$$-A(I-A^-A)M^-D^*D(B^-)^*A^*N^-(I-B^-B)^*A^*+AA^-C^*CN^-(I-B^-B)^*A^*$$
$$-A(I-A^-A)M^-BA^-C^*CN^-(I-B^-B)^*A^*+A(I-A^-A)M^-F^*FN^-(I-B^-B)^*A^*$$
$$=T+A(I-A^-A)M^-BA^-E^*E(B^-)^*A^*+A(I-A^-A)M^-D^*D(B^-)^*A^*-TN^-N$$
$$+A(I-A^-A)M^-BA^-E^*EB^-A^*N^-N-A(I-A^-A)M^-D^*D(B^-)^*A^*N^-N$$
$$+AA^-C^*CN^-N-A(I-A^-A)M^-BA^-C^*CN^-N+A(I-A^-A)M^-F^*FN^-N$$

By using the condition $C^*C-T+(T-AA^-C^*C)N^-N=0$ *we get;*

$$AX_0A^*=C^*C$$

Also

$$AX_0B^*=AA^-E^*E(B^-)^*B^*+A(I-A^-A)M^-BA^-E^*E(B^-)^*B^*+A(I-A^-A)M^-D^*D(B^-)^*B^*$$
$$-AA^-E^*E(B^-)^*A^*N^-(I-B^-B)^*B^*+A(I-A^-A)M^-BA^-E^*EB^-A^*N^-(I-B^-B)^*B^*$$
$$-A(I-A^-A)M^-D^*D(B^-)^*A^*N^-(I-B^-B)^*B^*+AA^-C^*CN^-(I-B^-B)^*B^*$$
$$-A(I-A^-A)M^-BA^-C^*CN^-(I-B^-B)^*B^*+A(I-A^-A)M^-F^*FN^-(I-B^-B)^*B^*$$

And by the condition $AA^-E^*E(B^-)^*B^*=E^*E$ *we get;*

$$AX_0B^*=E^*E$$

Finally

$$BX_0A^*=BA^-E^*E(B^-)^*A^*+B(I-A^-A)M^-BA^-E^*E(B^-)^*A^*+B(I-A^-A)M^-D^*D(B^-)^*A^*$$
$$-BA^-E^*E(B^-)^*A^*N^-(I-B^-B)^*A^*+B(I-A^-A)M^-BA^-E^*EB^-A^*N^-(I-B^-B)^*A^*$$
$$-B(I-A^-A)M^-D^*D(B^-)^*A^*N^-(I-B^-B)^*A^*+BA^-C^*CN^-(I-B^-B)^*A^*$$
$$-B(I-A^-A)M^-BA^-C^*CN^-(I-B^-B)^*A^*+B(I-A^-A)M^-F^*FN^-(I-B^-B)^*A^*$$
$$=Y+MM^-Y+MM^-D^*D(B^-)^*A^*-YN^-N+MM^-YN^-N-MM^-D^*D(B^-)^*A^*N^-N$$
$$+BA^-C^*CN^-N-MM^-BA^-C^*CN^-N+MM^-F^*FN^-N$$

And By using the condition;

$$F^*F-MM^-D^*D(B^-)^*A^*(I-N^-N)-MM^-F^*FN^-N-(I-MM^-)(Y(I-N^-N)$$
$$+(I-MM^-)BA^-C^*CN^-N=0,$$ *Can have* $BX_0A^*=F^*F$

Conversely; let X_0 *is a common solution to the system of operator equation (2),let* $U=\begin{bmatrix}A\\B\end{bmatrix}, V=\begin{bmatrix}B\\A\end{bmatrix}, W^*W=\begin{bmatrix}E^*E & C^*C\\D^*D & F^*F\end{bmatrix}$*,then the system of operator equations (2) have a common solution if and only if* $UXV^*=W^*W$ *has a solution , and by using thermo (2.1),we get the solution* X_0 *is in the form* $X_0=U^-W^*W(V^-)^*$ *with condition* $UU^-W^*W(V^-)^*V^*=W^*W$*, and from[5]one can have* $U^-=\left[A^- - (I-A^-A)M^-BA^- \quad (I-A^-A)M^-\right]$,

And $V^-=\left[\left((B^-)^*-(B^-)^*A^*N^-(I-B^-B)^*\right)^* \quad \left(N^-(I-B^-B)^*\right)^*\right]$ *therefore get;*

$$AA^-E^*E(B^-)^*B^*=E^*E,\ D^*D-W-MM^-(W-D^*D(B^-)^*B^*)=0\quad C^*C-T+(T-AA^-C^*C)N^-N=0$$

and

$$F^*F-MM^-D^*D(B^-)^*A^*(I-N^-N)-MM^-F^*FN^-N-(I-MM^-)(Y(I-N^-N)$$
$$+(I-MM^-)BA^-C^*CN^-N=0$$

Now we shows the general common solution of adjointable operator equations (2)with same condition above.

Theorem (2.3):

Let $A,B,D,C,E,F\in B(H,K)$ *where* A,B,N,M *have g-inverse operators, then the system adjointable operator equations (2) have a common solution if and only if*

$$AA^-E^*E(B^-)^*B^*=E^*E,$$
$$D^*D-W-MM^-(W-D^*D(B^-)^*B^*)=0,$$

$C^*C - T + (T - AA^-C^*C)N^-N = 0,$

$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$

$+ (I - MM^-)BA^-C^*CN^-N = 0,$ *Where;*

$W = BA^-E^*E(B^-)^*B^*$, $T = AA^-E^*E(B^-)^*A^*$,

$Y = BA^-E^*E(B^-)^*A^*$, $M = B(I - A^-A)$,

$N = (I - B^-B)^*A^*$, $Z \in B(K,H)$ *is arbitrary operator.*

Proof:

We clime

$X = X_0 + (I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*$

be a common solution for the system of adjointable operator equations(2), and we substitute in the left side of operator equations;

$BXB^* = B(X_0 + (I - A^-A)(I - M^-M)Z(I - NN^-)(I - B^-B)^*)B^*$

$= W + MM^-W + MM^-D^*D(B^-)^*B^*$ *and By using the condition* $D^*D - W - MM^-(W - D^*D(B^-)^*B^*) = 0$ *we get* $BXB^* = D^*D$

Also

$AXA^* = A(X_0 + (I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*)A^*$

$= T - TN^-N + AA^-C^*CN^-N$ *By using the condition* $C^*C - T + (T - AA^-C^*C)N^-N = 0$ *we get* $AXA^* = C^*C$

And

$AXB^* = AX_0B^* + A((I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*)B^*$

$AA^-E^*E(B^-)^*B^* = E^*E$ *and by the condition get* $AXB^* = E^*E$

Finally

$BXA^* = B(X_0 + (I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*)A^*$

$= Y + MM^-Y + MM^-D^*D(B^-)^*A^* - YN^-N + MM^-YN^-N - MM^-D^*D(B^-)^*A^*N^-N$

$+ BA^-C^*CN^-N - MM^-BA^-C^*CN^-N + MM^-F^*FN^-N$

And By using the condition

$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$

$+ BA^-C^*CN^-N) = 0$ *Can have* $BXA^* = F^*F$

Conversely; let X *is a common solution to the system of operator equation (2),let* $U = \begin{bmatrix} A \\ B \end{bmatrix}, V = \begin{bmatrix} B \\ A \end{bmatrix}, W^*W = \begin{bmatrix} E^*E & C^*C \\ D^*D & F^*F \end{bmatrix}$ *,then the system of operator equations (2) have a common solution if and only if* $UXV^* = W^*W$ *has a solution , and by using thermo (2.1),we get the solution* X *is in the form* $X = U^-W^*W(V^-)^* + (I - U^-U)Z(I - V^-V)^*$ *with condition* $UU^-W^*W(V^-)^*V^* = W^*W$ *and from[5]one can have* $U^- = \begin{bmatrix} A^- - (I - A^-A)M^-BA^- & (I - A^-A)M^- \end{bmatrix}$,

And $V^- = \begin{bmatrix} ((B^-)^* - (B^-)^*A^*N^-(I - B^-B)^*)^* & (N^-(I - B^-B)^*)^* \end{bmatrix}$ *therefore get;*

$AA^-E^*E(B^-)^*B^* = E^*E$, $D^*D - W - MM^-(W - D^*D(B^-)^*B^*) = 0$

$C^*C - T + (T - AA^-C^*C)N^-N = 0$ *and*

$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$

$+ (I - MM^-)BA^-C^*CN^-N) = 0.$

Now, to illustrate the above theorem, consider the following example.

Example (2.4):

let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $D^*D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $C^*C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

$E^*E = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}, F^*F = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, W = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, Y = \begin{bmatrix} 1.5 & 3 \\ 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} -8 & 4 \\ 0 & 0 \end{bmatrix}$

$, N = \begin{bmatrix} 0.5 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $Z = \begin{bmatrix} 10 & 4 \\ 0 & 3 \end{bmatrix}.$

thus from above theorem we get ;

$$X = A^{-}E^{*}E(B^{-})^{*} + (I - A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*} + (I - A^{-}A)M^{-}D^{*}D(B^{-})^{*} - A^{-}E^{*}E(B^{-})^{*}A^{*}N^{-}(I - B^{-}B)^{*}$$
$$+ (I - A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*}A^{*}N^{-}(I - B^{-}B)^{*} - (I - A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}(I - B^{-}B)^{*}$$
$$+ A^{-}C^{*}CN^{-}(I - B^{-}B)^{*} - (I - A^{-}A)M^{-}BA^{-}C^{*}CN^{-}(I - B^{-}B)^{*} + (I - A^{-}A)M^{-}F^{*}FN^{-}(I - B^{-}B)^{*}$$
$$+ (I - A^{-}A)(I - M^{-}M)Z(I\_NN^{-})(I - B^{-}B)^{*}.$$

$$X = \begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\4 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix} - \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}-0.1 & 0\\0.05 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\4 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}$$
$$+ \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}-0.1 & 0\\0.05 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix} - \begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\4 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$
$$+ \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}-0.1 & 0\\0.05 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\4 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$
$$- \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}-0.1 & 0\\0.05 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$
$$- \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}-0.1 & 0\\0.05 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}-0.1 & 0\\0.05 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$
$$+ \begin{bmatrix}-2 & 1\\-6 & 2\end{bmatrix}\begin{bmatrix}0.8 & -0.4\\-0.4 & 0.2\end{bmatrix}\begin{bmatrix}6 & 2\\8 & 3\end{bmatrix}\begin{bmatrix}0.5 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} - \begin{bmatrix}0.25 & 0.25\\0.7 & 0.7\end{bmatrix} + \begin{bmatrix}0.25 & 0.25\\0.7 & 0.7\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$
$$X = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

Now, we must to show $X = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ *is satisfy the system of adjontable operator equations (2).*

$$BXB^{*} = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} = D^{*}D$$
$$AXA^{*} = \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = C^{*}C$$
$$AXB^{*} = \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\4 & 0\end{bmatrix} = E^{*}E$$
$$BXA^{*} = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} = F^{*}F$$

Also, thus operators satisfy the conditions;

$$AA^{-}E^{*}E(B^{-})^{*}B^{*} = E^{*}E$$
$$\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\4 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\4 & 0\end{bmatrix} = E^{*}E$$

$$D^{*}D - W - MM^{-}(W - D^{*}D(B^{-})^{*}B^{*}) = 0$$
$$\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\left[\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\right]$$
$$\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$
$$C^{*}C - T + (T - AA^{-}C^{*}C)N^{-}N = 0$$
$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \left[\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\right]\begin{bmatrix}0.5 & 1\\0.25 & 0.5\end{bmatrix}$$
$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$$
$$+(I - MM^-)BA^-C^*CN^-N = 0$$

$$\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & -1\\-0.25 & 0.5\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 1\\0.25 & 0.5\end{bmatrix}$$
$$-\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1.5 & 3\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & -1\\-0.25 & 0.5\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}0.5 & 1\\0.25 & 0.5\end{bmatrix}$$
$$= \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

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