(k,d)–Mean Labeling of Some Family of Trees

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Abstract: ***Mean labeling of graphs was discussed in [24-25] and the concept of odd mean labeling was introduced in [22]. k–odd mean labeling and (k,d)–odd mean labeling are introduced and discussed in [1,6-8]. k–mean, k–even mean and (k,d)–even mean labeling are introduced and discussed in [9-17]. In this paper, we introduce (k,d)–mean labeling and we have obtained results for some family of trees.***

Keywords: (k,d)–mean labeling, (k,d)–mean graph

1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [20]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [23]. Labeled graphs serve as useful models for a broad range of applications such as X-ray, crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly interesting applications of graph labeling can be found in [2-5].

Mean labeling of graphs was discussed in [24,25]. Vaidya and et al. [28-31] have investigated several new families of mean graphs. Nagarajan and et al. [27] have found some new results on mean graphs.

Ponraj, Jayanthi and Ramya extended the notion of mean labeling to super mean labeling in [21]. Gayathri and Tamilselvi [18-19,26] extended super mean labeling to k–super mean, (k,d)–super mean, k–super edge mean and (k,d)–super edge mean labeling. Manickam and Marudai [22] introduced the concept of odd mean graphs. Gayathri and Amuthavalli [1,6-8] extended this concept to k–odd mean and (k,d)–odd mean graphs. Gayathri and Gopi [9-17] extended this concept to k–mean, k–even mean and (k,d)–even mean graphs.

In this paper, we extend k–mean graphs to (k,d)–mean graphs since there are graphs which are (k,d)–mean for all $k \geq 2$ and $d \geq 2$ but not $(k,1)$–mean for any $k \geq 1$. Here, we have found (k,d)–mean labeling of some family of trees.
Throughout this paper, k and d denote any positive integer greater than or equal to 1.

For brevity, we use (k,d)–ML for (k,d)–mean labeling and (k,d)–MG for (k,d)–mean graph.

2. Main Results

Definition 2.1

A (p,q) graph G is said to have a **mean labeling** if there is an injective function f from the vertices of G to $\{0,1,2,\ldots,q\}$ such that the induced map f^* defined on E by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection from E to $\{1,2,\ldots,q\}$.

A graph that admits a mean labeling is called a **mean graph**.

Definition 2.2

A (p,q) graph G is said to have a ***k*–mean labeling** if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, k+q-1\}$ such that the induced map f^* defined on E by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection from E to $\{k, k+1, k+2, \ldots, k+q-1\}$.

A graph that admits a k–mean labeling is called a ***k*–mean graph**.

Observation 2.3

Every 1–mean labeling is a mean labeling.

Definition 2.4

A (p,q) graph G is said to have a **(k,d)–mean labeling** if there exists an injective function f from the vertices of G to $\{0,1,2,\ldots,k+(q-1)d\}$ such that the induced map f^* defined on E by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection from E to $\{k, k+d, k+2d, \ldots, k+(q-1)d\}$.

A graph that admits a (k,d)–mean labeling is called a **(k,d)–mean graph**.

Observation 2.5

1) Every $(k,1)$–mean labeling is a k–mean labeling
2) Every $(1,1)$–mean labeling is a mean labeling.

Theorem 2.6

The path graph P_n is a (k,d)–mean graph for all k and d, when n is even.

Proof

Let $V(P_n)=\{v_1, v_2, \ldots, v_n\}$ and

$E(P_n)=\{v_iv_{i+1}, 1\le i\le n-1\}$ be denoted as in the Figure 2.1

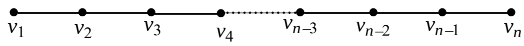

Figure 2.1: Ordinary labeling of P_n

First we label the vertices as follows:

Define $f: V\to\{0, 1, 2, \ldots, k+(q-1)d\}$ by

$$f(v_i) = k + d(i-1) - 1, \text{if } i \text{ is odd}$$
$$f(v_i) = k + d(i-2), \quad \text{if } i \text{ is even}$$

Then the induced edge labels are

$$f^*(v_iv_{i+1}) = k + d(i-1), \quad \text{for } 1 \le i \le n-1.$$

The above defined function f provides (k,d)–mean labeling of the graph.

So, the path graph P_n is a (k,d)–mean graph for all k and d, when n is even.

(k,d) mean labeling of P_n for different cases of k and d when n is even are shown in illustration 2.7.

Illustration 2.7

(100, 20)–mean labeling of the graph P_{10} is shown in Figure 2.2

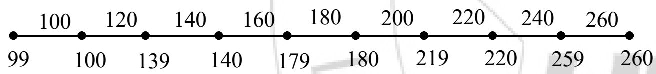

Figure 2.2: (100, 20)–*ML* of P_{10}

(55, 44)–mean labeling of the graph P_6 is shown in Figure 2.3

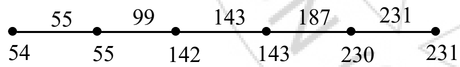

Figure 2.3: (55, 44)–*ML* of P_6

(33, 11)–mean labeling of the graph P_8 is shown in Figure 2.4

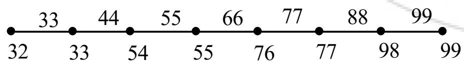

Figure 2.4: (33, 11)–*ML* of P_8

(60, 13)–mean labeling of the graph P_{10} is shown in Figure 2.5

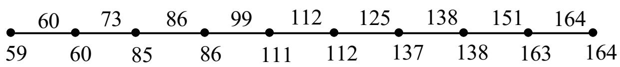

Figure 2.5: (60, 13)–*ML* of P_{10}

Definition 2.8

A **comb graph** P_n^+ is a tree obtained from a path by attaching exactly one pendant edge to each vertex of the path.

Theorem 2.9

The comb graph P_n^+ is a (k,d)–mean graph for all k and d.

Proof

Let $V(P_n^+) = \{u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n'\}$ and $E(P_n^+) = \{u_iu_i', 1\le i\le n \text{ and } u_iu_{i+1}, 1\le i\le n-1\}$ be denoted as in the Figure 2.6.

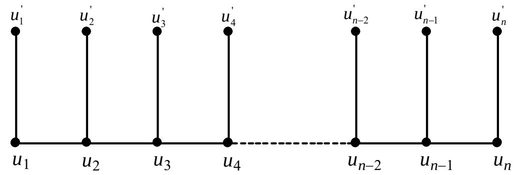

Figure 2.6: Ordinary labeling of P_n^+

First we label the vertices as follows:

Define $f: V\to\{0, 1, 2, \ldots, k+(q-1)d\}$ by

$$f(u_1) = k$$
$$f(u_1') = k-1$$
$$f(u_n) = k+(q-1)d$$
$$f(u_n') = k+(q-1)d-1$$
$$f(u_i) = k+2d(i-1), \quad \text{for } 2\le i\le n-1$$
$$f(u_i') = k+2d(i-1)-1, \text{for } 2\le i\le n-1$$

Then the induced edge labels are

$$f^*(u_iu_i') = k+2d(i-1), \quad \text{for } 1\le i\le n$$
$$f^*(u_iu_{i+1}) = k+d(2i-1), \quad \text{for } 1\le i\le n-1$$

The above defined function f provides (k,d)–mean labeling of the graph. So, the graph P_n^+ is a (k,d)–mean graph for all k and d. (k,d)–mean labeling of P_n^+ for different cases of k and d are shown in illustration 2.10

Illustration 2.10

(4,6)–mean labeling of the graph P_5^+ is shown in Figure 2.7

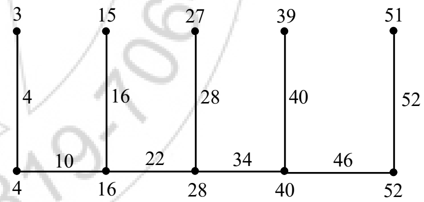

Figure 2.7: (4,6)–*ML* of P_5^+

(13,3)–mean labeling of the graph P_5^+ is shown in Figure 2.8

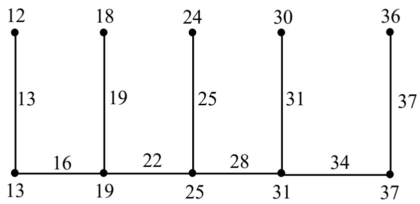

Figure 2.8: (13,3)–*ML* of P_5^+

(23,8)–mean labeling of the graph P_8^+ is shown in Figure 2.9

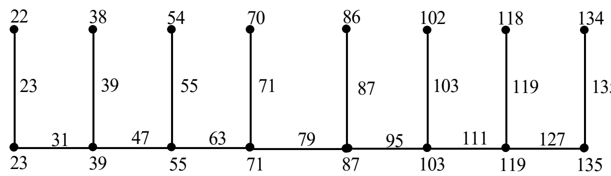

Figure 2.9: (23,8)–*ML* of P_8^+

(34,9)–mean labeling of the graph P_7^+ is shown in Figure 2.10

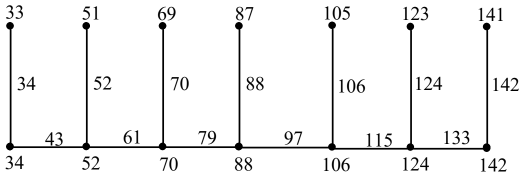

Figure 2.10: (34,9)–*ML* of P_7^+

Definition 2.11

A **twig** is a tree obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.

Theorem 2.12

The twig graph T_n is a (k,d)–mean graph for all k and d, when n is even.

Proof

Let $V[T_n] = \{v_i, 0 \le i \le n-1, u_i, w_i, 1 \le i \le n-2\}$ and $E(T_n) = \{v_iu_i, v_iw_i, 1 \le i \le n-2$ and $v_iv_{i+1}, 0 \le i \le n-2\}$ be denoted as in the Figure 2.11.

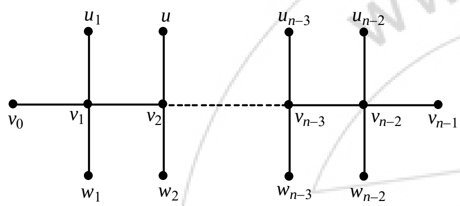

Figure 2.11: Ordinary labeling of T_n

First we label the vertices as follows:

Define $f : V \to \{0, 1, 2, \ldots, k+(q-1)d\}$ by

$$f(v_i) = \begin{cases} k+3di-1, & \text{if } i \text{ is even} \\ k+3d(i-1), & \text{if } i \text{ is odd} \end{cases}$$

$$f(u_i) = \begin{cases} k+d(3i-4)+1, & \text{if } i \text{ is even} \\ k+d(3i-1), & \text{if } i \text{ is odd} \end{cases}$$

$$f(w_i) = \begin{cases} k+d(3i-2)+1, & \text{if } i \text{ is even} \\ k+d(3i+1), & \text{if } i \text{ is odd} \end{cases}$$

Then the induced edge labels are

$f^*(v_iv_{i+1}) = k + 3di,$ for $0 \le i \le n-2$

$f^*(v_iu_i) = k + d(3i-2),$ for $1 \le i \le n-2$

$f^*(v_iw_i) = k + d(3i-1),$ for $1 \le i \le n-2$

The above defined function f provides (k,d)–mean labeling of the graph.

So, the twig graph T_n is a (k,d)–mean graph for all k and d, when n is even.

(k,d)–mean labeling of T_n for different cases of k and d when n is even are shown in illustration 2.13.

Illustration 2.13

(6,6)–mean labeling of the graph T_8 is shown in Figure 2.12

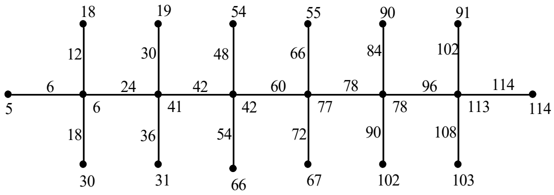

Figure 2.12: (6,6)–*ML* of T_8

(3,2)–mean labeling of the graph T_4 is shown in Figure 2.13

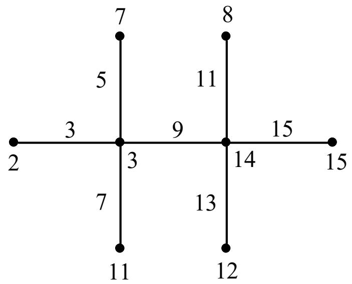

Figure 2.13: (3,2)–*ML* of T_4

(5,3)–mean labeling of the graph T_6 is shown in Figure 2.14

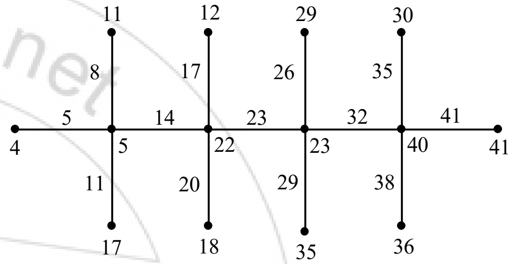

Figure 2.14: (5,3)–*ML* of T_6

(8,5)–mean labeling of the graph T_{10} is shown in Figure 2.15

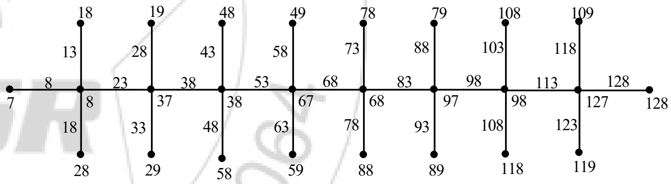

Figure 2.15: (8,5)–*ML* of T_{10}

Theorem 2.14

The star $K_{1,n}$ $(n \ge 4)$ is a (k,d)–mean graph for all $k \ge n-2$ and for all d satisfying $(q-1)d \le k+1$ except when n is odd and d is even.

Proof

Let $V(K_{1,n}) = \{u, v_1, v_2, \ldots, v_n\}$ and

$E(K_{1,n}) = \{uv_i, 1 \le i \le n\}$ be denoted as in Figure 2.16.

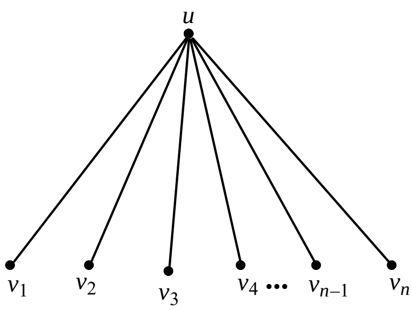

Figure 2.16: Ordinary labeling of $K_{1,n}$

First we label the vertices as follows:

Define $f : V \to \{0,\ 1,\ 2,..,k+(q-1)d\}$ by

Case (1): **when n is even**

Subcase (i): d is odd

$f(u) = k + (q-1)d - 1$

$f(v_i) = k - (q-1)d + 2i - 1,\quad$ for $1 \le i \le \frac{n}{2} - 1$

$f(v_i) = k,\quad$ for $i = \frac{n}{2}$

$f(v_i) = k + (q-1)d + (i-n) - 1,$ for $\frac{n}{2} + 1 \le i \le n-1$

$f(v_n) = k + (q-1)d$

Subcase (ii): d is even

$f(u) = k + (q-1)d - 1$

$f(v_i) = k - (q-1)d + 2i - 1,\quad$ for $1 \le i \le \frac{n}{2} - 1$

$f(v_i) = k - 1,\quad$ for $i = \frac{n}{2}$

$f(v_i) = k + (q-1)d + (i-n) - 2,$ for $\frac{n}{2} + 1 \le i \le n-1$

$f(v_n) = k + (q-1)d$

Then the induced edge labels are

$f^*(uv_i) = k + d(i-1),\quad$ for $1 \le i \le n$

Case (2): **when n is odd**

subcase (i): d is odd

$f(u) = k + (q-1)d - 1$

$f(v_i) = k - (q-1)d + 2i - 1,\quad$ for $1 \le i \le \frac{n-1}{2}$

$f(v_i) = k + d,\quad$ for $i = \frac{n+1}{2}$

$f(v_i) = k + (q-1)d + (i-n) - 1,$ for $\frac{n+3}{2} \le i \le n-1$

$f(v_n) = k + (q-1)d$

Then the induced edge labels are

$f^*(uv_i) = k + d(i-1),\quad$ for $1 \le i \le n$

The above defined function f provides (k,d)–mean labeling of the graph.

So, the star $K_{1,n}$ is a (k,d)–mean graph for all $k \ge n - 2$ and for all d satisfying $(q-1)d \le k + 1$ except when n is odd and d is even.

(k,d)–mean labeling of $K_{1,n}$ for different cases of k and d except when n is odd and d is even are shown in illustration 2.15.

Illustration 2.15

(6,1)–mean labeling of the graph $K_{1,8}$ is shown in Figure 2.17

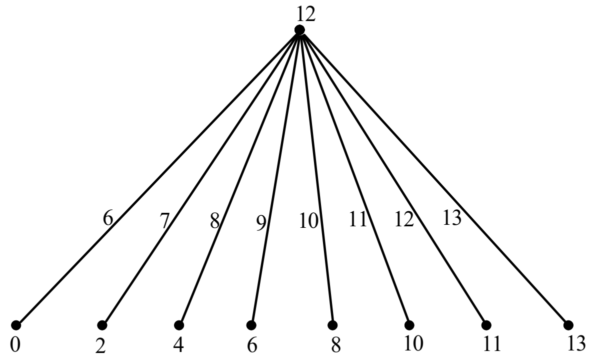

Figure 2.17: (6,1)–*ML* of $K_{1,8}$

(3,1)–mean labeling of the graph $K_{1,5}$ is shown in Figure 2.18

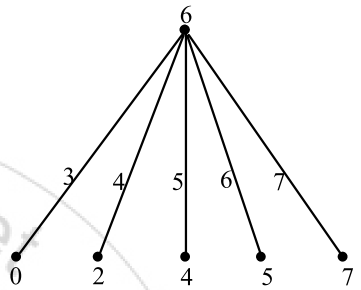

Figure 2.18: (3,1)–*ML* of $K_{1,5}$

(5,1)–mean labeling of the graph $K_{1,6}$ is shown in Figure 2.19

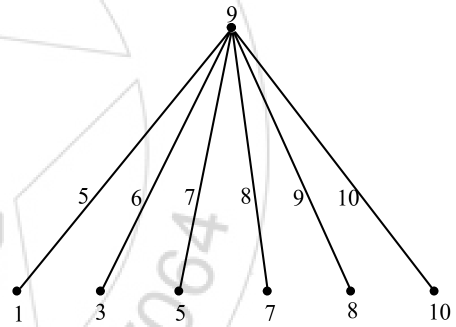

Figure 2.19: (5,1)–*ML* of $K_{1,6}$

(6,1)–mean labeling of the graph $K_{1,7}$ is shown in Figure 2.20

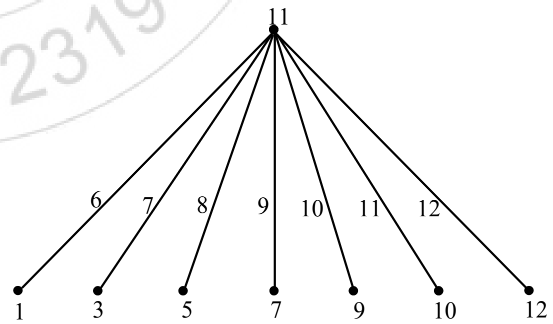

Figure 2.20: (6,1)–*ML* of $K_{1,7}$

Definition 2.16

A **bistar** $B_{m,n}$ is a tree obtained by joining the center vertices of the copies of $K_{1,m}$ and $K_{1,n}$ with an edge.

Theorem 2.17

The Bistar $B_{n,n}$ ($n \ge 2$) is a (k,d)–mean graph for all k and d.

Proof

Let $V(B_{n,n}) = \{u, v, u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$

and $E(B_{n,n})$ = {uv, uu_i, vv_i, $1 \le i \le n$}
be denoted as in the Figure 2.21.

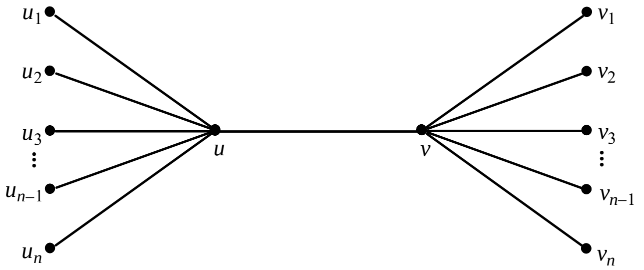

Figure 2.21: Ordinary labeling of $B_{n,n}$

First we label the vertices as follows:
Define $f: V \rightarrow \{0, 1, 2, \ldots, k+(q-1)d\}$ by

$f(u) = k + (q-1)d$
$f(u_i) = k + 2di - 1$, for $1 \le i \le n-1$
$f(u_n) = k + (q-1)d - 1$
$f(v) = k - 1$
$f(v_i) = k + 2d(i-1)$, for $1 \le i \le n$

Then the induced edge labels are
$f^*(vv_i) = k + d(i-1)$, for $1 \le i \le n$
$f^*(uv) = k + nd$
$f^*(uu_i) = k + (n+i)d$, for $1 \le i \le n$

The above defined function f provides (k,d)–mean labeling of the graph.
So, the graph $B_{n,n}$ is a (k,d)–mean graph for all k and d.
(k,d)–mean labeling of $B_{n,n}$ for different cases of k and d are shown in illustration 2.18.

Illustration 2.18
(6,7)–mean labeling of the graph $B_{7,7}$ is shown in Figure 2.22.

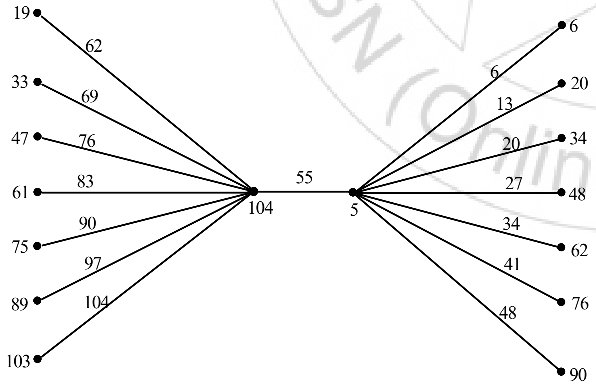

Figure 2.22: (6,7)–*ML* of $B_{7,7}$
(3,5)–mean labeling of the graph $B_{5,5}$ is shown in Figure 2.23

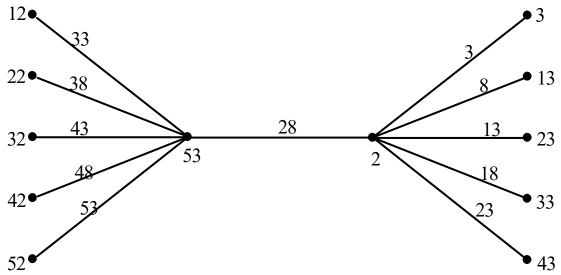

Figure 2.23: (3,5)–*ML* of $B_{5,5}$

(5,7)–mean labeling of the graph $B_{6,6}$ is shown in Figure 2.24

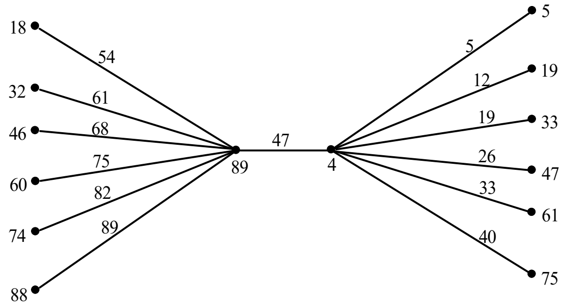

Figure 2.24: (5,7)–*ML* of $B_{6,6}$
(4,4)–mean labeling of the graph $B_{8,8}$ is shown in Figure 2.25

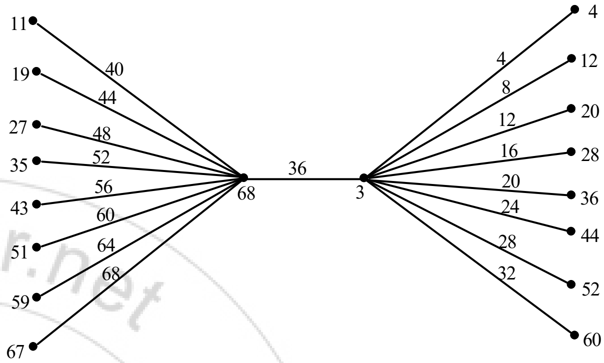

Figure 2.25: (4,4)–*ML* of $B_{8,8}$

Theorem 2.19
The Bistar $B_{n,n+1}$ $(n \ge 2)$ is a (k,d)–mean graph for all k and for all $d \le k+1$.

Proof
Let $V(B_{n,n+1})$ = {$u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n+1}$}
and $E(B_{n,n+1})$ = {uv, uu_i, $1 \le i \le n$, vv_i, $1 \le i \le n+1$}
be denoted as in the Figure 2.26.

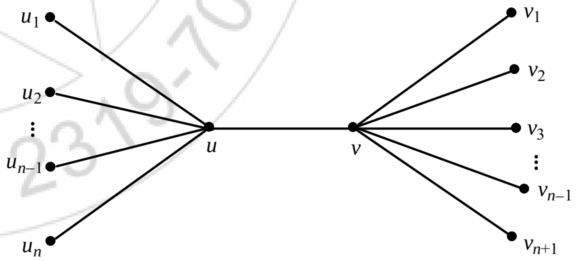

Figure 2.26: Ordinary labeling of $B_{n,n+1}$

First we label the vertices as follows:
Define $f: V \rightarrow \{0, 1, 2, \ldots, k+(q-1)d\}$ by

$f(u) = k - d + 1$
$f(u_1) = k + d - 2$
$f(u_i) = k + d(2i-1) - 1$, for $2 \le i \le n$
$f(v) = k + (q-1)d - 1$
$f(v_i) = k + d(2i-1)$, for $1 \le i \le n+1$

Then the induced edge labels are
$f^*(uu_i) = k + d(i-1)$, for $1 \le i \le n$
$f^*(uv) = k + nd$
$f^*(vv_i) = k + (n+i)d$, for $1 \le i \le n+1$

The above defined function f provides (k,d)–mean labeling of the graph.

So, the graph $B_{n,n+1}$ is a (k,d)–mean graph for all k and for all $d \le k + 1$.

(k,d)–mean labeling of $B_{n,n+1}$ for different cases of k and $d \le k + 1$ are shown in illustration 2.20.

Illustration 2.20

(4,5)–mean labeling of the graph $B_{5,6}$ is shown in Figure 2.27

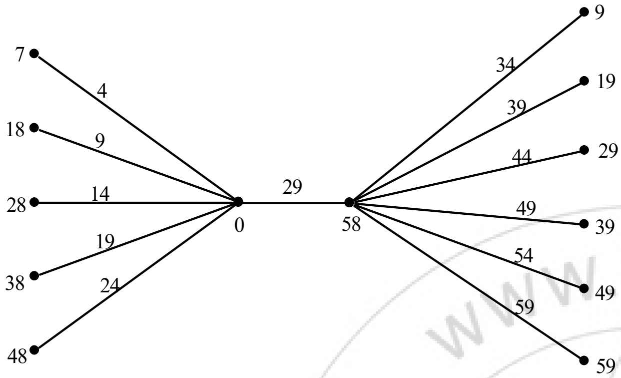

Figure 2.27: (4,5)–ML of $B_{5,6}$

(1,2)–mean labeling of the graph $B_{3,4}$ is shown in Figure 2.28

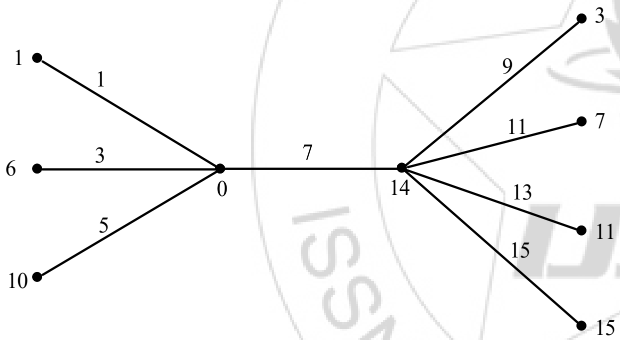

Figure 2.28: (1,2)–ML of $B_{3,4}$

(3,3)–mean labeling of the graph $B_{4,5}$ is shown in Figure 2.29

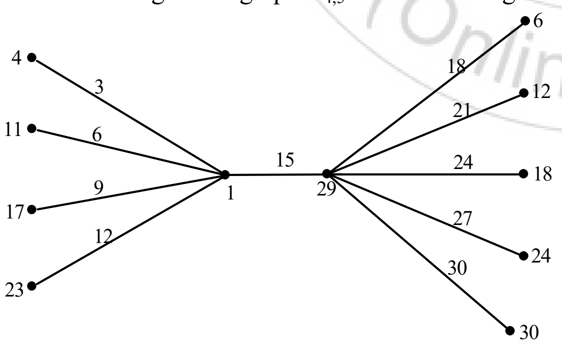

Figure 2.29: (3,3)–ML of $B_{4,5}$

(6,6)–mean labeling of the graph $B_{6,7}$ is shown in Figure 2.30

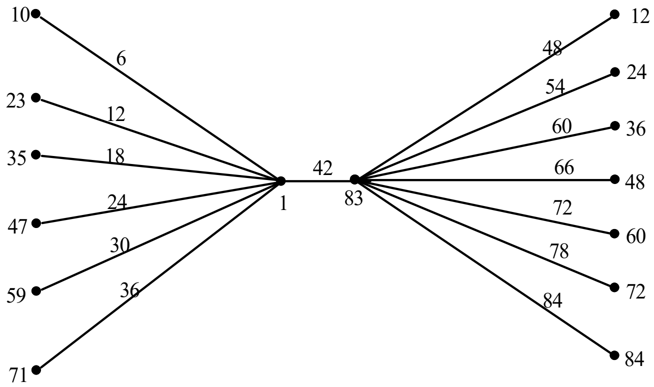

Figure 2.30: (6,6)–ML of $B_{6,7}$

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