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Some Curvatures of Indefinite Sasakian Manifold Admitting Quarter Symmetric Metric Connection

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Abstract: ***In this paper we have studied the some curvature properties of a quarter-symmetric metric connection on indefinite sasakian manifold.***

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1. Introduction

In 1924, Friedmann and Schouten[2] introduced the notion of semi-symmetric connection on a differentiable manifold. The quarter-symmetric connection generalized the semi-symmetric connection. 1975, Golab[15] introduced the quarter-symmetric linear connection on a differentiable manifold. Let T be the torsion tensor defined as

$$T(X,Y)=\overline{\nabla}_X Y-\overline{\nabla}_Y X-[X,Y]$$

A linear connection $\overline{\nabla}$ in an n-dimensional differentiable manifold is said to be quarter-symmetric connection if its torsion tensor is defined by

$$T(X,Y)=\eta(Y)\phi X-\eta(X)\phi Y$$

Where η is 1-form ϕ is tensor of type (1, 1) and X and Y is vector fields.

In particular, if $\phi X = X$ and $\phi Y = Y$ then the quarter-symmetric connection reduces to a semi-symmetric connection.

A quarter-symmetric linear connection $\overline{\nabla}$ is satisfies the condition $\overline{\nabla}_X g \neq 0$ for all $X \in TM$ then $\overline{\nabla}$ is said to be quarter-symmetric non metric connection otherwise it is metric connection where TM is the lie algebra of vector field of manifold. The quarter-symmetric non metric connection is studied by Prakash and Pandey[4], Prakash and Narain[3], Singh and Pandey[13], Yano and Imai[7], Biswas and De[14], Rastogi[17], Shukla and Jaiswal[8,9,10], Mishra and Pandey[12], Mukhopadhyay, Ray and Barua[16], Biswas and Sengupta[6] and many others.

Also in 1993, Bejancu and Duggal[1] introduced the concept of (ε)-Sasakian manifold. Many authors have studied the manifold with indefinite metrics. The study of manifold with indefinite metric is of interest from the standpoint of physics and relativity. Xufeng and Xiaoli[19] established that (ε)-Sasakian manifold are real hypersurface of Indefinite Kaehlerian manifold. Sasakian manifold with indefinite metric have been first studied by Takahashi[18]. Further Jin [5], Singh Pandey and Tiwari[13] and several authors have studied the indefinite sasakian manifold.

This paper is organized as follows:

After the introduction, in section2 we have the brief introduction of indefinite sasakian manifold admitting the quarter-symmetric metric connection. We studied the projective ricci tensor of quarter-symmetric metric Indefinite sasakian manifold in section3. We discuss the pseudo projective tensor and M-projective curvature in section4 and section5 respectively. At last in section6 we prove some results base on the some curvature tensor (concircular curvature and conharmonic curvature).

2. Preliminaries

An odd-dimensional semi-Riemannian manifold M is called an indefinite almost contact manifold if it satisfied an indefinite almost contact structure (ϕ,ξ,η) where ϕ is a tensor field of $(1,1)$ type, η is 1-form and ξ is the characteristic vector field satisfying the conditions:

$$\overline{\overline{X}}=-X+\eta(X)\xi \qquad (2.1)$$

$$\phi X=\overline{X},\ \eta(\phi X)=0,\ \phi\xi=0 \qquad (2.2)$$

If g is a semi-Riemannian metric with the almost contact structure (ϕ,ξ,η), that is,

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y) \qquad (2.3)$$

$$g(X,\xi)=\varepsilon\eta(X),\ g(\xi,\xi)=\varepsilon=\pm 1 \qquad (2.4)$$

$$g(X,\phi Y)=-g(\phi X,Y) \qquad (2.5)$$

Then (M,g) is called an indefinite almost contact metric manifold. For any X and Y are vector fields on M. An indefinite almost contact metric manifold M is called normal if

$$N_\phi+d\eta\otimes\xi=0 \qquad (2.6)$$

Where N_ϕ is the Nijenhuis tensor field.

An indefinite normal contact metric manifold M is called an indefinite sasakian manifold if it satisfied the following condition:

$$(\nabla_X \xi) = \varepsilon \phi X \tag{2.7}$$
$$(\nabla_X \eta) Y = \varepsilon g(X, \phi Y) \tag{2.8}$$
$$(\nabla_X \phi) Y = -g(X, Y)\xi + \varepsilon \eta(Y) X \tag{2.9}$$

In view of equation (2.4), ξ is never a light like vector field on *M*.
ξ is space like , $\varepsilon = 1$ and index of g is an even number(respectively ξ is time like, $\varepsilon = -1$ and the index of g is an odd number), then M is called a space like almost contact metric manifold (respectively, time like almost contact metric manifold).
Following relations hold in an indefinite sasakian manifold:

$$\eta(R(X,Y),Z) = \varepsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \tag{2.10}$$
$$R(X,Y)\xi = \varepsilon[\eta(Y)X - \eta(X)Y] \tag{2.11}$$
$$R(\xi, X)Y = \varepsilon[g(X,Y)\xi - \eta(Y)X] \tag{2.12}$$
$$S(X,\xi)X = -(n-\varepsilon)\eta(Y) \tag{2.13}$$
$$Q\xi = -(n-\varepsilon)\xi \tag{2.14}$$
$$S(\phi Y, \phi Z) = S(Y,Z) + \varepsilon(n-\varepsilon)\eta(Y)\eta(Z) \tag{2.15}$$

Where R is curvature tensor and S is ricci tensor of the indefinite sasakian manifold *M*.

Hence, the quarter-symmetric metric connection of indefinite sasakian manifold is defined by [11]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{2.16}$$

It also satisfies [11]:

$$\overline{R}(X,Y,Z) = R(X,Y,Z) + \varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi + \varepsilon\eta(Z)[\eta(Y)X - \eta(X)Y] - \varepsilon[g(\phi Y,Z)\phi X - g(\phi X, Z)\phi Y] \tag{2.17}$$
$$\overline{S}(Y,Z) = S(Y,Z) + \varepsilon(n-\varepsilon)\eta(Y)\eta(Z) \tag{2.18}$$
$$\overline{r} = r + \varepsilon(n-\varepsilon) \tag{2.19}$$
$$\overline{Q}Y = QY + \varepsilon(n-\varepsilon)\eta(Y)\xi \tag{2.20}$$
$$\overline{R}(X,Y,Z) + \overline{R}(Y,Z,X) + \overline{R}(Z,X,Y) = 0 \tag{2.21}$$
$$\eta(\overline{R}(X,Y)Z) = 0 \tag{2.22}$$

Where $\overline{R}$ and $\overline{S}$ are curvature tensor and ricci tensor of Indefinite sasakian-manifold admitting the quarter-symmetric metric connection respectively.

From (2.18), we have

$$\overline{S}(Y,Z) - \overline{S}(Z,Y) = S(Y,Z) - S(Z,Y) \tag{2.23}$$

Hence we state:

Corollary 1: The ricci tensor of indefinite sasakian manifold is symmetric if and only if the ricci tensor of indefinite sasakian manifold admitting the quarter symmetric metric manifold is symmetric.

3. Projective RICCI Tensor

Definition3.1: Let M be an n-dimensional indefinite sasakian manifold with the quarter-symmetric metric connection $\overline{\nabla}$. The projective ricci tensor tensor of M with respect to quarter-symmetric non metric connection $\overline{\nabla}$ is defined by

$$\overline{\hat{P}}(X,Y) = \frac{n}{(n-1)}\overline{S}(X,Y) - \frac{\overline{r}}{(n-1)}g(X,Y) \tag{3.1}$$

By using (2.18) and (2.19), we get

$$\overline{\hat{P}}(X,Y) = \hat{P}(X,Y) + \frac{\varepsilon(n-\varepsilon)}{n-1}[n\eta(X)\eta(Y) - g(X,Y)] \tag{3.2}$$

Interchanging X and Y in equation (3.2), we get

$$\overline{\hat{P}}(X,Y) - \overline{\hat{P}}(Y,X) = \hat{P}(X,Y) - \hat{P}(Y,X) \tag{3.3}$$

Hence we state:

Theorem1: Projective ricci tensor indefinite sasakian manifold with the Riemannian connection ∇ is symmetric if and only if the projective ricci tensor of indefinite sasakian manifold admitting the quarter-symmetric metric connection $\overline{\nabla}$.

Again if $\overline{\hat{P}} = 0$, then we have

$$\overline{S}(X,Y) = \frac{\overline{r}}{(n-1)}g(X,Y) \tag{3.4}$$

Therefore we can state:

Theorem2: If projective ricci tensor is flat, then the indefinite sasakian manifold M admitting the quarter-symmetric metric connection becomes the Einstein manifold.

4. Pseudo Projective Curvature Tensor

Definition4.1: Let M be an n-dimensional Indefinite sasakian manifold with the quarter-symmetric metric connection $\overline{\nabla}$, then the pseudo projective curvature tensor of M with respect to quarter-symmetric metric connection $\overline{\nabla}$ is defined by

$$\overline{\overline{P}}(X,Y)Z = a\overline{R}(X,Y)Z + b[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y] - \frac{\overline{r}}{n}\left\{\frac{a}{(n-1)} + b\right\}[g(Y,Z)X - g(X,Z)Y] \tag{4.1}$$

Where a and b are constant, such that $a, b \neq 0$
By using (4.1), (2.17), (2.18) and (2.19), we have

$$\tilde{\bar{P}}(X,Y)Z = a\begin{bmatrix} R(X,Y)Z + \varepsilon\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ + \varepsilon\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ -\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\} \end{bmatrix} + b\begin{bmatrix} S(Y,Z)X + \varepsilon(n-\varepsilon)\eta(Y)\eta(Z)X - \\ S(X,Z)Y - \varepsilon(n-\varepsilon)\eta(X)\eta(Z)Y \end{bmatrix} - \frac{(r+\varepsilon(n-\varepsilon))}{n}\left\{\frac{a}{(n-1)} + b\right\}[g(Y,Z)X - g(X,Z)Y] \quad (4.2)$$

By putting the $X=Y$, $Y=Z$ and $Z=X$ in (4.2), we get

$$\tilde{\bar{P}}(X,Y)Z + \tilde{\bar{P}}(Y,Z)X + \tilde{\bar{P}}(Z,X)Y = 0 \quad (4.3)$$

Hence we can state:

Theorem 4: In Indefinite sasakian manifold, admitting the quarter-symmetric metric connection, the pseudo projective curvature tensor is cyclic.

In (4.1) taking inner product with U and contracting with X and U, we get

$$\tilde{S}(Y,Z) = a\bar{S}(Y,Z) + (n-1)b\bar{S}(Y,Z) - \frac{\bar{r}}{n}\left\{\frac{a}{(n-1)} + b\right\}(n-1)g(Y,Z) \quad (4.4)$$

From (2.18), (2.19) and (4.4)

$$\tilde{S}(Y,Z) = \{a + b(n-1)\}\left[S(Y,Z) + \varepsilon(n-\varepsilon)\eta(Y)\eta(Z) - \frac{r+\varepsilon(n-\varepsilon)}{n}g(Y,Z)\right] \quad (4.5)$$

Interchanging the Y and Z in (4.5) **,** we can state

Theorem5: In indefinite sasakian manifold with quarter-symmetric metric connection, the ricci tensor of pseudo projective curvature is symmetric if and only if the ricci tensor of indefinite sasakian manifold is symmetric.

If pseudo projective e curvature is flat, from(4.1) we have

$$a\bar{R}(X,Y,Z,U) + b[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U)] = \frac{\bar{r}}{n}\left\{\frac{a}{(n-1)} + b\right\}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \quad (4.6)$$

In (4.5), we put $X = U = e_i$ and taking summation both side we obtain

$$\bar{S}(Y,Z) = \frac{\bar{r}}{n}g(Y,Z) \quad (4.7)$$

Hence from (4.7), we can state:

Theorem6: If the pseudo projective curvature tensor of quarter-symmetric metric indefinite sasakian manifold is flat then indefinite sasakian manifold admitting the quarter symmetric metric connection become Einstein manifold.

5. m-Projective Curvature Tensor

Definition5.1: Let M be an n-dimensional Indefinite sasakian manifold with the quarter-symmetric metric connection $\bar{\nabla}$, then the m-projective curvature tensor of M with respect to quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{W}^*(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \quad (5.1)$$

From (5.1) and (2.21), we obtain

$$\bar{W}^*(X,Y)Z + \bar{W}^*(Y,Z)X + \bar{W}^*(Z,X)Y = 0 \quad (5.2)$$

Hence we can state:

Theorem7: In Indefinite sasakian manifold admitting the quarter-symmetric metric connection, m-projective curvature tensor is cyclic.

If $\bar{S} = 0$, then from (5.1) we get

$$\bar{W}^*(X,Y)Z = \bar{R}(X,Y)Z \quad (5.3)$$

Colloary8: If ricci tensor of indefinite sasakian manifold with the quarter-symmetric metric connection is vanishes then m-projective curvature tensor and curvature tensor of quarter-symmetric metric connection are identical.

If $\bar{W}^* = 0$ and taking inner product with U in (5.1) we obtain

$$g(\bar{R}(X,Y)Z,U) = \frac{1}{(n-1)}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) + \bar{S}(X,U)g(Y,Z) - \bar{S}(Y,U)g(X,Z)] \quad (5.4)$$

From (5.4), we have

$$\bar{S}(Y,Z) = \frac{\bar{r}}{n}g(Y,Z) \quad (5.5)$$

Therefore we can state:

Theorem9: Let M be the indefinite sasakian manifold admitting the quarter-symmetric metric connection. If m-projective curvature of M is flat then M becomes Einstein manifold.

Again we take the inner product with U in (5.1) and put $X=U=e_i$, we get

$$S^{*}(Y,Z)=\bar{S}(Y,Z)-\frac{1}{2(n-1)}\left[(n-2)\bar{S}(Y,Z)+\bar{r}g(Y,Z)\right] \tag{5.6}$$

Theorem10: If M be indefinite sasakian manifold admitting a quarter-symmetric metric connection whose ricci tensor and scalar curvature both vanishes then the ricci tensor with respect to m-projective curvature also vanishes.

6. Some Curvature Property

Definition6.1: Let M be an odd n-dimensional Kenmostu manifold admitting the quarter-symmetric non metric connection $\bar{\nabla}$. The concircular curvature tensor of M with respect to quarter-symmetric non metric connection is defined by

$$\bar{Z}(X,Y)U=\bar{R}(X,Y)U-\frac{\bar{r}}{n(n-1)}\left[g(Y,U)X-g(X,U)Y\right] \tag{6.1}$$

From (2.17),(2.18) and (6.1), we have

$$\begin{aligned}&\bar{Z}(X,Y)U=\\&R(X,Y)U+\varepsilon\eta(U)[\eta(Y)X-\eta(X)Y]+\varepsilon[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\xi\\&-\varepsilon[g(\phi Y,U)\phi X-g(\phi X,U)\phi Y]-\frac{r+\varepsilon(n-\varepsilon)}{n(n-1)}[g(Y,U)X-g(X,U)Y]\end{aligned} \tag{6.2}$$

From (6.2), we obtain

$$\bar{Z}(X,Y)U+\bar{Z}(Y,U)X+\bar{Z}(U,X)Y=0 \tag{6.3}$$

Hence we can say,

Theorem11: The concircular curvature tensor of indefinite sasakian manifold M with respect to quarter-symmetric metric connection is cyclic. Now if concircular curvature tensor is flat and taking inner product with respect to vector field V in (6.1), we obtain

$$\bar{S}(Y,Z)=\frac{\bar{r}}{n}g(Y,Z)$$

Therefore we can state:

Theorem12: Let M be the indefinite sasakian manifold admitting the quarter-symmetric metric connection. If the concircular curvature tensor is flat then M gives the Einstein manifold.

Definition6.2: Let M be n-dimensional indefinite sasakian manifold admitting the quarter-symmetric metric connection $\bar{\nabla}$. The conharmonic curvature tensor of M with respect to quarter-symmetric non metric connection is defined by

$$\begin{aligned}&\bar{V}(X,Y)Z=\\&\bar{R}(X,Y)Z-\frac{1}{(n-2)}\left[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\right]\end{aligned} \tag{6.4}$$

From (6.4), (2.17),(2.18) and (2.20) we have

$$\begin{aligned}&\bar{V}(X,Y)Z=\\&V(X,Y)Z-\frac{1}{(n-2)}\eta(Z)[\eta(Y)X-\eta(X)Y]\\&-\frac{1}{(n-2)}[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi\\&+[g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y]\end{aligned} \tag{6.5}$$

From (6.5), we have

$$\bar{V}(X,Y)Z+\bar{V}(Y,Z)X+\bar{V}(Z,X)Y=V(X,Y)Z+V(Y,Z)X+V(Z,X)Y \tag{6.6}$$

Again we can state:

Theorem13: The conharmonic curvature tensor of indefinite sasakian manifold M with respect to quarter-symmetric non metric connection is cyclic if and only if the canharmonic curvature tensor of indefinite sasakian manifold is cyclic.

If $\bar{S}=0$, then from (6.4) we have

$$\bar{V}(X,Y)Z=\bar{R}(X,Y)Z \tag{6.7}$$

From (5.3) and (6.7), we obtain

$$\bar{V}(X,Y)Z=\bar{W}^{*}(X,Y)Z \tag{6.8}$$

Now from (6.8), we state:

Theorem14: If ricci tensor of indefinite sasakian manifold with the quarter-symmetric metric connection is vanishes then m-projective curvature tensor and conharmonic curvature tensor of quarter-symmetric metric connection are identical. If conharmonic curvature tensor is flat and taking inner product with respect to vector field U in (6.4), we obtain

$$\bar{r}=0$$

Hence finally we can state:

Theorem15: Let M be the indefinite sasakian manifold admitting the quarter-symmetric metric connection. If the conharmonic curvature tensor of M is flat then scalar curvature of indefinite sasakian manifold admitting the quarter-symmetric metric connection is vanishes.

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