Fractional Calculus of Saigo Fractional Integral Operator on Univalent and P-Valent Function

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Abstract: In this paper, we apply the fractional calculus technique for the class \( UT(n,p; \eta, \delta) \) and \( Q_m(\alpha, \gamma, \mu) \) which consists of p-valent and univalent functions with Negative coefficients by using the definition of fractional differentiation, and Integration, we establish four distortion theorems. The theorems 1 to 4 are established for the Saigo Fractional Integral Operator \( I_{0,x}^{\alpha, \eta, \delta} \) applying on the classes of p-valent functions.

Keywords: Fractional Calculus, Univalent Function, p-valent function, Special Function, Saigo Fractional Integral Operator.

1. Introduction and Definitions

In this paper, we apply the fractional calculus technique for the class \( UT(n,p; \eta, \delta) \) and \( Q_m(\alpha, \gamma, \mu) \) which consists of p-valent and univalent functions with Negative coefficients by using the definition of fractional differentiation, and Integration, we establish four distortion theorems here.

1.1 Definitions:

(i) Saigo’s Fractional Integral Operator:

The fractional integral operator \( I_{0,x}^{\alpha, \eta, \delta} \) is defined by

\[
I_{0,x}^{\alpha, \eta, \delta} f(x) = \frac{x^{-\eta}}{\Gamma(\eta)} \int_0^x (x - t)^{\alpha-1} f(t) t^{-\eta} dt, \quad x > 0, \quad \eta > 0, \quad \alpha > 0.
\]

Where \( f(x) \) is analytic and \( f(z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \) is the Gauss hypergeometric function.

(ii) Gauss hypergeometric function:

The Gauss hypergeometric function is defined as follows \([4, p. 2, eq. (1.1.3)]\):

\[
_2 F_1 (a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad |x| < 1
\]

(iii) Univalent and p-Valent Functions

Let S denote the class of function \([2, p. 2807, eq. (1.1)]\):

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0)
\]

Which are analytic and univalent in \( U = \{z; |z| < 1\} \).

Let T denote the subclass of S consisting of functions of the form \([2, p. 2808, eq. (1.4)]\):

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \geq 0)
\]

A function \( f \) of T is in \( Q_m(\gamma), 0 \leq \gamma < 1, m \in N \) if \([2, p. 2808, eq. (1.5)]\):

\[
Re \left( D^m f(z) \right) > \gamma, \quad z \in U
\]

Where \( D^m f(z) \) denotes usual mth order derivative introduced by Rushevey.

The class \( Q_m(\gamma) \) was introduced and studied by Uralegaddi and Sarangi \([5]\).

We aim to study the class \( Q_m(\alpha, \gamma, \mu) \) which consists of functions \( f \in T \), and satisfied the conditions

\[
\left| \frac{((m^n f(z))^\gamma - 1)}{m^n f(z) + (1 - \gamma)} \right| < \mu, \quad z \in U
\]

For \( 0 \leq \gamma < 1, \ 0 \leq \alpha < 1, 0 < \mu < 1 \).

Let \( S(n, p) \) be the class of functions \( f(z) \) of the form \([1, p. 1090, eq. (1.1)]\):

\[
f(z) = z^n + \sum_{k=n+p}^{\infty} a_k z^k, \quad (a_k \geq 0, n, p \in N, \ n = \{1, 2, 3, \ldots\})
\]

And

\[
f(z) = z^n - \sum_{k=n+p}^{\infty} a_k z^k, \quad (a_k \geq 0, n, p \in N, \ n = \{1, 2, 3, \ldots\})
\]

This is analytic p-valent in the open unit disk \( U = \{z \in c; |z| < 1\} \).

Let \( S^*WA(n, p, \beta) \) denotes the subclass of \( S(n, p) \) consisting of p-valent star like functions of order \( \beta, 0 \leq \beta \leq p, \) if it also satisfies the inequality

\[
Re \left\{ \frac{f'(z)}{f(z)} \right\} > \beta, \quad z \in U
\]

And also, let \( CWA(n, p, \beta) \) denotes the subclass of \( S(n, p) \) consisting of p-valent star like functions of order \( \beta, 0 \leq \beta \leq p, \) if it also satisfies the inequality (for more details see \([1, p. 1090, eq. (1.3)]\))

\[
Re \left\{ 1 + \frac{f''(z)}{f'(z)} \right\} > \beta, \quad z \in U
\]

Then, we see that \( f(z) \) is in \( CWA(n, p, \beta) \) if and only if \( f(z) \) is in \( S^*WA(n, p, \beta). \)

2. Results Required

The following results are required here

The following Saigo Fractional Integral Operator formulas are also required here

Let \( \beta, k > \eta - \delta - 1, \) then \([2, p. 2811]\)
The result (2.2.11) is sharp for the function

$$f(z) = z^p - \frac{\mu(\alpha + (1 - \gamma))}{k(1 + \mu \alpha)(m, k)} z^k, k \geq n + p$$

(2.5)

Lemma 2 [5, p. 72, eq. (3.16), p. 64, eq. (1.2)]:
A function $f(z) \in UT(n, p, \beta, \eta, \delta)$ for $\gamma \leq 1 < \Gamma \land k(k \geq 0)$ and only if $k = n + \rho \sigma(y, m)k \leq k \leq n + \rho \sigma(y, m, k)ak \leq k \leq n + \rho \sigma(y, m, k)ak \leq 1 - \gamma$

(2.6)

The result is sharp for

$$f(z) = z^p - \frac{1 - \gamma}{n(\gamma, \eta, k)} z^k, k \geq n + p$$

(2.7)

3. Main Results

The following four distortion theorems for p-valent function concerning to Saigo Fractional Integral Operator are established as the main results.

**Theorem 1:** Let the function $f(z)$ is a p-valent function and defined in the class $Q_m(\alpha, \gamma, \mu)$, then we have

$$|f^{\beta, \eta, \delta}_{0,z} f(z)| \geq \frac{\Gamma(p + 1)}{\Gamma(p + 1)(p + \beta + \delta + 1)} |z|^{p-\eta}$$

$$\left[1 - \frac{\Gamma(p + 1)(n + \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)}{(n + \eta + 1)(p + \beta + \delta + 1)(p + 1)(p + 1)(m + n + p)(1 + \mu \alpha)(n + p)}\right] z^n$$

(3.1)

For $z \in U_0$ where $U_0 = \{U, \eta \leq 1\}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(p + 1)(n + p - \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)(m + n + p)}{(n + p - \eta + 1)(p + \beta + \delta + 1)(p + 1)(m + n + p)(1 + \mu \alpha)(n + p)} z^n$$

(3.2)

**Theorem 2:** Let the function $f(z)$ is a p-valent function and defined in the class $Q_m(\alpha, \gamma, \mu)$, then we have

$$|f^{\beta, \eta, \delta}_{0,z} f(z)| \leq \frac{\Gamma(p + 1)}{\Gamma(p + 1)(p + \beta + \delta + 1)} |z|^{p-\eta}$$

$$\left[1 + \frac{\Gamma(n + p + 1)(n + p - \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)}{(n + p - \eta + 1)(p + \beta + \delta + 1)(p + 1)(p + 1)(m + n + p)(1 + \mu \alpha)(n + p)}\right] z^n$$

(3.3)

For $z \in U_0$ where $U_0 = \{U, \eta \leq 1\}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(n + p + 1)(n + p - \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)(m + n + p)}{(n + p - \eta + 1)(p + \beta + \delta + 1)(p + 1)(p + 1)(m + n + p)(1 + \mu \alpha)(n + p)} z^n$$

(3.4)

**Theorem 3:** Let the function $f(z)$ is a p-valent function and defined in the class $UT(n, p, \beta, \eta, \delta)\{\sigma(y, m, k)\}_{k=n+p}$, then we have

$$|f^{\beta, \eta, \delta}_{0,z} f(z)| \geq \frac{\Gamma(p + 1)}{\Gamma(p + 1)(p + \beta + \delta + 1)} |z|^{p-\eta}$$

$$\left[1 - \frac{\Gamma(n + p + 1)(n + p - \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)}{(n + p - \eta + 1)(p + \beta + \delta + 1)(p + 1)(p + 1)(\sigma(y, m, n) + p)}\right] z^n$$

(3.5)

For $z \in U_0$ where $U_0 = \{U, \eta \leq 1\}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(n + p + 1)(n + p - \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)(m + n + p)}{(n + p - \eta + 1)(p + \beta + \delta + 1)(p + 1)(p + 1)(\sigma(y, m, n) + p)} z^n$$

(3.6)

**Theorem 4:** Let the function $f(z)$ is a p-valent function and defined in the class $UT(n, p, \beta, \eta, \delta)\{\sigma(y, m, k)\}_{k=n+p}$, then we have

$$|f^{\beta, \eta, \delta}_{0,z} f(z)| \geq \frac{\Gamma(p + 1)}{\Gamma(p + 1)(p + \beta + \delta + 1)} |z|^{p-\eta}$$

$$\times \left[1 + \frac{\Gamma(n + p + 1)(n + p - \eta + 1)(p + \beta + \delta + 1)\mu(\alpha + (1 - \gamma))\Gamma(m, k)}{(n + p - \eta + 1)(p + \beta + \delta + 1)(p + 1)(p + 1)(\sigma(y, m, n) + p)}\right] z^n$$

(3.7)
Since for $G(z)$ where $U_0 = \{U, \eta \leq 1\}$
the result sharp for the function $f(z)$ given by the following form
\[
f(z) = z^p - \frac{\Gamma(\alpha+1)\Gamma(\beta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(\beta+\delta+1)}z^{\eta+p}z^p \quad (3.8)\]

**Proof of Theorem – 1:**
To establish the theorem – 1, we first apply the operator $I_{0,2}^{\beta,\eta,\delta}$ on both sides of (1.1.7), we have
\[
I_{0,2}^{\beta,\eta,\delta}f(z) = I_{0,2}^{\beta,\eta,\delta}z^p \]
On making use of (2.1), we obtain
\[
I_{0,2}^{\beta,\eta,\delta}f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p+\beta+\delta+1)}z^{\eta-p} - \sum_{k+n+p} a_k z^k \phi(k, \eta, \beta, \delta) \quad \text{where} \quad \phi(k, \eta, \beta, \delta) = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(k+\beta+\delta+1)}z^{\eta-k} \]
Therefore, (3.9)
Since for $k \geq n + p$, $\phi(k, \eta, \beta, \delta)$ is a decreasing function of $k$.
We have
\[
\phi(k, \eta, \beta, \delta) \leq \phi(n + p, \eta, \beta, \delta) \quad \text{i.e.} \\
\phi(k, \eta, \beta, \delta) \leq \phi(n + p, \eta, \beta, \delta) \quad \text{on setting} \\
G(z) = I_{0,2}^{\beta,\eta,\delta}f(z) = I_{0,2}^{\beta,\eta,\delta}z^p \quad \text{(3.10)} \]
\[
G(z) = \int_{0,2}^{\beta,\eta,\delta}f(z) - \int_{0,2}^{\beta,\eta,\delta}z^p \quad \text{equation in view of (3.9), (3.10), reduce in to the following inequality} \\
|G(z)| \geq |z|^p - \phi(n + p, \eta, \beta, \delta) \]
\[
\times |z|^{n+p} \sum_{k=n+p} a_k \geq |z|^{p} - \phi(n + p, \eta, \beta, \delta)|z|^{n+p}a_{n+p} \]
Now from the Lemma – 1 on using (2.2), (2.3), (2.5), we have
\[
\sum_{k=n+p} a_k \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(\beta+\delta+1)}z^{\eta+p} \quad (3.12)\]

The result in (3.12) in view of (3.13) takes the following form
\[
|G(z)| \leq |z|^p - \phi(n + p, \eta, \beta, \delta) \quad \text{on setting} \\
G(z) = I_{0,2}^{\beta,\eta,\delta}f(z) = I_{0,2}^{\beta,\eta,\delta}z^p \quad \text{on making use of (2.1), we obtain} \\
I_{0,2}^{\beta,\eta,\delta}f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p+\beta+\delta+1)}z^{\eta-p} - \sum_{k+n+p} a_k z^k \phi(k, \eta, \beta, \delta) \quad \text{where} \quad \phi(k, \eta, \beta, \delta) = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(k+\beta+\delta+1)}z^{\eta-k} \]
It can be written in the following form
\[
I_{0,2}^{\beta,\eta,\delta}f(z) = \int_{0,2}^{\beta,\eta,\delta}f(z) - \int_{0,2}^{\beta,\eta,\delta}z^p \quad \text{on setting} \\
G(z) = \int_{0,2}^{\beta,\eta,\delta}f(z) = \int_{0,2}^{\beta,\eta,\delta}z^p \quad \text{on making use of (2.1), we obtain} \\
I_{0,2}^{\beta,\eta,\delta}f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p+\beta+\delta+1)}z^{\eta-p} - \sum_{k+n+p} a_k z^k \phi(k, \eta, \beta, \delta) \quad \text{where} \quad \phi(k, \eta, \beta, \delta) = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(k+\beta+\delta+1)}z^{\eta-k} \]

\[
\sum_{k=n+p} a_k \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(\beta+\delta+1)}z^{\eta+p} \quad (3.14)\]

Since for $k \geq n + p$, $\phi(k, \eta, \beta, \delta)$ is a decreasing function of $k$.
We have
\[
\phi(k, \eta, \beta, \delta) \leq \phi(n + p, \eta, \beta, \delta) \quad \text{i.e.} \\
\phi(k, \eta, \beta, \delta) \leq \phi(n + p, \eta, \beta, \delta) \]
\[
\begin{align*}
\Gamma(n+p+1)\Gamma(n+p-q+\delta+1)&=\Gamma(p-n+1)/\Gamma(p+\beta+\delta+1) \\
\Gamma(n+p-q+1)&=\Gamma(p-\eta+\delta+1)\Gamma(p+\beta+\delta+1)
\end{align*}
\]…(3.15)

on setting
\[
G(z) = \sum_0^\infty f(z) = \sum_0^\infty \frac{\phi(n+p, \eta, \beta, \delta)}{\Gamma(m+n+1)}(1+\mu n)(1+\mu n + p)
\]

…(3.16)There for this above equation in view of (2.3.42), (2.3.43), reduce in to the following inequality
\[
[|G(z)|] \leq |z|^p + \phi(n+p, \eta, \beta, \delta)|z|^{n+p}
\]

…(3.17)

Now from the Lemma – 1 on using (2.2), (2.3), (2.5), we have
\[
(n+p)(1+\mu n)a_k \leq \sum_0^\infty \frac{\phi(n+p, \eta, \beta, \delta)}{\Gamma(m+n+1)}(1+\mu n)(1+\mu n + p)
\]

…(3.18)

The result in (3.17) in view of (3.18) takes the following form
\[
|G(z)| \leq |z|^p + \phi(n+p, \eta, \beta, \delta)|z|^{n+p}
\]

Now on using the result in (3.15) and (3.16), we at once arise at the derived result in (3.3) for the function defined in (3.4).

**Proof of Theorem – 3:**

To establish the theorem – 3, we first apply the operator \( I_{\beta, \eta, \delta}^{\beta, \eta, \delta} \) on both sides of (1.1.8), we have
\[
I_{\beta, \eta, \delta}^{\beta, \eta, \delta} f(z) = I_{\beta, \eta, \delta}^{\beta, \eta, \delta} z^p
\]

…(3.21)

\[
\begin{align*}
\phi(k, \eta, \beta, \delta) &= \frac{\Gamma(k+1)\Gamma(k-n+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)} z^{k-n} \\
\end{align*}
\]…(3.19)

Since for \( k \geq n+ \phi(k, \eta, \beta, \delta) \) is a decreasing function of \( k \).

We have
\[
\phi(k, \eta, \beta, \delta) \leq \phi(n+p, \eta, \beta, \delta)
\]

i.e.
\[
\phi(k, \eta, \beta, \delta) \leq \phi(n+p, \eta, \beta, \delta)
\]

on setting
\[
G(z) = \sum_0^\infty f(z) = \sum_0^\infty \frac{\phi(n+p, \eta, \beta, \delta)}{\Gamma(m+n+1)}(1+\mu n)(1+\mu n + p)
\]

…(3.20)

The result in (3.21) in view of (3.19), (3.20), reduce in to the following inequality
\[
\sum_0^\infty a_k \geq |z|^p - \phi(n+p, \eta, \beta, \delta)|z|^{n+p}
\]

…(3.22)

Now from the Lemma – 2 on using (2.6), (2.7), we have
\[
\sigma(y, m+n+1) \sum_0^\infty a_k
\]

\[
\leq \sum_0^\infty \frac{\phi(n+p, \eta, \beta, \delta)}{\Gamma(m+n+1)}(1+\mu n)(1+\mu n + p)
\]

…(3.23)

The result in (3.22) in view of (3.23) takes the following form
\[
|G(z)| \geq |z|^p - \phi(n+p, \eta, \beta, \delta)|z|^{n+p}
\]

Now on using the result in (3.20) and (3.21), we at once arise at the derived result in (3.5) for the function defined in (3.6).

**Proof of Theorem – 4:**

To establish the theorem – 4, we first apply the operator \( I_{\beta, \eta, \delta}^{\beta, \eta, \delta} \) on both sides of (1.1.7), we have
\[
I_{\beta, \eta, \delta}^{\beta, \eta, \delta} f(z) = I_{\beta, \eta, \delta}^{\beta, \eta, \delta} z^p
\]

…(3.24)

\[
\begin{align*}
\phi(k, \eta, \beta, \delta) &= \frac{\Gamma(k+1)\Gamma(k-n+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)} z^{k-n} \\
\end{align*}
\]…(3.25)

It can be written in the following form
\[
G(z) = \sum_0^\infty f(z) = \sum_0^\infty \frac{\phi(n+p, \eta, \beta, \delta)}{\Gamma(m+n+1)}(1+\mu n)(1+\mu n + p)
\]

\[
\leq \sum_0^\infty \frac{\phi(n+p, \eta, \beta, \delta)}{\Gamma(m+n+1)}(1+\mu n)(1+\mu n + p)
\]

…(3.26)

Now on using the result in (3.20) and (3.21), we at once arise at the derived result in (3.5) for the function defined in (3.6).
\[
I_{\theta,\alpha,\beta,\gamma}^{\rho,\eta,\delta} f(z) = \frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^n
= z^n + \sum_{k=n+p} a_k z^k \phi(k,\eta,\beta,\delta)
\]

Where
\[
\phi(k,\eta,\beta,\delta) = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(k+\delta+1)\Gamma(k+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)}
\]

Since for \( k \geq n + p \), \( \phi(k,\eta,\beta,\delta) \) is a decreasing function of \( k \).

We have
\[
\phi(k,\eta,\beta,\delta) \leq \phi(n + p, \eta, \beta, \delta)
\]

i.e.
\[
\frac{\Gamma(n+p+1)}{\Gamma(n+p-\eta+\delta+1)} \frac{\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)} \phi(n+p, \eta, \beta, \delta) \leq \phi(n, \eta, \beta, \delta)
\]

...(3.24)

Now from the Lemma – 2 on using (2.6), (2.7), we have
\[
\sigma(y,m,n+p) \sum_{k=n+p} a_k \leq (1 - \gamma) a_{n+p}
\]

\[
\frac{1 - \gamma}{\sigma(y,m,n+p)}
\]

...(3.27)

Now on using the result in (3.25) and (3.26), we at once arise at the derived result in (3.7) for the function defined in (3.8).

4. Special Cases

(1) If in (3.1) and (3.3), we take \( n = p = 1 \), then these results reduce to the following known results [2, p. 2811, eqns. (3.7), (3.8)] i.e.
\[
\frac{I_{\theta,\alpha,\beta,\gamma}^{\rho,\eta,\delta} f(z)}{\Gamma(2-\eta+\delta)(2-p+\beta)} \geq \left| \frac{1}{2-\eta+\delta} \right| |z|^{1-\eta} \left| 1 - 2\eta + \delta \right| \left| z^m + Lz \right| \n
m = n, m, n \in N_0
\]

References


