Coincidence and Fixed Point Theorem in 2-Metric Spaces

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Abstract: In this Paper, we have proved coincidence and fixed point theorem in 2-metric space. The result in this paper are extend, generalized the Parsai V. and Singh B.[8], Fisher [1], Pathak[10].

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1. Introduction

The concept of 2-metric space has been investigated initially by Gähler in a series of papers and has been developed extensively by Gähler and many others. A 2-metric space is a set X with a real-valued function d on XxXxX satisfying the following conditions:

(M1) For two distinct point x, y ∈ X, there is a point z ∈ X such that d(x, y, z) ≠ 0
(M2) d(x, y, z) = 0 if at least two of x, y, z are equal,
(M3) d(x, y, z) = d(x, z, y) = d(y, z, x) ∀ x, y, z ∈ X,
(M4) d(x, y, z) ≤ d(x, y, u) + d(x, u, z) + d(u, y, z) ∀ x, y, z, u ∈ X ,

The function d is called a 2-metric for the space X and ( X, d) denotes a 2-metric space. It has been shown by Gähler in a series of papers and has been developed extensively by Gähler and many others. A 2-metric space is called a Cauchy sequence if d(xn, xn, z) → 0 for all z ∈ X, whenever {xn} is a sequence in X.

Definition [1]: A sequence {xn} in a 2-metric space (X, d) is called to be convergent to a point x ∈ X, denoted by limn→∞ xn = x, if limn→∞ d(xn, x, z) = 0 for all z ∈ X. The point x is said to be limit of sequence {xn} in X.

Definition [2]: A sequence {xn} in a 2-metric space (X,d) is called a Cauchy sequence if d(xn, xm, z) → 0 as n, m → ∞ for all z ∈ X.

Definition [3]: A 2-metric space in which every Cauchy sequence is convergent is called complete.

Definition [4]: A mapping S from a 2-metric space (X, d) into itself is said to be sequentially continuous at a point x ∈ X if for every sequence { xn } in X such that limn→∞ d(xn, x, z) = 0 for all z ∈ X, limn→∞ d(Sxn, Sx, z) = 0.

Definition [5]: Let S and T be mappings from a 2-metric space (X, d) into itself. The mappings S and T are said to be compatible of type (P) if for every sequence { xn } in X, limn→∞ d(Sxn, TTxn, z) = 0 for all z ∈ X, whenever {xn} is a sequence in X.

Proposition [1]: Let S and T be sequentially continuous mappings of a 2-metric space (X, d) into itself. If S and T are compatible if and only if they are compatible of type (P).

Proof: Let {xn} be sequence in X such that limn→∞ Sxn = limn→∞ Txn = t for some t ∈ X. Suppose that the mappings S and T are compatible.

By (M4), we have

d(SSxn, TTxn, z) ≤ d(SSxn, TTxn, STxn) + d(SSxn, STxn, z) + d(STxn, TTxn, z)

≤ d(SSxn, TTxn, STxn) + d(SSxn, STxn, z) + d(STxn, TTxn, z)

letting n → ∞ since S and T are compatible and sequentially continuous, we have

limn→∞ d(SSxn, TTxn, z) = 0 for all z ∈ X. Conversely, suppose that S and T are compatible of type (P). By (M4), we have

d(STxn, TTxn, z) ≤ d(STxn, TTxn, SSxn) + d(STxn, SSxn, z) + d(SSxn, TTxn, z)

≤ d(STxn, TTxn, SSxn) + d(STxn, SSxn, z) + d(SSxn, TTxn, z)

letting n → ∞ since S and T are compatible of type (P) and sequentially continuous, we have

limn→∞ d(STxn, TTxn, z) = 0 for all z ∈ X. This completes the proof.

Proposition [2]: Let S and T be compatible mappings of type (P) from a 2-metric space (X, d) into itself. If St = Tt for some t in X, Then STt = SSTt = TSt = Tt.

Proof: Suppose that {xn} is a sequence in X defined by x0 = t , n = 1,2,3,... and St = Tt. Then we have limn→∞ Sxn = limn→∞ Txn = St. Since S and T are compatible mappings of type (P), we have d (SSt, Tt, z) = limn→∞ d(SSxn, TTxn, z) = 0.
Hence we have $SS_t = TT_t$. Therefore, $ST_t = SST_t = TT_t = ST_t$.

**Proposition [3]:** Let $S$ and $T$ be compatible mappings of type $(P)$ from a 2-metric spaces $(X,d)$ into itself. Suppose $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$. Then we have the following.

(i) $\lim_{n \to \infty} TTx_n = \text{St}$ if $S$ is sequentially continuous at $t$;

(ii) $\lim_{n \to \infty} SSSx_n = \text{St}$ if $T$ is sequentially continuous at $t$;

(iii) $ST_t = \text{St}$ and $ST_t = t$ if $S$ and $T$ are Sequentially continuous at $t$.

**Proof:**

(i) Suppose that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$. Since $S$ is sequentially continuous we have $\lim_{n \to \infty} STx_n = \text{St}$. We have

$$d(TTx_n, z) \leq d(TTx_n, St) + d(St, z)$$

Therefore, since $S$ and $T$ are compatible mappings of type $(P)$, we have $\lim_{n \to \infty} TTx_n = \text{St}$.

(ii) The proof of $\lim_{n \to \infty} SSSx_n = \text{St}$ follows on the similar lines as argued in (i).

(iii) Since $T$ is sequentially continuous at $t$, we have $TTx_n = \text{Tt}$. By (i) since $S$ is sequentially continuous at $t$, we have also $\lim_{n \to \infty} TTx_n = \text{St}$. Hence by the uniqueness of the limit, we have $ST_t = \text{St}$ and so **PROPOSITION[2]** $ST_t = \text{St}$.

Let $R$ denote the set of all non-negative real numbers and $F$ be the family of mappings $\phi : (R^2)^n \to R^n$ such that each $\phi$ is upper-semi-continuous, non-decreasing in each coordinate variable, and for any $t > 0$, $\gamma(t) = (t, t, \ldots, t) < t$, where $\gamma : \mathbb{R} \to \mathbb{R}$ is a mapping with $\gamma(0) = 0$ and $a_1 + a_2 = 3$.

We have proved the following theorems:

**Theorem [1]:** Let $A$, $B$, $S$ and $T$ be mappings from a complete 2-metric space $(X, d)$ into itself, satisfying the following conditions:

[1.1] $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

[1.2] $S(X) \cap T(X)$ is a complete subspace of $X$,

[1.3] $[1 + p(d(Ax, Sx) + d(By, Ty))] d(Ax, By) \leq p[d^2(Ax, Sx) + d^2(By, Ty)] + \phi(d(Sx, Ty), d(Ax, Ty), d(By, Sx))$

for all $x, y \in X$, where $\phi \in F$. Then the pairs $A, S$ and $B, T$ have a coincidence point in $X$.

For our theorems, we need the following LEMMAS:

**LEMMA [1]:** For every $t > 0$, $\gamma(t) < t$ and only if $\lim_{n \to \infty} \gamma^n(t) = 0$, where $\gamma^n$ denotes the $n$-times composition of $\gamma$.

**LEMMA [2]:** Let $A$, $B$, $S$ and $T$ be mappings from a complete 2-metric space $(X, d)$ into itself, satisfying the conditions [1.1], [1.3].

Then we have the following:

(a) For every $n \in \mathbb{N}$, $d(y_n y_{n+1}, y_{n+2}) = 0$.

(b) For every $i, j, k \in \mathbb{N}$, $d(y_i, y_j, y_k) = 0$, where $(y_n)$ is the sequence in $X$ defined by $[1.4]$.

**Proof of the Lemma:**

(a) By (1.1) since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = TX_1$. Since $B(X) \subset S(X)$, for any arbitrary point $x_1 \in X$, there exists a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $(y_n)$ in $X$ such that

$[1.4] y_{2n} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for $n = 0, 1, 2, \ldots$

In [1.3], taking $x = x_{2n+2}$, $y = x_{2n+1}$, $z = x_{2n}$ we have,

$$[1 + p(d(Ax_{2n+2}, Sx_{2n+2}, y_{2n+2}) + d(Bx_{2n+1}, Tx_{2n+1}, y_{2n+1}))]

\leq p[d^2(Ax_{2n+2}, Sx_{2n+2}, y_{2n+2}) + d^2(Bx_{2n+1}, Tx_{2n+1}, y_{2n+1})] + \phi(d(Sx_{2n+2}, Ty_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}, y_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1}, y_{2n+1}),

\leq p[d^2(Ax_{2n+2}, Sx_{2n+2}, y_{2n+2}) + d^2(Bx_{2n+1}, Tx_{2n+1}, y_{2n+1})] + \phi(0, 0, 0, 0, 0, 0, 0) < d(y_{2n+2}, y_{2n+1}, y_{2n})$$

Which is a contradiction. Thus we have $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$.

Similarly, we have $d(y_{2n+1}, y_{2n+1}, y_{2n}) = 0$.

Hence, for $n = 0, 1, 2, \ldots$ we have $[1.4]$ $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$.

(b) For all $z \in X$, let $d(z) = d(y_{2n+1}, y_{2n})$ for $n = 0, 1, 2, \ldots$.

By (a), we have

$$d(y_{2n+1}, z) \leq d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, z) = d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, z)$$

$$d(y_{2n+2}, z) \leq d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, z)$$

$$d(y_{2n+1}, z) \leq \phi(d(z), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}))$$

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in [3.1.3], we have

$$[1 + p(d(Ax_{2n+2}, Sx_{2n+2}, y_{2n+2}) + d(Bx_{2n+1}, Tx_{2n+1}, y_{2n+1}))]

\leq p[d^2(Ax_{2n+2}, Sx_{2n+2}, y_{2n+2}) + d^2(Bx_{2n+1}, Tx_{2n+1}, y_{2n+1})] + \phi(0, 0, 0, 0, 0, 0) < d(y_{2n+2}, y_{2n+1}, y_{2n})$$

Therefore, by repetition of the above inequality, we have
Once again, by using LEMMA [2], since \( d_0(z) \) is a non increasing sequence in \( R^+ \), by [1.3], we have,

\[
1 + \sum_{\ell = 1}^{\infty} \left( d(y_{\ell}, y_{\ell+1}) + d(y_{\ell+1}, y_{\ell+2}) \right) \leq \lim_{\ell \to \infty} \sum_{i = 0}^{\ell} d(y_i, y_{i+1}) = \lim_{\ell \to \infty} \sum_{i = 0}^{\ell} (y_{i+1} - y_i).
\]

Thus, by [1.3], we have,

\[
\lim_{\ell \to \infty} \sum_{i = 0}^{\ell} |y_{i+1} - y_i| = 0.
\]

This completes the proof.

**Proof of the Lemma**: In the proof of LEMMA [2], since \( d_0(z) \) is a non increasing sequence in \( R^+ \), by [1.3], we have,

\[
1 + \sum_{\ell = 1}^{\infty} \left( d(y_{\ell}, y_{\ell+1}) + d(y_{\ell+1}, y_{\ell+2}) \right) \leq \lim_{\ell \to \infty} \sum_{i = 0}^{\ell} d(y_i, y_{i+1}) = \lim_{\ell \to \infty} \sum_{i = 0}^{\ell} (y_{i+1} - y_i).
\]

Thus, by [1.3], we have,

\[
\lim_{\ell \to \infty} \sum_{i = 0}^{\ell} |y_{i+1} - y_i| = 0.
\]

This completes the proof.

**Proof of the Theorem**: By LEMMA [3], the sequence \( \{y_n\} \) defined by [1.2] is a Cauchy sequence in \( R \). Since \( |X| = T(X) \) is a complete subspace of \( X \), \( \{y_n\} \) converges to a point \( w \) in \( S(X) \cap T(X) \). On the other hand, since the subsequences \( \{y_{2n}\} \) and \( \{y_{2n+1}\} \) of \( \{y_n\} \) are also Cauchy sequences in \( S(X) \cap T(X) \), they also converge to the same limit \( w \). Hence there exist two points \( u, v \) in \( X \) such that \( S(u) = w \) and \( T(v) = w \), respectively.

By [1.3], we have

\[
1 + \sum_{\ell = 1}^{\infty} \left( d(u_{\ell}, u_{\ell+1}) + d(u_{\ell+1}, u_{\ell+2}) \right) \leq \lim_{\ell \to \infty} \sum_{i = 0}^{\ell} d(u_i, u_{i+1}) = \lim_{\ell \to \infty} \sum_{i = 0}^{\ell} (u_{i+1} - u_i).
\]

Thus, by [1.3], we have

\[
\lim_{\ell \to \infty} \sum_{i = 0}^{\ell} |u_{i+1} - u_i| = 0.
\]

Since \( d(u, v) = 0 \) by [1.3], we have

\[
\lim_{\ell \to \infty} \sum_{i = 0}^{\ell} d(u_i, v_i) = 0.
\]

This completes the proof.

**Theorem [2]**: Let \( A, B, S \) and \( T \) be mappings from a 2-dimensional space \( (X, d) \) to itself satisfying the conditions [1.1], [1.3], [1.10] and the following:

[2.1] the pairs \( A, S \) and \( B, T \) are compatible mappings of type (P).

[2.2] the pairs \( A, S \) and \( B, T \) are sequentially continuous at their coincidence points.

Then, \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).
\[ p[d^2(Aw,Sw,z)+d^2(y_{2n+1},y_{2n},z)] + \phi(d(Sw,y_{2n},z),
\]
\[ d(Aw,Sw,z), d(y_{2n+1},y_{2n},z),
\]
\[ d(Aw,y_{2n},z), d(y_{2n+1},Sw,z))
\]

Since \( \lim_{n \to \infty} d(z) = 0 \) in the proof of Lemma2, letting
\[ n \to \infty, \]
we have
\[ [1+p\{d(Aw, w,z) + d(w, w,z)}] d(Aw, w,z)
\]
\[ \leq p[d^2(Aw, w,z) +d^2(w, w, z)] + \phi( d(w, w,z), d(Aw, w,z),
\]
\[ d(w ,w,z),
\]
\[ d(Aw, w,z), d(w, w,z))
\]
\[ d(Aw, w,z) \leq \phi( 0, d(Aw, w,z), 0,d(Aw, w,z),0) < \gamma (d(Aw, w,z)) < d(Aw, w,z)
\]
which is contradiction.

Hence \( Aw = w = Sw \).

Similarly, we have \( Bw = Tw = w \).

This means that \( w \) is a common fixed point of \( A, B, S \) and \( T \).

The uniqueness of the fixed point \( w \) follows from [3.1.3].

This complete the proof.

References

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