HOPF Bifurcation Analysis of a Food Web Consisting of Two Logistic Prey and a Harvesting Predator with Modified Leslie- Gower scheme

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Abstract: ***This paper deals with the dynamics of food web consisting of two logistic preys and a variable harvesting predator of Modified Leslie- Gower type. The existence of Hopf bifurcation analysis with varying key parameter is investigated.***

Keywords: Harvesting food web model, Hopf bifurcation, numerical simulation.

1. Introduction

The Lotka-Volterra type predator-prey model is the usual model on which multi-species food web and food chain built in which predator takes food as prey species. It is more difficult to study the food web and food chain to be biologically feasibility. In underlying food web the modified Leslie-Gower type predator getting food as Holling's type second functional response. The predator population is subjected to exploitation under the catch-per-unit-effort hypothesis h_1X_3;harvest rate is being used as control effort of dynamics of food web. The Hopf bifurcation refers to the change in the stability as key parameter is varied. In research paper [5], this mathematical model is studied without harvesting in predator.

2. The Mathematical Model

The Mathematical model for the food web consisting two logistic prey and one Modified Leslie- Gower predator with harvesting effort being proportional to predator density is given by the following non-linear system of equations:

$$\frac{dX_1}{dt}=r_1X_1(1-\frac{X_1}{K})-\frac{A_1X_1X_3}{1+B_1X_1+B_2X_2}$$

$$\frac{dX_2}{dt}=r_2X_2(1-\frac{X_2}{K})-\frac{A_2X_3X_2}{1+B_1X_1+B_2X_2}$$

$$\frac{dX_3}{dt}=r_3X_3{}^2(1-\frac{1}{S_3+S_1X_1+S_2X_2})-h_1X_3$$

Where $X_i\geq 0, i=1,2$ represent the population density of two preys and $X_3>0$ is the population density of the predator. The constants K, r_i, A_i, B_i and S_i, are model parameters assuming only positive values. The term h_1X_3 indicates the harvesting in the predator population. The constant h_1 is the harvesting effort. The model does not consider any direct competition between the two prey populations, but they are in apparent competition through the shared predation. Indeed, this apparent competition appears, as both prey types are included in predators diet. In the model, the third equation is written according to the Leslie- Gower scheme in which the conventional carrying capacity term is being replaced by the renewable resources for the predator as $S_1X_1+S_2X_2$. The additional constant S_3 normalizes the residual reductions in the predator population in case of severe scarcity of food. Further, the square term signifies the fact that mating frequency is proportional to the number of males as well as that of females.

Rescaling model we get non-dimensionalised form:

$$\begin{aligned}
\frac{dy_1}{dt}&=y_1(1-y_1-\frac{w_1y_3}{1+w_2y_1+w_3y_2})=y_1f_1(y_1,y_2,y_3)\\
\frac{dy_2}{dt}&=y_2[(1-y_2)w_4-\frac{w_5y_3}{1+w_3y_2+w_2y_1}]=y_2f_2(y_1,y_2,y_3)\\
\frac{dy_3}{dt}&=w_6y_3^2(1-\frac{w_7}{1+\alpha_1w_2y_1+\alpha_2w_3y_2})-w_{10}y_3=y_3f_3(y_1,y_2,y_3)
\end{aligned}\qquad(2)$$

$w_i>0, i=1,2,3,4,5,6,7$; $y_i\geq 0, i=1,2,3$; $\alpha_i>0, i=1,2$.

$$t = r_1 T,\ y_i = X_i / K,\ y_3 = X_3 / K, w_1 = A_1 K / r_1,\ w_2 = B_1 K\ , w_3 = B_2 K$$
$$w_4 = r_2 / r_1,\ w_5 = A_2 K / r_1, w_6 = r_3 K / r_1, w_7 = 1/ S_3,\ w_8 = S_1 K / S_3\ , w_9 = S_2 K / S_3,$$
$$w_{10} = h_1 / K$$

In the above, for simplicity, we have further assumed that B_i and S_i, $i = 1,2$ are in same proportion i.e. $w_8 = \alpha_1 w_2$, $w_9 = \alpha_2 w_2$.

3. The existence of Positive Equilibrium Point:

The existence of positive equilibrium point is established in the following theorem:

Theorem 3.1 The system (2) has positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ provided the following conditions are satisfied:

$$w_5 > w_1 w_4 \text{ and } 1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2 > w_7 \quad (3)$$

Proof. For nonzero equilibrium point, equating the three equations to zero and solving them we get,

$$f_1(y_1, y_2, y_3) = 0;\ f_2(y_1, y_2, y_3) = 0;\ f_3(y_1, y_2, y_3) = 0 \quad (4)$$

by solving first two isoclines of (4); we get

$$y_2 = \frac{w_5}{w_1 w_4}\left(y_1 - (1 - \frac{w_1 w_4}{w_5} \right) > 0 \Rightarrow y_1 - (1 - \frac{w_1 w_4}{w_5}) > 0 \Rightarrow w_5 > w_1 w_4 \quad (5)$$

$$\hat{y}_3 = \frac{w_{10}}{w_6}\left(\frac{1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2}{1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2 - w_7} \right) > 0 \text{ when } 1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2 > w_7$$

Since $0 < \hat{y}_1 < 1$, $0 < \hat{y}_2 < 1$, therefore, the system will have a positive equilibrium point.
Thus the system will have a positive nonzero solution under condition (3) in this case.
This proves the theorem.

4. HOPF's Analysis of the Harvested Food Web

Assume $y_1 = \hat{y}_1 + u, y_2 = \hat{y}_2 + v, y_3 = \hat{y}_3 + w$, where u, v and w small perturbations are. The variational matrix about $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -\hat{y}_1 + \dfrac{w_1 w_2 \hat{y}_1 \hat{y}_3}{(1 + w_2 \hat{y}_1 + w_3 \hat{y}_2)^2} & \dfrac{w_1 w_3 \hat{y}_1 \hat{y}_3}{(1 + w_2 \hat{y}_1 + w_3 \hat{y}_2)^2} & -\dfrac{w_1 \hat{y}_1}{(1 + w_2 \hat{y}_1 + w_3 \hat{y}_2)} \\ \dfrac{w_2 w_5 \hat{y}_2 \hat{y}_3}{(1 + w_2 \hat{y}_1 + w_3 \hat{y}_2)^2} & \hat{y}_2(- w_4 + \dfrac{w_5 w_3 \hat{y}_3}{(1 + w_2 \hat{y}_1 + w_3 \hat{y}_2)^2}) & -\dfrac{w_5 \hat{y}_2}{(1 + w_2 \hat{y}_1 + w_3 \hat{y}_2)} \\ \dfrac{\alpha_1 w_7 w_2 w_6 \hat{y}_3 \hat{y}_3}{(1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2)^2} & \dfrac{\alpha_2 w_7 w_3 w_6 \hat{y}_3 \hat{y}_3}{(1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2)^2} & w_6 \hat{y}_3 \left(1 - \dfrac{w_7}{(1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2)} \right) \end{bmatrix}$$

The characteristic equation of variational matrix is

$$\lambda^3 + a_0 \lambda^2 + a_1 \lambda + a_2 = 0 \quad (6)$$

where

$$a_0 = -(a_{11} + a_{22} + a_{33}); a_1 = (a_{11} a_{33} + a_{22} a_{33} + a_{11} a_{22} - a_{12} a_{21} - a_{23} a_{32} - a_{13} a_{31});$$
$$a_2 = (a_{13} a_{31} a_{22} + a_{23} a_{32} a_{11} + a_{21} a_{33} a_{12} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32} - a_{11} a_{22} a_{33}).$$

Let $a_{11} = -m_1$; $a_{22} = -m_2; a_{33} = -m_3$; $a_{13} = -m_4$; $a_{23} = -m_5$. Since

$$a_0 = m_1 + m_2 + m_3 > 0; a_1 = (m_1 m_3 + m_2 m_3 + m_1 m_2 - a_{12} a_{21} + m_5 a_{32} + m_4 a_{31}) > 0;$$
$$a_2 = (m_4 a_{31} m_2 + m_5 a_{32} m_1 - a_{21} m_3 a_{12} + a_{12} m_5 a_{31} + m_4 a_{21} a_{32} + m_1 m_2 m_3) > 0.$$

Then applying Routh's criteria $a_0 > 0$ provided $(a_{11} + a_{22} + a_{33}) < 0$, that is, $a_{11} < 0$, $a_{22} < 0, a_{33} < 0$. Also $a_2 > 0, a_1 > 0$ and $a_1 a_0 - a_2 > 0$. Therefore positive nonzero equilibrium point is locally asymptotically stable. None of the roots of equation (3) is zero as $a_0 \neq 0$.

Substituting $\lambda = \pm i\omega$ into (3), the real and imaginary partitions of the results lead to the following conditions: (i) $\omega = \pm\sqrt{a_1}$ (ii) $a_0 \omega^2 = a_2$
(i) and (ii) and (3) results that a pair of purely imaginary roots $\pm i\sqrt{a_1}$ and a real root " $-a_0$ ".

Transversality condition: - Let the characteristic equation be such that it contains a real root, say c_1, and a pair of purely imaginary roots $\lambda_1 = \lambda_1' \pm i\,\lambda_2'$:

$$(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)(\lambda - c_1) = 0.$$

Or

$$\lambda^3 - (2\lambda_1' + c_1)\lambda^2 + (|\lambda_1|^2 + 2\lambda_1' c_1) - |\lambda_1|^2 c_1 = 0 \tag{7}$$

Comparing the coefficients of (3) and (4) gives

$$a_1(-a_0 - 2\lambda_1') = -a_2 + 2\lambda_1'(2\lambda_1' + a_0)^2 \quad (8)$$

Differentiating (5)with respect to bifurcation parameter w_7, and substituting $w_7 = w_7^*$ and $\lambda_1'(w_7^*) = 0$ yields:

$$\left.\frac{\partial \lambda_1'}{\partial w_7}\right|_{w_7 = w_7^*} = -\frac{(a_0 \frac{\partial a_1}{\partial w_7} + a_1 \frac{\partial a_0}{\partial w_7} - \frac{\partial a_2}{\partial w_7})}{2(a_0^2 + a_1)} \Rightarrow (9)$$

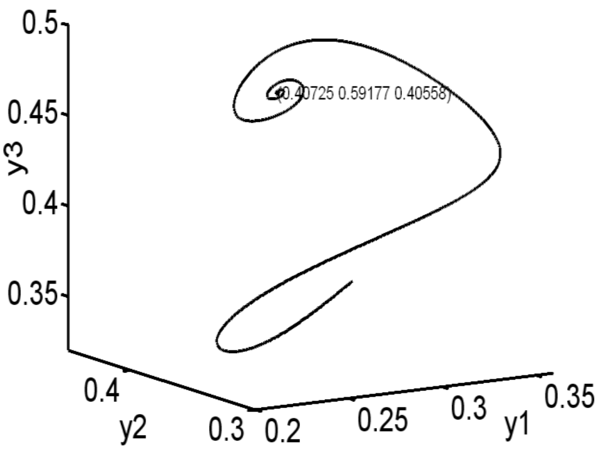
$$\left.\frac{\partial \lambda_1'}{\partial w_7}\right|_{w_7 = w_7^*} \neq 0 .$$

Thus the transversality condition is satisfied. So there exists a family of periodic solutions bifurcating from non zero equilibrium in the neighborhood of w_7^*, that is, the Hopf bifurcation will occur when $w_7 \in (w_7^* - \delta, w_7^* + \delta)$.

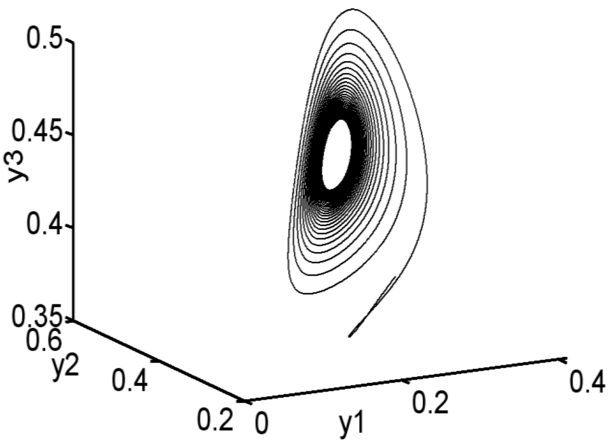
5. Numerical Simulation for HOPF bifurcation

In this section, the numerical analysis of underlying harvested food web is carried out under the biological feasible conditions. It is observed that numerically the other parameters are fixed at biologically feasible values. Only key parameter w_7 is varied.

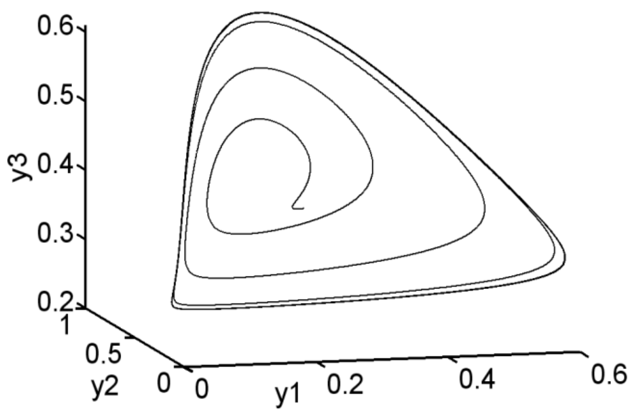
$$w_1 = 3.3, w_2 = 1.2, w_3 = 1.3, w_4 = 1.1, w_5 = 2.5, w_6 = 1.0, \alpha_1 = 1.5, \alpha_2 = 0.5, w_{10} = 0.08.$$



$w_7 = 1.7$, Asymptotically behavior



$w_7 = 1.5$, Quasiperiodic behavior



$w_7 = 1.3$ Limit cycle

6. Conclusion

It is concluded that for a set of parameter values with varying key parameter food web harvesting model shows hopf bifurcation. In this paper hopf bifurcation analysis of food web harvesting model is shown analytically as well as numerically.

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