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Coincidence and Fixed Point Theorem in 2-Metric Spaces

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Abstract: ***In this Paper, we have prove coincidence and fixed point theorem in 2-metric space. The result in this paper are extend, generalized the Parsai V. and Singh B.[8], Fisher [1], Pathak[10].***

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1. Introduction

The concept of 2-metric space has been investigated initially by Gähler in a series of papers and has been developed extensively by Gähler and many others. A 2-metric space is a set X with a real-valued function d on XxXxX satisfying the following conditions:

(M_1) For two distinct point x, y $\in$ X, there is a point z $\in$ X such that $d(x, y ,z) \neq 0$

(M_2) $d(x, y ,z) = 0$ if at least two of x, y, z are equal,

(M_3) $d(x, y ,z) = d(x, z, y) = d(y, z, x)\ \forall\ x, y, z \in X$,

(M_4) $d(x, y, z) \leq d(x, y, u) + d(x, u ,z) + d(u, y ,z)\ \forall\ x, y, z, u \in X$,

The function d is called a 2-metric for the space X and (X, d) denotes a 2-metric space. It has been shown by Gähler [18] that a 2-metric d is non-negative and although d is continuous function in any one of its three arguments, it need not be continuous in two arguments . A 2-metric d which is continuos in all of its arguments is said to be continuous. we use the concept of compatible mappings of type (P) in 2-metric spaces.

In the last three decades, a many authors have studied the aspects of fixed point theory in the setting of 2-metric spaces. They have been motivated by various concepts already known for metric space and have thus introduced analogous of various concepts in the framework of the 2-metric spaces.

Definitions [1]: A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point x $\in$ X, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n,x,z) = 0$ for all $z \in X$. The point x is said to be limit of sequence $\{x_n\}$ in X.

Definition [2]: A sequence $\{x_n\}$ in a 2-metric space (X,d) is called a Cauchy sequence if $d(x_m,x_n,z) \to 0$ as $n, m \to \infty$ for all $z \in X$.

Definition [3]: A 2-metric space in which every Cauchy sequence is convergent is called complete.

Definition [4]: A mapping S from a 2-metric space (X, d) into itself is said to be sequentially continuous at a point $x \in X$ if every sequence $\{ x_n\}$ in X such that $\lim_{n\to\infty} d(x_n,x,z) = 0$ for all $z \in X$, $\lim_{n\to\infty} d(Sx_n,Sx,z) = 0$.

Definition [5]: Let S and T be mappings from a 2-metric space (X,d) into itself. The mappings S and T are said to be compatible of type (P) if $\lim_{n\to\infty} d(SSx_n, TTx_n,z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X

Proposition [1]: Let S and T be sequentially continuous mappings of a 2-metric space (X, d) into itself. If S and T are compatible if and only if they are compatible of type (P).

Proof: Let $\{x_n\}$ be sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Suppose that the mappings S and T are compatible.

By (M_4), we have

$d(SSx_n,TTx_n,z) \leq d(SSx_n, TTx_n, STx_n) + d(SSx_n, STx_n,z) + d(STx_n, TTx_n,z)$

$\leq d(SSx_n, TTx_n, STx_n) + d(SSx_n, STx_n,z) + d(STx_n, TSx_n,z) + d(STx_n, TTx_n, TSx_n) + d(TSx_n, TTx_n,z).$

letting $n \to\infty$ since S and T are compatible and sequentially continuous, we have $\lim_{n\to\infty} d(SSx_n, TTx_n,z) = 0$ for all $z \in X$. Conversely, suppose that S and T are compatible of type(P). By (M_4), we have

$d(STx_n,TSx_n,z) \leq d(STx_n, TSx_n, SSx_n) + d(STx_n, SSx_n,z) + d(SSx_n, TSx_n,z)$

$\leq d(STx_n, TSx_n, SSx_n) + d(STx_n, SSx_n,z) + d(SSx_n, TTx_n,z) + d(SSx_n, TSx_n, TTx_n) + d(TTx_n, TSx_n,z).$

letting $n \to\infty$ since S and T are compatible of type(P) and sequentially continuous, we have $\lim_{n\to\infty} d(STx_n, TSx_n,z) = 0$ for all $z \in X$. This completes the proof.

Proposition [2]: Let S and T be compatible mappings of type(P) from a 2-metric space (X, d) into itself. If St = Tt for some t in X, Then STt = SSt = TTt = TSt.

Proof : Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, n = 1,2,3,... and St = Tt. Then we have $\lim_{n\to\infty} Sxn = \lim_{n\to\infty} Txn$ = St. Since S and T are compatible mappings of type (P), we have $d(SSt, TTt, z) = \lim_{n\to\infty} d(SSx_n, TTx_n, z) = 0$.

Hence we have SSt = TTt. Therefore, STt = SSt = TTt = TSt.

Proposition [3]: Let S and T be compatible mappings of type(P) from a 2-metric spaces (X,d) into itself. Suppose $\lim_{n\to\infty}Sx_n = \lim_{n\to\infty}Tx_n = t$ for some $t\in X$. Then we have the following.
(i) $\lim_{n\to\infty}TTx_n = St$ if S is sequentially continuous at t;
(ii) $\lim_{n\to\infty}SSx_n = Tt$ if T is sequentially continuous at t;
(iii) STt = TSt and St = Tt if S and T are Sequentially continuous at t.

Proof: (i) Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty}Tx_n = t$ for some $t\in X$. Since S is sequentially continuous we have $\lim_{n\to\infty}STx_n = St$. We have
$d(TTx_n,St,z) \le d(TTx_n, St, SSx_n) + d(TTx_n, SSx_n,z) + d(STx_n, St,z)$
Therefor, since S and T are compatible mappings of type(P), we have
$\lim_{n\to\infty} TTx_n = St$.
(ii) The proof of $\lim_{n\to\infty}SSx_n = Tt$ follows on the similar lines as argued in (i).
(iii) Since T is sequentially continuous at t,we have $TTx_n = Tt$. By (i) since S is sequentially continuous at t. we have also $\lim_{n\to\infty}TTx_n = St$. Hence by the uniqueness of the limit, we have St = Tt and so **PROPOSITION[2]** STt = TSt.
Let R^+ denote the set of all non-negative real numbers and F be the family of mappings $\phi : (R^+)^5 \to R^+$ such that each ϕ is upper-semi-continuous, non-decreasing in each coordinate variable, and for any $t > 0$, $\gamma(t) = \phi(t,t,a_1t,a_2t,t) < t$, where $\gamma : R^+ \to R^+$ is a mapping with $\gamma(0) = 0$ and $a_1+ a_2 = 3$.
We have prove the following theorems:

Theorem [1]: Let A, B, S and T be mappings from a complete 2-metric space (X, d) into itself , satisfying the following conditions:
[1.1] $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
[1.2] $S(X) \cap T(X)$ is a complete subspace of X.
[1.3] $[1+p\{d(Ax,Sx,z) + d(By,Ty,z)\}]\, d(Ax,By,z) \le p[d^2(Ax,Sx,z) +d^2(By,Ty,z)] + \phi(d(Sx,Ty,z), d(Ax,Sx,z), d(By,Ty,z), d(Ax,Ty,z), d(By,Sx,z))$
for all $x,y,z \in X$, where $\phi \in F$. Then the pairs A, S and B, T have a coincidence point in X.
For our theorems, we need the following **LEMMAS**:
LEMMA [1]: For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n-times composition of γ.
LEMMA [2] : Let A, B, S and T be mappings from a complete 2-metric space (X, d) into itself , satisfying the conditions [1.1], [1.3].

Then we have the following :
(a) For every $n \in N0$, $d(y_n,y_{n+1},y_{n+2}) = 0$,
(b) For every $i, j, k \in N0$, $d(y_i, y_j, y_k) = 0$, where $\{y_n\}$ is the sequence in X defined by [1.4].

Proof of the Lemma: (a) By(1.1) since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for any arbitrary point $x_1 \in X$, there exists a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that
[1.4] $y_{2n} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for n = 0,1,2, …

In [1.3], taking $x = x_{2n+2}$, $y = x_{2n+1}$, $z = x_{2n}$ we have, $[1+p\{d(Ax_{2n+2},Sx_{2n+2},y_{2n}) + d(Bx_{2n+1},Tx_{2n+1},y_{2n})\}]\, d(Ax_{2n+2},Bx_{2n+1},y_{2n})$
$\le p[d^2(Ax_{2n+2},Sx_{2n+2},y_{2n}) +d^2(Bx_{2n+1},Tx_{2n+1},y_{2n})] + \phi(d(Sx_{2n+2},Tx_{2n+1},y_{2n}), d(Ax_{2n+2},Sx_{2n+2},y_{2n}), d(Bx_{2n+1},Tx_{2n+1},y_{2n}), d(Ax_{2n+2},Tx_{2n+1},y_{2n}), d(Bx_{2n+1},Sx_{2n+2},y_{2n}))$ $[1+p\{d(y_{2n+2},y_{2n+1},y_{2n}) + d(y_{2n+1},y_{2n},y_{2n})\}]\, d(y_{2n+2},y_{2n+1},y_{2n}) \le p[d^2(y_{2n+2},y_{2n+1},y_{2n}) +d^2(y_{2n+1},y_{2n},y_{2n})] + \phi(d(y_{2n+1},y_{2n},y_{2n}), d(y_{2n+2},y_{2n+1},y_{2n}), d(y_{2n+1},y_{2n},y_{2n}), d(y_{2n+2},y_{2n},y_{2n}), d(y_{2n+1},y_{2n+1},y_{2n}))$
$[1+p\{d(y_{2n+2},y_{2n+1},y_{2n}) + 0\}]\, d(y_{2n+2},y_{2n+1},y_{2n}) \le p[d^2(y_{2n+2},y_{2n+1},y_{2n}) + 0] + \phi(0, d(y_{2n+2},y_{2n+1},y_{2n}), 0, 0, 0)$
$d(y_{2n+2},y_{2n+1},y_{2n}) \le \phi(0, d(y_{2n+2},y_{2n+1},y_{2n}), 0, 0, 0) < d(y_{2n+2},y_{2n+1},y_{2n})$.

Which is a contradiction. Thus we have $d(y_{2n+2},y_{2n+1},y_{2n}) = 0$,

Similarly, we have $d(y_{2n+1},y_{2n},y_{2n-1}) = 0$.
Hence , for n = 0,1,2 …., we have [3.1.4] $d(y_{n+2},y_{n+1},y_n) = 0$.
(b) For all $z \in X$, let $d_n(z) = d(y_n, y_{n+1},z)$ for n = 0,1,2,….. .
By (a), we have
$d(y_n, y_{n+2},z) \le d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1},z) + d(y_{n+1}, y_{n+2},z)$
$d(y_n, y_{n+2},z) \le d(y_n, y_{n+1},z) + d(y_{n+1}, y_{n+2},z)$
$d(y_n, y_{n+2},z) \le d_n(z) + d_{n+1}(z)$
Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in [3.1.3],
we have

$[1+p\{d(Ax_{2n+2},Sx_{2n+2},z) + d(Bx_{2n+1},Tx_{2n+1},z)\}]\, d(Ax_{2n+2},Bx_{2n+1},z)$
$\le p[d^2(Ax_{2n+2},Sx_{2n+2},z) +d^2(Bx_{2n+1},Tx_{2n+1},z)]$
$+ \phi(d(Sx_{2n+2},Tx_{2n+1},z), d(Ax_{2n+2},Sx_{2n+2},z), d(Bx_{2n+1},Tx_{2n+1},z),$
$d(Ax_{2n+2},Tx_{2n+1},z), d(Bx_{2n+1},Sx_{2n+2},z))$
$[1+p\{d(y_{2n+2},y_{2n+1},z) + d(y_{2n+1},y_{2n},z)\}]\, d(y_{2n+2},y_{2n+1},z)$
$\le p[d^2(y_{2n+2},y_{2n+1},z) +d^2(y_{2n+1},y_{2n},z)] + \phi(d(y_{2n+1},y_{2n},z),$
$d(y_{2n+2},y_{2n+1},z), d(y_{2n+1},y_{2n},z), d(y_{2n+2},y_{2n},z), d(y_{2n+1},y_{2n+1},z))$
[1.5] $[1+p\{d_{2n+1}(z) + d_{2n}(z)\}]\, d_{2n+1}(z)$
$\le p[d^2_{2n+1}(z) +d^2_{2n}(z)] + \phi(d_{2n}(z),d_{2n+1}(z), d_{2n}(z),$
$\{d_{2n}(z)+d_{2n+1}(z)\}, 0)$
Now, we shall show that $\{ d_n(z)\}$ is a non increasing sequence in R^+ . In fact, let $d_{n+1}(z) > d_n(z)$ for some n.
By [1.5] we have, $d_{2n+1}(z) < d_{2n+1}(z)$, which is a contradiction in R^+.
Now, we claim that $d_n(y_m) = 0$ for all non negative integers m, n.
Case 1. $n \ge m$. Then we have $0 = d_m(y_m) \ge d_n(y_m)$.
Case 2. $n < m$. By (M_4), we have
$d_n(y_m) \le d_n(y_{m-1}) +d_{m-1}(y_n) \le d_n(y_{m-1}) + d_n(y_n) = d_n(y_{m-1})$
By using the above inequality repeatedly, we have
$d_n(y_m) \le d_n(y_{m-1}) \le d_n(y_{m-2}) \le \ldots\ldots \le d_n(y_n) = 0$,
which completes the proof of our claim.
Finally, let i, j, and k be arbitrary non-negative integers. We may assume that $i < j$. By (M_4),
we have $d(y_i,y_j,y_k) \le d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k) = d(y_{i+1},y_j, y_k)$.
Therefore, by repeatition of the above inequality, we have

$d(y_i,y_j,y_k) \le d(y_{i+1},y_j, y_k) \le \ldots\ldots \le d(y_i,y_j, y_k) = 0.$
This completes the proof.

LEMMA [3]: Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying the following conditions [1.1] and [1.3]. Then the sequence $\{y_n\}$ defined by [1.4] is a Cauchy sequence in X.

PROOF OF THE LEMMA : In the proof of LEMMA [2], since $d_n(z)$ is a non increasing sequence in R^+ , by [1.3], we have ,

$[1+p\{d(Ax_2,Sx_2,z) + d(Bx_1,Tx_1,z)\}]\, d(Ax_2,Bx_1,z)$
$\le p[d^2(Ax_2,Sx_2,z) +d^2(Bx_1,Tx_1,z)]$
$+ \phi(d(Sx_2,Tx_1,z), d(Ax_2,Sx_2,z),d(Bx_1,Tx_1,z), d(Ax_2,Tx_1,z), d(Bx_1,Sx_2,z))$
$[1+p\{d(y_2,y_1,z) + d(y_1,y_0,z)\}]\, d(y_2,y_1,z)$
$\le p[d^2(y_2,y_1,z) +d^2(y_1,y_0,z)]+ \phi(d(y_1,y_0,z), d(y_2,y_1,z), d(y_1,y_0,z), d(y_2,y_0,z),d(y_1,y_1,z))$
$[1+p\{d_1(z) + d_0(z)\}]\, d_1(z)$
$\le p[d^2_1(z) +d^2_0(z)]$
$+ \phi(d_0(z), d_1(z), d_0(z), \{d_0(z)+d_1(z)\}, 0)$
$d_1(z) \le \phi(d_0(z), d_0(z), d_0(z), \{d_0(z)+d_0(z)\}, 0)$
$d_1(z) \le \gamma(d_0(z))$
and $d_2(z) \le \gamma(d_1(z)) \le \gamma(\gamma(d_0(z)) = \gamma^2(d_0(z))$.
In general, we have $d_n(z) \le \gamma^n(d_0(z))$.
Thus, if $d_0(z) > 0$, by LEMMA [1] $\lim_{n\to\infty} d_n(z) = 0$. If $d_0(z) = 0$, we have clearly $\lim_{n\to\infty} d_n(z) = 0$ since $d_n(z) = 0$ for n = 1,2, …

Now, we shall prove that $\{y_n\}$ is a Cauchy sequence in X. Since $\lim_{n\to\infty} d_n(z) = 0$, it is sufficient to show that a subsequence $\{ y_{2n}\}$ of $\{ y_n \}$ is a Cauchy sequence in X. Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence in X. Then there exist a point $z \in X$, an $\varepsilon > 0$ and strictly increasing sequences $\{m(k)\}$, $\{n(k)\}$ of positive integers such that $k \le n(k) < m(k)$,

[1.6] $(y_{2n(k)},y_{2m(k)},z) \ge \varepsilon$ and $d(y_{2n(k)},y_{(2m-2)(k)}, z) < \varepsilon$
for all k = 1,2,….. By LEMMA[2] and (M_4), we have
$d(y_{2n(k)}, y_{2m(k)}, z) - d(y_{2n(k)}, y_{2m(k-2)}, z) \le d(y_{2m(k-2)},y_{2m(k)}, z) \le d_{2m(k-2)}(z) + d_{2m(k-1)}(z)$
Since $\{d(y_{2n(k)}, y_{2m(k)}, z) - \varepsilon\}$ and $\{\varepsilon- d(y_{2n(k)}, y_{2m(k-2)}, z)\}$ are sequences in R^+ and $\lim_{n\to\infty} d_n(z) = 0$, we have
[1.7] $\lim_{k\to\infty} d(y_{2n(k)},y_{2m(k)}, z) = \varepsilon$ and $\lim_{k\to\infty} d(y_{2n(k)}, y_{2mk-2}, z) = \varepsilon$
Note that, by (M_4), we have
[1.8] $| d(x,y,a) - d(x,y,b)| \le d(a,b,x) + d(a,b,y)$
for all $x, y, a, b \in X$. Taking $x = y_{2n(k)}$, $y = a$, $a = y_{2m(k-1)}$ and $b = y_{2m(k)}$ in [1..8] and using LEMMA [2] and [1.7], we have
[1.9] $\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k-1)}, z) = \varepsilon$.
Once again, by using LEMMA[2], [1..7] and [1.8], we have
[1.10] $\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}, z) = \varepsilon$ and $\lim_{k\to\infty} d(y_{2n(k-1)}, y_{2m(k-1)}, z) = \varepsilon$.
Thus, by [1.3], we have,
[1.13]
$[1+p\{d(Ax_{2m(k)},Sx_{2m(k)},z)+d(Bx_{2n(k+1)},Tx_{2n(k+1)},z)\}]d(Ax_{2m(k)}, Bx_{2n(k+1)},z)$
$\le p[d^2(Ax_{2m(k)},Sx_{2m(k)},z) +d^2(Bx_{2n(k+1)},Tx_{2n(k+1)},z)] + \phi(d(Sx_{2m(k)},Tx_{2n(k+1)},z), d(Ax_{2m(k)},Sx_{2m(k)},z), d(Bx_{2n(k+1)},Tx_{2n(k+1)},z),d(Ax_{2m(k)},Tx_{2n(k+1)},z), d(Bx_{2n(k+1)},Sx_{2m(k)},z))$
$[1+p\{d(y_{2m(k)},y_{2m(k-1)},z) + d(y_{2n(k+1)},y_{2n(k)},z)\}]\, d(y_{2m(k)},y_{2n(k+1)},z)$
$\le p[d^2(y_{2m(k)},y_{2m(k-1)},z) +d^2(y_{2n(k+1)},y_{2n(k)},z)] + \phi(d(y_{2m(k-1)},y_{2n(k)},z), d(y_{2m(k)},y_{2m(k-1)},z), d(y_{2n(k+1)},y_{2n(k)},z),d(y_{2m(k)},y_{2n(k)},z), d(y_{2n(k+1)},y_{2m(k-1)},z))$.
As $k \to\infty$ in [1.11] and noting that d is continuous, we have
$\varepsilon \le \phi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \gamma(\varepsilon) < \varepsilon$
which is a contradiction. Therefore, $\{ y_{2n}\}$ is a Cauchy sequence in X and so the sequence $\{ y_n\}$ is a Cauchy sequence in X. This completes the proof.

Proof of the Theorem : By LEMMA[3], the sequence $\{ y_n\}$ defined by [1.2] is a Cauchy sequence in $S(X) \cap T(X)$. Since $S(X) \cap T(X)$ is a complete subspace of X, $\{y_n\}$ converges to a point w in $S(X) \cap T(X)$. On the other hand, since the subsequences $\{ y_{2n}\}$ and $\{ y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequences in $S(X) \cap T(X)$, they also converge to the same limit w. Hence there exist two points u, v in X such that Su = w and Tv = w, respectively.
By [1.3], we have
$[1+p\{d(Au,Su,z) + d(Bx_{2n+1},Tx_{2n+1},z)\}]\, d(Au,Bx_{2n+1},z)$
$\le p[d^2(Au,Su,z) +d^2(Bx_{2n+1},Tx_{2n+1},z)] + \phi(d(Su,Tx_{2n+1},z), d(Au,Su,z), d(Bx_{2n+1},Tx_{2n+1},z),d(Au,Tx_{2n+1},z), d(Bx_{2n+1},Su,z))$
$[1+p\{d(Au,Su,z) + d(y_{2n+1},y_{2n},z)\}]\, d(Au,y_{2n+1},z)$
$\le p[d^2(Au,Su,z) + d^2(y_{2n+1},y_{2n},z)] + \phi(d(Su,y_{2n},z), d(Au,Su,z), d(y_{2n+1},y_{2n},z), d(Au,y_{2n},z),d(y_{2n+1},Su,z))$
Since $\lim_{n\to\infty} d_n(z) = 0$ in the proof of LEMMA[2], letting $n\to\infty$, we have
$[1+p\{d(Au, w,z) + d(w, w,z)\}]\, d(Au, w,z)$
$\le p[d^2(Au, w,z) +d^2(w, w, z)] + \phi(d(w, w,z), d(Au, w,z),d(w ,w,z), d(Au, w,z),d(w, w,z))$
$d(Au, w,z) \le \phi(0, d(Au, w,z), 0,d(Au, w,z),0) < \gamma (d(Au, w,z)) < d(Au, w,z)$
which is contradiction . Hence Au = w = Su , that is u is a coincidence of A and S.
Similarly, we can show that v is a coincidence point of B and T.

Theorem [2] : Let A, B, S and T be mappings from a 2-metric spaces (X,d) into itself satisfying the conditions [1.1], [1.3], [1..10] and the following:
[2.1] the pairs A, S and B,T are compatible mappings of type (P).
[2.2] the pairs A,S and B, T are sequentially continuous at their coincidence points.
Then A, B, S and T have a unique common fixed point in X.

Roof of Theorem : By THEOREM [1], there exist two points u, v in X such that Au = Su = w and Bv = Tv = w, respectively, since A and S are compatible mappings of type(P), by PROPOSITION[3], ASu = SSu = SAu = AAu, which implies that Aw = Sw, Similarly B and T are compatible mapping of type(P) we have Bw = Tw. Now, we prove that Aw = w. If $Aw \ne w$, then by [1.3], we have

$[1+p\{d(Aw,Sw,z) + d(Bx_{2n+1},Tx_{2n+1},z)\}]\, d(Aw,Bx_{2n+1},z)$
$\le p[d^2(Aw,Sw,z) +d^2(Bx_{2n+1},Tx_{2n+1},z)] + \phi(d(Sw,Tx_{2n+1},z),d(Aw,Sw,z), d(Bx_{2n+1},Tx_{2n+1},z), d(Aw,Tx_{2n+1},z), d(Bx_{2n+1},Sw,z))$
$[1+p\{d(Aw,Sw,z) + d(y_{2n+1},y_{2n},z)\}]\, d(Aw,y_{2n+1},z)$

$\leq p[d^2(Aw,Sw,z)+d^2(y_{2n+1},y_{2n},z)] + \phi(d(Sw,y_{2n},z), d(Aw,Sw,z), d(y_{2n+1},y_{2n},z), d(Aw,y_{2n},z), d(y_{2n+1},Sw,z))$

Since $\lim_{n\to\infty} d_n(z) = 0$ in the proof of Lemma2, letting $n\to\infty$, we have

$[1+p\{d(Aw, w,z) + d(w, w,z)\}]\, d(Aw, w,z)$

$\leq p[d^2(Aw, w,z) + d^2(w, w, z)] + \phi(d(w, w,z), d(Aw, w,z), d(w ,w,z), d(Aw, w,z), d(w, w,z))$

$d(Aw, w,z) \leq \phi(0, d(Aw, w,z), 0,d(Aw, w,z),0) < \gamma (d(Aw, w,z)) < d(Aw, w,z)$

which is contradiction .

Hence Aw = w = Sw.

Similarly, we have Bw = Tw = w.

This means that w is a common fixed point of A, B, S and T.

The uniqueness of the fixed point w follows from [3.1.3].

This complete the proof.

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