$$y_n(t)=\phi(0)+\sum_{0<t_k<t} S(t-t_k)I_k\left(y_n\left(t_k^-\right)\right)+\int_0^t v_n(s)ds$$

Consider the linear and continuous operator $K:L^1\left(J,\mathbb{R}^n\right)\to C\left(J,\mathbb{R}^n\right)$ defined by

$$Kv(t)=\int_0^t v_n(s)ds$$

Now

$$\left\|y_n(t)-\phi(0)-\sum_{o<t_k<t} S(t-t_k)I_k\left(y_n\left(t_k^-\right)\right)-\left(y_*(t)-\phi(0)-\sum_{o<t_k<t} S(t-t_k)I_k\left(y_*\left(t_k^-\right)\right)\right)\right\|\to 0,$$

As $n\to\infty$. From Lemma 3.2 it follows that $\left(\kappa\circ S_G^1\right)$ is a closed graph operator and from the definition of κ one has

$$y_n(t)-\phi(0)-\sum_{0<t_k<t} S(t-t_k)I_k\left(y_n\left(t_k^-\right)\right)\in\left(\kappa\circ S_F^1\left(y_n\right)\right)$$

.

As $x_n\to x_*$ and $y_n\to y_*$,there is a $v\in S_G^1\left(x_*\right)$ such that

$$y_*(t)=\phi(0)+\sum_{0<t_k<t} S(t-t_k)I_k\left(y_*\left(t_k^-\right)\right)+\int_0^t v_*(s)ds.$$

Hence the multi-valued operator B is an upper semi-continuous operator on Ω.

Step VI:

Finally we show that the set

$$\varepsilon=\left\{u\in\Omega:\lambda u\in Au+Bu\ for\ some\ \lambda>1\right\}$$

is bounded. Let $u\in\varepsilon$ be any element. Then there exist $v\in S_G^1(u)$ such that

$$u(t)=\lambda^{-1}S(t)\left[\phi(0)-f\left(0,\phi(0)\right)\right]+\lambda^{-1}f\left(t,x_t\right)$$
$$+\lambda^{-1}\int_0^t AS(t-s)f\left(s,x_s\right)ds+\lambda^{-1}\int_0^t S(t-s)v(s)ds$$
$$+\lambda^{-1}\int_0^t s(t-s)\left(\int_0^s g\left(s,\tau,x_\tau\right)d\tau\right)ds$$
$$+\lambda^{-1}\sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right)$$

Then,

$$\|u(t)\|\le\left\|\lambda^{-1}\right\|\|S(t)\|\|\phi(0)\|-\left\|\lambda^{-1}\right\|\|S(t)\|\left\|f\left(0,\phi(0)\right)\right\|$$
$$+\left\|\lambda^{-1}\right\|\left\|f\left(t,x_t\right)\right\|+\left\|\lambda^{-1}\right\|\int_0^t\|AS(t-s)\|\left\|f\left(t,x_s\right)\right\|ds$$
$$+\left\|\lambda^{-1}\right\|\int_0^t\|S(t-s)\|\|v(s)\|ds+\left\|\lambda^{-1}\right\|\sum_{0<t_k<t}\|S(t-t_k)\|\left\|I_k\left(x\left(t_k^-\right)\right)\right\|$$

$$\le M\|\phi\|_D\left\{1+c_1\left\|(-A)^{-\beta}\right\|\right\}+c_2\left\|(-A)^{-\beta}\right\|\{M+1\}$$
$$+\frac{c_{1-\beta}c_2T^\beta}{\beta}+c_1\left\|(-A)^{-\beta}\right\|\|u_t\|_D$$
$$+\int_0^t\frac{c_{1-\beta}}{(t-s)^{1-\beta}}\|u_s\|_D\,ds+M\int_0^t q(s)\psi\left(\|u_s\|_D\right)ds$$
$$+\frac{c_{1-\beta}T^\beta}{\beta}\int_0^t\left[\rho^2c_1+\rho c_2+c_3\right]d\tau+M\sum_{k=1}^m c_k$$
$$\le F+c_1\left\|(-A)^{-\beta}\right\|\|u_t\|_D$$
$$+\int_0^t\frac{c_{1-\beta}}{(t-s)^{1-\beta}}\|u_s\|_D\,ds+M\int_0^t q(s)\psi\left(\|u_s\|_D\right)ds$$
$$+\frac{c_{1-\beta}T^\beta}{\beta}\int_0^t\left[\rho^2c_1+\rho c_2+c_3\right]d\tau$$

Where,

$$F=M\|\phi\|_D\left\{1+c_1\left\|(-A)^{-\beta}\right\|\right\}+c_2\left\|(-A)^{-\beta}\right\|\{M+1\}$$
$$+\frac{c_{1-\beta}c_2T^\beta}{\beta}+M\sum_{k=1}^m c_k$$

Put $\omega(t)=\max\{\|u(s)\|:-r\le s\le t\}, t\in J$ .Then $\|u_t\|_D\le\omega(t)$ for all $t\in J$ and there is a point $t^*\in[-r,t]$ such that $\omega(t)=\|u(t^*)\|$.Hence we have

$$\omega(t)=\|u(t^*)\|$$

$$\le F+c_1\|(-A)^{-\beta}\|\|u_{t^*}\|_D+C_{1-\beta}c_1\int_0^{t^*}\frac{\|u_s\|_D}{(t-s)^{1-\beta}}ds$$

$$+M\int_0^{t^*}q(s)\psi(\|u_s\|_D)ds+\frac{c_{1-\beta}T^\beta}{\beta}\int_0^t[\rho^2c_1+\rho c_2+c_3]d\tau$$

$$\le F+c_1\|(-A)^{-\beta}\|\omega(t)+C_{1-\beta}c_1\int_0^t\frac{\omega(s)}{(t-s)^{1-\beta}}ds+M\int_0^t q(s)\psi(\omega(s))ds,$$

$$+\frac{c_{1-\beta}T^\beta}{\beta}\int_0^t[\rho^2c_1+\rho c_2+c_3]d\tau$$

$$\omega(t)\le\frac{F}{1-c_1\|(-A)^{-\beta}\|}+\frac{c_{1-\beta}c_1}{1-c_1\|(-A)^{-\beta}\|}\int_0^t\frac{\omega(s)}{(t-s)^{1-\beta}}ds$$

$$+\frac{M}{1-c_1\|(-A)^{-\beta}\|}\int_0^t q(s)\psi(\omega(s))ds$$

$$+\frac{c_{1-\beta}T^\beta}{\beta(1-c_1\|(-A)^{-\beta}\|)}\int_0^t[\rho^2c_1+\rho c_2+c_3]d\tau$$

$$\le K_0+K_1\int_0^t\frac{\omega(s)}{(t-s)^{1-\beta}}ds+K_2\int_0^t q(s)\psi(\omega(s))ds+K_3\int_0^t[\rho^2c_1+\rho c_2+c_3]d\tau,\quad t\in I,$$

Where,

$$K_0=\frac{F}{1-c_1\|(-A)^{-\beta}\|}, K_1=\frac{c_{1-\beta}c_1}{1-c_1\|(-A)^{-\beta}\|}$$

$$K_2=\frac{M}{1-c_1\|(-A)^{-\beta}\|}$$

and

$$K_3=\frac{c_{1-\beta}T^\beta}{\beta(1-c_1\|(-A)^{-\beta}\|)}$$

From Lemma 3.3 we have

$$\omega(t)\le b\left(K_0+K_2\int_0^t q(s)\psi(\omega(s))ds\right),$$

Where

$$b=e^{K_1^n(\Gamma(\beta))^nT^{n\beta}/\Gamma(n\beta)}\sum_{j=0}^{n-1}\left(\frac{K_1T^\beta}{\beta}\right)^j$$

Let $m(t)=b\left(K_0+K_2\int_0^t q(s)\psi(\omega(s))ds\right),\ t\in J$.

Then we have $\omega(t)\le m(t)$ for all $t\in J$ .Differentiating with respect to t, we obtain

$$m'(t)=bK_2q(t)\psi(\omega(t)),\quad a.e.\ t\in J, m(0)=K_0.$$

This implies

$m'(t)\le bK_2q(t)\psi(m(t)),\quad a.e.\ t\in J$; that is,

$$\frac{m'(t)}{\psi(m(t))}\le bK_2q(t),\quad a.e.\ t\in J$$

Integrating from 0 to t, we obtain

$$\int_0^t\frac{m'(t)}{\psi(m(s))}ds\le bK_2\int_0^t q(s)ds.$$

By the change of variable,

$$\int_{K_0}^{m(t)}\frac{ds}{\omega(s)}\le bK_2\int_0^T q(t)ds<\int_{K_0}^{\infty}\frac{ds}{\psi(s)}$$

Hence there exists a constant M such that $m(t)\le M$ for all $t\in J$, and therefore

$\omega(t)\le m(t)\le M$ for all $t\in J$.

Now from the definition of ω it follows that

$$\|u\|=\sup_{t\in[-r,T]}\|u(t)\|=\omega(T)\le m(T)\le M,$$

For all $u \in \varepsilon$. This shows that the set ε is bounded in Ω. As a result the conclusion (ii) of Theorem 2.2 does not hold. Hence the conclusion (i) holds consequently the initial value problem (1.1)-(1.3) has a solution x on $[-r, T]$. This completes the proof.

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