \sigma_n^2 \ne 1$, then for each l, we have $\sigma_l^1,\sigma_l^2 \in (A_1^{II} \cup A_2^{II})$, and $(A_1 \cup A_2)$ can be written as a sum of two unitary bimatrices.

Theorem 3.3

Let $(A_1 \cup A_2) \in C_{n\times n}$ be given. Let k be the least (positive) integer so that there exist $(U_1^1 \cup U_2^1),\ (U_1^2 \cup U_2^2),\ldots,(U_1^k \cup U_2^k) \in \mathcal{U}_{n\times n}$ satisfying $(U_1^1 \cup U_2^1) + \ldots + (U_1^k \cup U_2^k) = (A_1 \cup A_2)$.
1. If $A_1 \cup A_2$ is unitary bimatrix, then k=1.
2. If $A_1 \cup A_2$ is not unitary bimatrix and $\sigma_1^1(A_1) \cup \sigma_1^2(A_2) \le 2$ then k=2.
3. If $m \ge 2$ is an integer such that $m < \sigma_1^1(A_1) \cup \sigma_1^2(A_2) \le m+1$ then k=m+1.

For positive integers $m \ge k,$ we have $(A_1^k \cup A_2^k) \subseteq (A_1^m \cup A_2^m)$. Hence, every $(U_1 \cup U_2) \in \mathcal{U}_{n\times n}$ can be written as a sum of two or more elements of $\mathcal{U}_{n\times n}$.
It follows that every $(A_1 \cup A_2) \in C_{n\times n}$ that can be written as a sum of k elements of $\mathcal{U}_{n\times n}$ can be written as a sum of m elements of $\mathcal{U}_{n\times n}$.

II. The case $O_{n\times n}$

Let n=2. Let $\alpha_1,\alpha_2,\beta_1,\beta_2 \in C$ be given.

Set $$(A_1(\alpha_1,\beta_1) \cup A_2(\alpha_2,\beta_2)) \equiv \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \cup \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix} \quad (2)$$

Choose β_1,β_2 such that $\alpha_1^2 + \beta_1^2 = 1; \alpha_2^2 + \beta_2^2 = 1$

and notice that

$$\left(A_1(\pm\alpha_1,\pm\beta_1)\cup A_2(\pm\alpha_2,\pm\beta_2)\right)\in O_{2\times 2}$$

Set $\left(A_1^I\cup A_2^I\right)\equiv\left(A_1(\alpha_1,\beta_1)\cup A_2(\alpha_2,\beta_2)\right)$ and

$$\left(A_1^{II}\cup A_2^{II}\right)\equiv\left(A_1(\alpha_1,-\beta_1)\cup A_2(\alpha_2,-\beta_2)\right)$$

Then

$$\left(A_1^I\cup A_2^I\right)+\left(A_1^{II}\cup A_2^{II}\right)=2\left[\alpha_1 I_1^{II}\cup\alpha_2 I_2^{II}\right].$$

Lemma 2.7 guarantees that every $\left(A_1\cup A_2\right)\in C_{2\times 2}$ can be written as a sum of bimatrices from $O_{2\times 2}$.

We look at the case when $n=3$. Let $\alpha_1,\alpha_2,\beta_1,\beta_2\in F$ be given.

Set

$$\left[B_1(\alpha_1,\beta_1)\cup B_2(\alpha_2,\beta_2)\right]\equiv\begin{bmatrix}1&0&0\\0&\alpha_1&\beta_1\\0&-\beta_1&\alpha_1\end{bmatrix}\cup\begin{bmatrix}1&0&0\\0&\alpha_2&\beta_2\\0&-\beta_2&\alpha_2\end{bmatrix}, \quad (3)$$

set

$$\left[C_1(\alpha_1,\beta_1)\cup C_2(\alpha_2,\beta_2)\right]\equiv\begin{bmatrix}\alpha_1&0&\beta_1\\0&1&0\\-\beta_1&0&\alpha_1\end{bmatrix}\cup\begin{bmatrix}\alpha_2&0&\beta_2\\0&1&0\\-\beta_2&0&\alpha_2\end{bmatrix} \quad (4)$$

and set

$$\left[D_1(\alpha_1,\beta_1)\cup D_2(\alpha_2,\beta_2)\right]\equiv\begin{bmatrix}\alpha_1&\beta_1&0\\-\beta_1&\alpha_1&0\\0&0&1\end{bmatrix}\cup\begin{bmatrix}\alpha_2&\beta_2&0\\-\beta_2&\alpha_2&0\\0&0&1\end{bmatrix} \quad (5)$$

Choose β_1,β_2 so that $\alpha_1^2+\beta_1^2=1;\ \alpha_2^2+\beta_2^2=1$.

Then $\left[B_1(\pm\alpha_1,\pm\beta_1)\cup B_2(\pm\alpha_2,\pm\beta_2)\right]$, $\left[C_1(\pm\alpha_1,\pm\beta_1)\cup C_2(\pm\alpha_2,\pm\beta_2)\right]$, and $\left[D_1(\pm\alpha_1,\pm\beta_1)\cup D_2(\pm\alpha_2,\pm\beta_2)\right]$, are all elements of $O_{3\times 3}$

Set

$$\left(B_1^I\cup B_2^I\right)\equiv\left[B_1(\alpha_1,\beta_1)\cup B_2(\alpha_2,\beta_2)\right],$$

$$\left(B_1^{II}\cup B_2^{II}\right)\equiv\left[B_1(-\alpha_1,\beta_1)\cup B_2(-\alpha_2,\beta_2)\right],$$

$$\left(C_1^I\cup C_2^I\right)\equiv\left[C_1(\alpha_1,\beta_1)\cup C_2(\alpha_2,\beta_2)\right],$$

$$\left(C_1^{II}\cup C_2^{II}\right)\equiv\left[C_1(-\alpha_1,\beta_1)\cup C_2(-\alpha_2,\beta_2)\right],$$

$$\left(D_1^I\cup D_2^I\right)\equiv\left[D_1(\alpha_1,\beta_1)\cup D_2(\alpha_2,\beta_2)\right],$$

And

$$\left(D_1^{II}\cup D_2^{II}\right)\equiv\left[D_1(-\alpha_1,\beta_1)\cup D_2(-\alpha_2,\beta_2)\right].$$

Then,

$$\left(B_1^I\cup B_2^I\right)-\left(B_1^{II}\cup B_2^{II}\right)+\left(C_1^I\cup C_2^I\right)-\left(C_1^{II}\cup C_2^{II}\right)$$

$$+\left(D_1^I\cup D_2^I\right)-\left(D_1^{II}\cup D_2^{II}\right)=2\left[\alpha_1 I_1^{III}\cup\alpha_2 I_2^{III}\right]$$

Lemma 2.7 now guarantees that every $\left(A_1\cup A_2\right)\in C_{3\times 3}$ can be written as a sum of bimatrices in $O_{3\times 3}$.

Let $n=2m$ be a positive even integer, and let $\delta_1,\delta_2\in C$ be given. Choose $\left(A_1^I\cup A_2^I\right),\left(A_1^{II}\cup A_2^{II}\right)\in O_{2\times 2}$ so that

$$\left[\left(A_1^I\cup A_2^I\right)+\left(A_1^{II}\cup A_2^{II}\right)\right]=\left[\delta_1 I_1^{II}\cup\delta_2 I_2^{II}\right].$$

Set $\left(E_1^I\cup E_2^I\right)=\left(A_1^I\cup A_2^I\right)\oplus\ldots\oplus\left(A_1^I\cup A_2^I\right)(m\ copies)$

and set $\left(E_1^{II}\cup E_2^{II}\right)=\left(A_1^{II}\cup A_2^{II}\right)\oplus\ldots\oplus\left(A_1^{II}\cup A_2^{II}\right)(m\ copies)$.

Then $(E_1^I \cup E_2^I),(E_1^{II} \cup E_2^{II}) \in O_{2m\times 2m}$, and $\left[(E_1^I + E_1^{II}) \cup (E_2^I + E_2^{II})\right] = (\delta_1 I_1^{2m} \cup \delta_2 I_2^{2m})$

Let $n = 2m+1 \geq 3$ be an odd integer, and let $\delta_1, \delta_2 \in C$ be given. Choose $(A_1^I \cup A_2^I),(A_1^{II} \cup A_2^{II}) \in O_{2\times 2}$ so that $\left((A_1^I + A_1^{II}) \cup (A_2^I + A_2^{II})\right) = (\delta_1 I_1^{II} \cup \delta_2 I_2^{II})$.

Also, choose

$$(B_1^I \cup B_2^I),(B_1^{II} \cup B_2^{II}),(C_1^I \cup C_2^I),(C_1^{II} \cup C_2^{II}),(D_1^I \cup D_2^I),(D_1^{II} \cup D_2^{II}) \in O_{3\times 3}$$

Such that

$$\left[(B_1^I - B_1^{II}) \cup (B_2^I - B_2^{II})\right] + \left[(C_1^I - C_1^{II}) \cup (C_2^I - C_2^{II})\right] +$$

$$\left[(D_1^I - D_1^{II}) \cup (D_2^I - D_2^{II})\right] = (\delta_1 I_1^{III} + \delta_2 I_2^{III})$$

Set

$(E_1^I \cup E_2^I) = (A_1^I \cup A_2^I) \oplus \ldots \oplus (A_1^I \cup A_2^I) \oplus (B_1^I \cup B_2^I)$ $\left[m-1 \text{ copies of } (A_1^I \cup A_2^I)\right]$,

Set

$(E_1^{II} \cup E_2^{II}) = (A_1^{II} \cup A_2^{II}) \oplus \ldots \oplus (A_1^{II} \cup A_2^{II}) \oplus (-B_1^{II} \cup -B_2^{II})$ $\left[m-1 \text{ copies of } (A_1^{II} \cup A_2^{II})\right]$,

Set

$(E_1^{III} \cup E_2^{III}) = (I_1^{2m-2} \cup I_2^{2m-2}) \oplus (C_1^I \cup C_2^I)$,

Set

$(E_1^{IV} \cup E_2^{IV}) = -(I_1^{2m-2} \cup I_2^{2m-2}) \oplus -(C_1^{II} \cup C_2^{II})$,

Set

$(E_1^{V} \cup E_2^{V}) = (I_1^{2m-2} \cup I_2^{2m-2}) \oplus (D_1^I \cup D_2^I)$,

and set

$(E_1^{VI} \cup E_2^{VI}) = -(I_1^{2m-2} \cup I_2^{2m-2}) \oplus -(D_1^{II} \cup D_2^{II})$.

Then each $(E_1^j \cup E_2^j) \in O_{2m+1}$, and

$$(E_1^I \cup E_2^I) + \ldots + (E_1^{VI} \cup E_2^{VI}) = (\delta_1 I_1^{2m+1} \cup \delta_2 I_2^{2m+1}).$$

Hence, for every $\alpha_1, \alpha_2 \in C$ and for every integer $n \geq 2, (\alpha_1 I_1 \cup \alpha_2 I_2)$ can be written as a sum of bimatrices from $O_{n\times n}$. Lemma 3.2 guarantees that every $(A_1 \cup A_2) \in C_{n\times n}$ can be written as a sum of bimatrices from $O_{n\times n}$.

Theorem 3.4

Let $n \geq 2$ be a given integer. Then every $(A_1 \cup A_2) \in C_{n\times n}$ can be written as a sum of bimatrices from $O_{n\times n}$.

Proof

Suppose that $(A_1 \cup A_2) = \left[(Q_1^I + Q_1^{II}) \cup (Q_2^I + Q_2^{II})\right]$, where $(Q_1^I \cup Q_2^I),(Q_1^{II} \cup Q_2^{II}) \in O_{n\times n}$.

Then one checks that $(A_1 A_1^T \cup A_2 A_2^T) = (Q_1^I A_1^T \cup Q_2^I A_2^T)(A_1 Q_1^{I^T} \cup A_2 Q_2^{I^T})$, so that $(A_1 A_1^T \cup A_2 A_2^T)$ and $(A_1^T A_1 \cup A_2^T A_2)$ are similar. Theorem 13 of **[3]** ensures that $(A_1 \cup A_2) + (Q_1 S_1 \cup Q_2 S_2)$, where $(Q_1 \cup Q_2)$ is orthogonal bimatrix and $(S_1 \cup S_2)$ is symmetric bimatrix (or that $(A_1 \cup A_2)$ has a $(Q_1 S_1 \cup Q_2 S_2)$ bidecomposition).

Suppose now that has a $(Q_1 S_1 \cup Q_2 S_2)$ bidecomposition. Is it true that $(A_1 \cup A_2)$ can be written as a sum of two (complex) orthogonal bimatrices?

Take the case $n=1$, and notice that every $(A_1 \cup A_2) \in C_{n\times n}$ is a scalar and has a $(Q_1 S_1 \cup Q_2 S_2)$ bidecomposition.

However, only the integers can be written as a sum of orthogonal bimatrices in this case.

Lemma 3.5

Let an integer $n \geq 2$ and $0 \neq \alpha_1; 0 \neq \alpha_2 \in \square$ be given. If $(\alpha_1 I_1 \cup \alpha_2 I_2) = (Q_1 \cup Q_2) + (V_1 \cup V_2)$

is a sum of two bimatrices from $O_{n\times n}$, then there exists a skew-symmetric bimatrix $\left(D_1 \cup D_2\right) \in C_{n\times n}$ such that

$$\left(Q_1 \cup Q_2\right) = \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) + \left(D_1 \cup D_2\right),$$

$$\left(V_1 \cup V_2\right) = \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) - \left(D_1 \cup D_2\right),$$

and

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right].$$

Conversely, if there exists a skew-symmetric bimatrix $\left(D_1 \cup D_2\right) \in C_{n\times n}$ such that $\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right]$, then $\left(Q_1 \cup Q_2\right) \equiv \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) + \left(D_1 \cup D_2\right)$ and $\left(V_1 \cup V_2\right) \equiv \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) - \left(D_1 \cup D_2\right)$ are in $O_{n\times n}$ and $\left(Q_1 \cup Q_2\right) + \left(V_1 \cup V_2\right) = \left(\alpha_1 I_1 \cup \alpha_2 I_2\right)$

Proof

Let an integer $n \geq 2$ and $0 \neq \alpha_1; 0 \neq \alpha_2 \in$ □ be given. Suppose that $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) \in C_{n\times n}$ can be written as a sum of two orthogonal bimatrices, say $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) = \left(Q_1 \cup Q_2\right) + \left(V_1 \cup V_2\right)$.

Write

$$\left(Q_1 \cup Q_2\right) = \left[a_{ij}^1 \cup a_{ij}^2\right] = \left[q_1^1 \ldots q_n^1\right] \cup \left[q_1^2 \ldots q_n^2\right]$$

and

$$\left(V_1 \cup V_2\right) = \left[b_{ij}^1 \cup b_{ij}^2\right] = \left[v_1^1 \ldots v_n^1\right] \cup \left[v_1^2 \ldots v_n^2\right].$$

Then $b_{ij}^1 = -a_{ij}^1$ and $b_{ij}^2 = -a_{ij}^2$ for $i \neq j$.

Now, for each $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} a_{ij}^{1^2} = q_i^{1^T} q_i^1 = 1 = v_i^{1^T} v_i^1 = \sum_{j=1}^{n} b_{ij}^{1^2} = b_{ii}^{1^2} + \sum_{j\neq i, j=1}^{n} a_{ij}^{1^2}$$

and

$$\sum_{j=1}^{n} a_{ij}^{2^2} = q_i^{2^T} q_i^2 = 1 = v_i^{2^T} v_i^2 = \sum_{j=1}^{n} b_{ij}^{2^2} = b_{ii}^{2^2} + \sum_{j\neq i, j=1}^{n} a_{ij}^{2^2}$$

Hence, $b_{ii}^1 = \pm a_{ii}^1$ and $b_{ii}^2 = \pm a_{ii}^2$.

Because $\left(Q_1 \cup Q_2\right) + \left(V_1 \cup V_2\right) = \left(\alpha_1 I_1 \cup \alpha_2 I_2\right)$ and $\alpha_1 \neq 0; \alpha_2 \neq 0$ we have $b_{ii}^1 = a_{ii}^1 = \frac{\alpha_1}{2}; b_{ii}^2 = a_{ii}^2 = \frac{\alpha_2}{2}$. Set $\left(D_1 \cup D_2\right) = \left[d_{ij}^1\right] \cup \left[d_{ij}^2\right]$, with $d_{ij}^1 = a_{ij}^1; d_{ij}^2 = a_{ij}^2$ if $i \neq j$, and $d_{ii}^1 = 0;\ d_{ii}^2 = 0$, so that $\left(Q_1 \cup Q_2\right) = \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) + \left(D_1 \cup D_2\right)$ and $\left(V_1 \cup V_2\right) = \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) - \left(D_1 \cup D_2\right)$.

Now, since $\left(Q_1 \cup Q_2\right)$ and $\left(V_1 \cup V_2\right)$ are orthogonal bimatrices, we have

$$\left(Q_1 Q_1^T \cup Q_2 Q_2^T\right) = \left(\frac{\alpha_1^2}{4} I_1 \cup \frac{\alpha_2^2}{4} I_2\right) + \left[\frac{\alpha_1}{2}\left(D_1 + D_1^T\right) \cup \frac{\alpha_2}{2}\left(D_2 + D_2^T\right)\right] + \left(D_1 D_1^T \cup D_2 D_2^T\right) = \left(I_1 \cup I_2\right) \tag{6}$$

And

$$\left(V_1 V_1^T \cup V_2 V_2^T\right) = \left[\frac{\alpha_1^2}{4} I_1 \cup \frac{\alpha_2^2}{4} I_2\right] - \left[\frac{\alpha_1}{2}\left(D_1 + D_1^T\right) \cup \frac{\alpha_2}{2}\left(D_2 + D_2^T\right)\right] + \left(D_1 D_1^T \cup D_2 D_2^T\right) = \left(I_1 \cup I_2\right) \tag{7}$$

Subtracting equation (7) from equation (6), we get $\left(D_1 \cup D_2\right) = -\left(D_1^T \cup D_2^T\right)$, so that $\left(D_1 \cup D_2\right)$ is skew-symmetric bimatrix.

Moreover

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right].$$

For the converse, suppose that $\left(D_1 \cup D_2\right) \in C_{n\times n}$ is skew-symmetric bimatrix and satisfies,

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right].$$

Set $\left(Q_1 \cup Q_2\right) \equiv \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) + \left(D_1 \cup D_2\right)$

and set

$$\left(V_1 \cup V_2\right) \equiv \left(\frac{\alpha_1}{2} I_1 \cup \frac{\alpha_2}{2} I_2\right) - \left(D_1 \cup D_2\right)$$

Then one checks that both $\left(Q_1 \cup Q_2\right)$ and $\left(V_1 \cup V_2\right)$ are orthogonal bimatrices and $\left[\left(Q_1 + V_1\right) \cup \left(Q_2 + V_2\right)\right] = \left(\alpha_1 I_1 \cup \alpha_2 I_2\right).$

Remark 3.6

When $\alpha_1 = 0;\ \alpha_2 = 0,$ then for any orthogonal bimatrix $\left(Q_1 \cup Q_2\right)$, notice that $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) = \left[\left(Q_1 + \left(-Q_1\right)\right) \cup \left(Q_2 + \left(-Q_2\right)\right)\right]$ is a sum of two orthogonal bimatrices.

Let n=2 and $\alpha_1 \neq 0;\ \alpha_2 \neq 0$. Set $\beta_1 \equiv \sqrt{1 - \frac{\alpha_1^2}{4}};\ \beta_2 \equiv \sqrt{1 - \frac{\alpha_2^2}{4}}$ (either square root).

Then $\left(D_1 \cup D_2\right) \equiv \begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & \beta_2 \\ -\beta_2 & 0 \end{bmatrix}$ is skew-symmetric bimatrix and satisfies

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right]$$

.

Lemma 3.5 guarantees that $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right)$ can be written as a sum of two orthogonal bimatrices.

If $n=2k$ and $\alpha_1 \neq 0;\ \alpha_2 \neq 0,$ set $\left(E_1 \cup E_2\right) = \left(D_1 \cup D_2\right) \oplus \ldots \oplus \left(D_1 \cup D_2\right)$ (k copies) and notice that $\left(E_1 \cup E_2\right)$ is skew-symmetric bimatrix and satisfies

$$\left(E_1 E_1^T \cup E_2 E_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right]$$

.

Hence, if $n=2k$ and if $\alpha_1,\ \alpha_2$ be a scalar, then $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right)$ can be written as a sum of two orthogonal bimatrices.

Theorem 3.7

Let n be a given positive integer. For each $\alpha_1, \alpha_2 \in \square$ and each orthogonal bimatrix $\left(Q_1 \cup Q_2\right) \in C_{2n},\ \left(\alpha_1 Q_1 \cup \alpha_2 Q_2\right)$ can be written as a sum of two orthogonal bimatrices.

Remark 3.8

Let an integer $n \geq 2$ be given. If $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$ then one checks that $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) \in C_{n \times n}$ can be written as a sum of two orthogonal bimatrices.

Theorem 3.9

Let $\alpha_1, \alpha_2 \in \square$ and let a positive integer n be given. Then $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) \in C_{2n+1}$ can be written as a sum of two bimatrices from $O_{n \times n}$ if and only if $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$.

Proof

For the forward implication, let $\alpha_1, \alpha_2 \in \square$ and let a positive integer n be given. Suppose that $\left(\alpha_1 I_1 \cup \alpha_2 I_2\right) \in C_{2n+1}$ can be written as a sum of two orthogonal bimatrices. Then $\alpha_1 = 0;\ \alpha_2 \neq 0$ (or) $\alpha_1 \neq 0;\ \alpha_2 \neq 0.$ If $\alpha_1 = 0;\ \alpha_2 = 0,$ then $\alpha_1, \alpha_2 \in \{-2, 0, 2\}$. If $\alpha_1 \neq 0;\ \alpha_2 \neq 0,$ we show that $\alpha_1 = \alpha_2 = \pm 2$. Lemma 3.5 guarantees that there exists a skew-symmetric bimatrix $\left(D_1 \cup D_2\right) \in C_{n \times n}$ satisfying

$$\left(D_1 D_1^T \cup D_2 D_2^T\right) = \left[\left(1 - \frac{\alpha_1^2}{4}\right) I_1 \cup \left(1 - \frac{\alpha_2^2}{4}\right) I_2\right].$$

Now, since n is odd, the skew-symmetric bimatrix $\left(D_1 \cup D_2\right)$ is singular. Hence, $\left(D_1 D_1^T \cup D_2 D_2^T\right)$ is singular and $\alpha_1 = \alpha_2 = \pm 2.$

The backward implication can be show by direct computation.

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