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The General Common Solutions for Adjointable Bounded Operator Equations

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Abstract: ***The aim of this work, given the general common solutions of some types of adjointable bounded operator equations define on complex Hilbert space, also introduce the general common solutions for system of operator equations upon some necessary and sufficient conditions for existence these solutions via g-inverse operator. <u>Introduction</u>: The subject of common solution studied in initial on matrix equations after that this concept introduce on some types of bounded operator equations such as Dajic and koliha, [1] in 2007 given the general common Hermition and positive solutions for bounded linear operator equations AX = C and XB = D also given the represented of this solution in 2005 Xian Zhang, [4] introduced the general common Hermition nonnegative solutions for the matrix equations, in this work, we introduce the generalized to matrix equation appear in [4] and give the general common solutions for the operator equations*** $AXA^* = C^*C$ ***and*** $CXC^* = D^*D$, ***by using g-inverse, also, the general common solutions for system of adjointable operator equations*** $BXB^* = D^*D,\ AXA^* = C^*C,\ AXB^* = E^*E,\ BXA^* = F^*F$, ***At first we must introduce some basic concept of operator such as, adjoint of operator was define by: let*** $A: H \to K$ ***be a bounded operator from a Hilbert spaces H onto K then adjoint operator denoted by*** A^* ***and*** $A^*: K \to H$ ***such that*** $< Ax, y >=< x, A^*y >$, ***where for all*** $x \in H$ ***and*** $y \in K$, ***[2], also, the generalize inverse of operator*** $A: H \to K$ ***is denoted by*** $A^- \in B(K, H)$ ***satisfying the condition*** $AA^-A = A$ ***[3],***

1. General Common Solutions for some types of adjointable Operator Equations.

In this section, we introduce the common solution of adjointable bounded operator equations, $AXA^* = C^*C,\ CXC^* = D^*D$ (1)via g-inverse operator this equations is generalized of operator equations appeared in [4],

Theorem (1.1):
Let $A, B, D \in B(H, K)$ *where* A, B *and* M *have g-inverse operators, then the adjointable operator equations (1) have a common solution if and only if*

$$AA^-B^*B = B^*B,$$

$$D^*D - S - MM^-(S - D^*D(C^-)^*C^*) = 0, \text{ Where,}$$

$$S = CA^-B^*B(A^-)^*C^*,\ \ M = C(I - A^-A)$$

Proof:

We claim $X_0 = A^-B^*B(A^-)^* + (I - A^-A)M^-CA^-B^*B(A^-)^* + (I - A^-A)M^-D^*D(C^-)^*$ *be a common solution to the operator equations(1) and substitute in the left side of equation (1) we get;*

$$AX_0A^* = AA^-B^*B(A^-)^*A^* + A(I - A^-A)M^-CA^-B^*B(A^-)^*A^* + A(I - A^-A)M^-D^*D(C^-)^*A^*$$

$= AA^-B^*B(A^-)^*A^*$, *and by using the condition* $AA^-B^*B = B^*B$ *we get;* $AX_0A^* = B^*B$

And

$$CX_0C^* = CA^-B^*B(A^-)^*C^* + C(I - A^-A)M^-CA^-B^*B(A^-)^*C^* + C(I - A^-A)M^-D^*D(C^-)^*C^*$$

$= S + MM^-S + MM^-D^*D(C^-)^*C^*$, *also from the condition* $D^*D - S - MM^-(S - D^*D(C^-)^*C^*) = 0$

we get; $CX_0C^* = D^*D$. *Therefore;* X_0 *is a common solution of operator equations (1).*

Conversely; let X_0 *is a common solution of operator equations (1), then its satisfy* $AX_0A^* = B^*B$

$$AA^-B^*B(A^-)^*A^* + A(I - A^-A)M^-CA^-B^*B(A^-)^*A^* + A(I$$

$AA^-B^*B(A^-)^*A^* = B^*B$, So, $AA^-B^*B = D^*D.$

$$CX_0C^* = D^*D$$

$$CA^-B^*B(A^-)^*C^* + C(I - A^-A)M^-BA^-B^*B(A^-)^*B^* + C(I$$

$$S + MM^-S + MM^-D^*D(C^-)^*C^* = D^*D$$

$$D^*D - S - MM^-(S - D^*D(C^-)^*C^*) = 0.$$

Thus, satisfy the sufficient conditions.

Now, we shows the general common solution of adjointable operator equations (1) with same condition of above theorem.

Theorem (1.2):
Let $A, B, D \in B(H, K)$ *where* A, B, M *have g-inverse operators, then the adjointable operator equations (1) have a common solution if and only if*

$$AA^-B^*B = B^*B,$$

$$D^*D - S - MM^-(S - D^*D(C^-)^*C^*) = 0, \text{ Where,}$$

$S = CA^- B^* B(A^-)^* C^*$, $M = C(I - A^- A)$,

$Z \in B(K,H)$ is arbitrary operator.

Proof:

We claim $X = X_0 + (I - A^- A)(I - M^- M)Z$ *be a common solution of adjointable operator equations(1),we get;*

$$AXA^* = A(X_0 + (I - A^- A)(I - M^- M)Z)A^*$$
$$= AX_0 A^* + A((I - A^- A)(I - M^- M)Z)A^*$$

$$= AX_0 A^* + AZA^* - AM^- MZA^* - AZA^* + AM^- MZA^*$$
$$= AX_0 A^*,\ \text{then}\ AX\ A^* = B^* B.$$

and

$$CXC^* = C(X_0 + (I - A^- A)(I - M^- M)Z)C^*$$
$$= CX_0 C^* + C((I - A^- A)(I - M^- M)Z)C^*$$
$$= CX_0 C^* + M(I - M^- M)ZC^*$$
$$= CX_0 C^*,\ \text{then}\ CXC^* = D^* D.$$

Conversely; let X *is a common solution of operator equations (1); so is satisfy* $AXA^* = B^* B$

$$A(X_0 + (I - A^- A)(I - M^- M)Z)A = B^* B$$

$$AX_0 A^* = B^* B,\ \text{then}\ AAB^* B = B^* B.$$

Also ;

$$CXC^* = D^* D$$
$$C(X_0 + (I - A^- A)(I - M^- M)Z)C^* = D^* D$$
$$CX_0 C^* = D^* D,\ \text{then}$$
$$D^* D - S - MM^-(S - D^* D(C^-)^* C^*) = 0$$

Now, to illustrate the above theorem consider the following example.

Example (1.3):

Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B^* B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, D^* D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$, S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $Z = \begin{bmatrix} 6 & 7 \\ 5 & 9 \end{bmatrix}$

thus from above theorem we get ;

$$X = A^- B^* B(A^-)^* + (I - A^- A)M^- CA^- B^* B(A^-)^* + (I - A^- A)M^- D^* D(C^-)^* + (I - A^- A)(I - M^- M)Z$$

$$X = \begin{bmatrix} 0.4 & 0.2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0 \\ 0.2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0 \\ 0.2 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 05. \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 6 & 7 \\ 5 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, we must to show $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *is satisfy the adjontable operator equations (1.6).*

$$AXA^* = \begin{bmatrix} 2 & 0. \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = B^* B$$

$$CXC^* = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = D^* D$$

Also, thus operators satisfy the conditions;

$$AA^- B^* B = B^* B$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = B^* B$$

$$D^* D - S - MM^-(S - D^* D(C^-)^* C^*) = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. General Common Solutions for System of Adjointable Operator Equations.

Here we, give the common solution of the system of adjointable bounded operator equations $BXB^* = D^*D,\ AXA^* = C^*C,\ AXB^* = E^*E,$ $BXA^* = F^*F$ *(2), where A and B are bounded noninvertible operators and (A,B,C,D,E,F) defines on Hilbert space* H.

It first we gave the solution of adjointable operator equation $AXB^* = D^*D$ *(3) when* A, B *invertible operators we will need in this section.*

Theorem (2.1):

let $A \in B(H,K)$ *and* $B \in B(K,H)$ *have g-inverse operators,* $D \in B(K)$*.then the equation(3) has a general solution if and only if*

$$AA^-D^*D(B^-)^*B^* = D^*D.$$

proof:

Let $AA^-D^*D(B^-)^*B^* = D^*D$*, We claim*

$X = A^-D^*D(B^-)^* + (I - A^-A)Z(I - B^-B)^*$ *be a solution of equation (3). To do this substitute in the left side of operator equation (3) we get;*

$$AXB^* = A(A^-D^*D(B^-)^*)B^* + A((Z - ZA^-A)(I - B^-B)^*)B^*$$

$$= A(A^-D^*D(B^-)^*)B^* + AZB^* - AZB^* - AZB^* + AZB^*$$

$= AA^-D^*D(B^-)^*B^*$ *by using the condition we get;*

$$AXB^* = D^*D$$

Conversely, since X *is a solution of equation (3) therefore,*

$$AXB^* = D^*D$$

$$AA^-D^*D(B^-)^*B^* + AZB^* - AZB^* = D^*D.$$

$$AA^-D^*D(B^-)^*B^* = D^*D$$

The following theorem we give the necessary and sufficient condition to get the common solution of the system of adjointable operator equations (2).

Theorem (2.2):

Let $A, B, D, C, E, F \in B(H,K)$ *where* A, B, N, M *have g-inverse operators, then the system adjointable operator equations (2) have a common solution if and only if*

$$AA^-E^*E(B^-)^*B^* = E^*E,\ D^*D - W - MM^-(W - D^*D(B^-)^*B^*) = 0,$$

$$C^*C - T + (T - AA^-C^*C)N^-N = 0,$$

$$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$$

$+ (I - MM^-)BA^-C^*CN^-N = 0,$ *Where;* $W = BA^-E^*E(B^-)^*B^*,\ T = AA^-E^*E(B^-)^*A^*,$

$$Y = BA^-E^*E(B^-)^*A^*,\ \ M = B(I - A^-A),\ \ N = (I - B^-B)^*A^*.$$

Proof:

It first claim

$$X_0 = A^-E^*E(B^-)^* + (I - A^-A)M^-BA^-E^*E(B^-)^* + (I - A^-A)M^-D^*D(B^-)^* - A^-E^*E(B^-)^*AN^-(I - B^-B)^*$$
$$+ (I - A^-A)M^-BA^-E^*EB^-A^*N^-(I - B^-B)^* - (I - A^-A)M^-D^*D(B^-)^*A^*N^-(I - B^-B)^*$$
$$+ A^-C^*CN^-(I - B^-B)^* - (I - A^-A)M^-BA^-C^*CN^-(I - B^-B)^* + (I - A^-A)M^-F^*FN^-(I - B^-B)^*$$

be a common solution for system of adjointable operator equations(2),and we substitute in the left side of operator equations;

$$BX_0B^* = BA^-E^*E(B^-)^*B^* + B(I - A^-A)M^-BA^-E^*E(B^-)^*B^* + B(I - A^-A)M^-D^*D(B^-)^*B^*$$
$$- BA^-E^*E(B^-)^*A^*N^-(I - B^-B)^*B^* + B(I - A^-A)M^-BA^-E^*EB^-A^*N^-(I - B^-B)^*B^*$$
$$- B(I - A^-A)M^-D^*D(B^-)^*A^*N^-(I - B^-B)^*B^* + BA^-C^*CN^-(I - B^-B)^*B^*$$
$$- B(I - A^-A)M^-BA^-C^*CN^-(I - B^-B)^*B^* + B(I - A^-A)M^-F^*FN^-(I - B^-B)^*B^*$$
$$= W + MM^-W + MM^-D^*D(B^-)^*B^* - BA^-E^*E(B^-)^*A^*N^-(I - B^-B)^*B^*$$
$$+ MM^-BA^-E^*EB^-A^*N^-(I - B^-B)^*B^* - MM^-D^*D(B^-)^*A^*N^-(I - B^-B)^*B^*$$
$$+ BA^-C^*CN^-(I - B^-B)^*B^* - MM^-BA^-C^*CN^-(I - B^-B)^*B^* + MM^-F^*FN^-(I - B^-B)^*B^*$$

By using the condition $D^*D - W - MM^-(W - D^*D(B^-)^*B^*) = 0$ *we get;*

$$BX_0B^* = D^*D$$

And

$$AX_0A^* = AA^-E^*E(B^-)^*A^* + A(I - A^-A)M^-BA^-E^*E(B^-)^*A^* + A(I - A^-A)M^-D^*D(B^-)^*A^*$$
$$- AA^-E^*E(B^-)^*A^*N^-(I - B^-B)^*A^* + A(I - A^-A)M^-BA^-E^*EB^-A^*N^-(I - B^-B)^*A^*$$

$$-A(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}(I-B^{-}B)^{*}A^{*}+AA^{-}C^{*}CN^{-}(I-B^{-}B)^{*}A^{*}$$
$$-A(I-A^{-}A)M^{-}BA^{-}C^{*}CN^{-}(I-B^{-}B)^{*}A^{*}+A(I-A^{-}A)M^{-}F^{*}FN^{-}(I-B^{-}B)^{*}A^{*}$$
$$=T+A(I-A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*}A^{*}+A(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}-TN^{-}N$$
$$+A(I-A^{-}A)M^{-}BA^{-}E^{*}EB^{-}A^{*}N^{-}N-A(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}N$$
$$+AA^{-}C^{*}CN^{-}N-A(I-A^{-}A)M^{-}BA^{-}C^{*}CN^{-}N+A(I-A^{-}A)M^{-}F^{*}FN^{-}N$$

By using the condition $C^{*}C-T+(T-AA^{-}C^{*}C)N^{-}N=0$ *we get;*

$$AX_{0}A^{*}=C^{*}C$$

Also

$$AX_{0}B^{*}=AA^{-}E^{*}E(B^{-})^{*}B^{*}+A(I-A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*}B^{*}+A(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}B^{*}$$
$$-AA^{-}E^{*}E(B^{-})^{*}A^{*}N^{-}(I-B^{-}B)^{*}B^{*}+A(I-A^{-}A)M^{-}BA^{-}E^{*}EB^{-}A^{*}N^{-}(I-B^{-}B)^{*}B^{*}$$
$$-A(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}(I-B^{-}B)^{*}B^{*}+AA^{-}C^{*}CN^{-}(I-B^{-}B)^{*}B^{*}$$
$$-A(I-A^{-}A)M^{-}BA^{-}C^{*}CN^{-}(I-B^{-}B)^{*}B^{*}+A(I-A^{-}A)M^{-}F^{*}FN^{-}(I-B^{-}B)^{*}B^{*}$$

And by the condition $AA^{-}E^{*}E(B^{-})^{*}B^{*}=E^{*}E$ *we get;*

$$AX_{0}B^{*}=E^{*}E$$

Finally

$$BX_{0}A^{*}=BA^{-}E^{*}E(B^{-})^{*}A^{*}+B(I-A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*}A^{*}+B(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}$$
$$-BA^{-}E^{*}E(B^{-})^{*}A^{*}N^{-}(I-B^{-}B)^{*}A^{*}+B(I-A^{-}A)M^{-}BA^{-}E^{*}EB^{-}A^{*}N^{-}(I-B^{-}B)^{*}A^{*}$$
$$-B(I-A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}(I-B^{-}B)^{*}A^{*}+BA^{-}C^{*}CN^{-}(I-B^{-}B)^{*}A^{*}$$
$$-B(I-A^{-}A)M^{-}BA^{-}C^{*}CN^{-}(I-B^{-}B)^{*}A^{*}+B(I-A^{-}A)M^{-}F^{*}FN^{-}(I-B^{-}B)^{*}A^{*}$$
$$=Y+MM^{-}Y+MM^{-}D^{*}D(B^{-})^{*}A^{*}-YN^{-}N+MM^{-}YN^{-}N-MM^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}N$$
$$+BA^{-}C^{*}CN^{-}N-MM^{-}BA^{-}C^{*}CN^{-}N+MM^{-}F^{*}FN^{-}N$$

And By using the condition;

$$F^{*}F-MM^{-}D^{*}D(B^{-})^{*}A^{*}(I-N^{-}N)-MM^{-}F^{*}FN^{-}N-(I-MM^{-})(Y(I-N^{-}N)$$
$$+(I-MM^{-})BA^{-}C^{*}CN^{-}N=0,$$ *Can have* $BX_{0}A^{*}=F^{*}F$

Conversely; let X_{0} *is a common solution to the system of operator equation (2),let* $U=\begin{bmatrix}A\\B\end{bmatrix}, V=\begin{bmatrix}B\\A\end{bmatrix}, W^{*}W=\begin{bmatrix}E^{*}E & C^{*}C\\ D^{*}D & F^{*}F\end{bmatrix}$ *,then the system of operator equations (2) have a common solution if and only if* $UXV^{*}=W^{*}W$ *has a solution , and by using thermo (2.1),we get the solution* X_{0} *is in the form* $X_{0}=U^{-}W^{*}W(V^{-})^{*}$ *with condition* $UU^{-}W^{*}W(V^{-})^{*}V^{*}=W^{*}W$*, and from[5]one can have* $U^{-}=\left[A^{-}-(I-A^{-}A)M^{-}BA^{-}\quad (I-A^{-}A)M^{-}\right]$,

And $V^{-}=\left[\left((B^{-})^{*}-(B^{-})^{*}A^{*}N^{-}(I-B^{-}B)^{*}\right)^{*}\quad \left(N^{-}(I-B^{-}B)^{*}\right)^{*}\right]$ *therefore get;*

$AA^{-}E^{*}E(B^{-})^{*}B^{*}=E^{*}E$, $D^{*}D-W-MM^{-}(W-D^{*}D(B^{-})^{*}B^{*})=0$ $C^{*}C-T+(T-AA^{-}C^{*}C)N^{-}N=0$

and

$$F^{*}F-MM^{-}D^{*}D(B^{-})^{*}A^{*}(I-N^{-}N)-MM^{-}F^{*}FN^{-}N-(I-MM^{-})(Y(I-N^{-}N)$$
$$+(I-MM^{-})BA^{-}C^{*}CN^{-}N=0$$

Now we shows the general common solution of adjointable operator equations (2)with same condition above.

Theorem (2.3):

Let $A,B,D,C,E,F\in B(H,K)$ *where* A,B,N,M *have g-inverse operators, then the system adjointable operator equations (2) have a common solution if and only if*

$AA^{-}E^{*}E(B^{-})^{*}B^{*}=E^{*}E$,

$D^{*}D-W-MM^{-}(W-D^{*}D(B^{-})^{*}B^{*})=0$,

$C^*C - T + (T - AA^-C^*C)N^-N = 0,$

$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$

$+ (I - MM^-)BA^-C^*CN^-N = 0,$ *Where;*

$W = BA^-E^*E(B^-)^*B^*$, $T = AA^-E^*E(B^-)^*A^*$,

$Y = BA^-E^*E(B^-)^*A^*$, $M = B(I - A^-A)$,

$N = (I - B^-B)^*A^*$, $Z \in B(K,H)$ *is arbitrary operator.*

Proof:

We clime

$X = X_0 + (I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*$

be a common solution for the system of adjointable operator equations(2), and we substitute in the left side of operator equations;

$BXB^* = B(X_0 + (I - A^-A)(I - M^-M)Z(I - NN^-)(I - B^-B)^*)B^*$

$= W + MM^-W + MM^-D^*D(B^-)^*B^*$ *and By using the condition* $D^*D - W - MM^-(W - D^*D(B^-)^*B^*) = 0$ *we get* $BXB^* = D^*D$

Also

$AXA^* = A(X_0 + (I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*)A^*$

$= T - TN^-N + AA^-C^*CN^-N$ *By using the condition* $C^*C - T + (T - AA^-C^*C)N^-N = 0$ *we get* $AXA^* = C^*C$

And

$AXB^* = AX_0B^* + A((I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*)B^*$

$AA^-E^*E(B^-)^*B^* = E^*E$ *and by the condition get* $AXB^* = E^*E$

Finally

$BXA^* = B(X_0 + (I - A^-A)(I - M^-M)Z(I\_NN^-)(I - B^-B)^*)A^*$

$= Y + MM^-Y + MM^-D^*D(B^-)^*A^* - YN^-N + MM^-YN^-N - MM^-D^*D(B^-)^*A^*N^-N$

$+ BA^-C^*CN^-N - MM^-BA^-C^*CN^-N + MM^-F^*FN^-N$

And By using the condition

$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$

$+ BA^-C^*CN^-N) = 0$ *Can have* $BXA^* = F^*F$

Conversely; let X *is a common solution to the system of operator equation (2),let* $U = \begin{bmatrix} A \\ B \end{bmatrix}, V = \begin{bmatrix} B \\ A \end{bmatrix}, W^*W = \begin{bmatrix} E^*E & C^*C \\ D^*D & F^*F \end{bmatrix}$ *,then the system of operator equations (2) have a common solution if and only if* $UXV^* = W^*W$ *has a solution , and by using thermo (2.1),we get the solution* X *is in the form* $X = U^-W^*W(V^-)^* + (I - U^-U)Z(I - V^-V)^*$ *with condition* $UU^-W^*W(V^-)^*V^* = W^*W$ *and from[5]one can have* $U^- = \begin{bmatrix} A^- - (I - A^-A)M^-BA^- & (I - A^-A)M^- \end{bmatrix}$,

And $V^- = \left[\left((B^-)^* - (B^-)^*A^*N^-(I - B^-B)^* \right)^* \quad \left(N^-(I - B^-B)^* \right)^* \right]$ *therefore get;*

$AA^-E^*E(B^-)^*B^* = E^*E$, $D^*D - W - MM^-(W - D^*D(B^-)^*B^*) = 0$

$C^*C - T + (T - AA^-C^*C)N^-N = 0$ *and*

$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$

$+ (I - MM^-)BA^-C^*CN^-N) = 0.$

Now, to illustrate the above theorem, consider the following example.

Example (2.4):

let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $D^*D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $C^*C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

$E^*E = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$, $F^*F = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $W = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 1.5 & 3 \\ 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} -8 & 4 \\ 0 & 0 \end{bmatrix}$

, $N = \begin{bmatrix} 0.5 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $Z = \begin{bmatrix} 10 & 4 \\ 0 & 3 \end{bmatrix}$.

thus from above theorem we get ;

$$X = A^{-}E^{*}E(B^{-})^{*} + (I - A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*} + (I - A^{-}A)M^{-}D^{*}D(B^{-})^{*} - A^{-}E^{*}E(B^{-})^{*}A^{*}N^{-}(I - B^{-}B)^{*}$$
$$+ (I - A^{-}A)M^{-}BA^{-}E^{*}E(B^{-})^{*}A^{*}N^{-}(I - B^{-}B)^{*} - (I - A^{-}A)M^{-}D^{*}D(B^{-})^{*}A^{*}N^{-}(I - B^{-}B)^{*}$$
$$+ A^{-}C^{*}CN^{-}(I - B^{-}B)^{*} - (I - A^{-}A)M^{-}BA^{-}C^{*}CN^{-}(I - B^{-}B)^{*} + (I - A^{-}A)M^{-}F^{*}FN^{-}(I - B^{-}B)^{*}$$
$$+ (I - A^{-}A)(I - M^{-}M)Z(I \_ NN^{-})(I - B^{-}B)^{*}.$$

$$X = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -0.1 & 0 \\ 0.05 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -0.1 & 0 \\ 0.05 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -0.1 & 0 \\ 0.05 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -0.1 & 0 \\ 0.05 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -0.1 & 0 \\ 0.05 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -0.1 & 0 \\ 0.05 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0.25 & 0.25 \\ 0.7 & 0.7 \end{bmatrix} + \begin{bmatrix} 0.25 & 0.25 \\ 0.7 & 0.7 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Now, we must to show $X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *is satisfy the system of adjontable operator equations (2).*

$$BXB^{*} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = D^{*}D$$
$$AXA^{*} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = C^{*}C$$
$$AXB^{*} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = E^{*}E$$
$$BXA^{*} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = F^{*}F$$

Also, thus operators satisfy the conditions;

$$AA^{-}E^{*}E(B^{-})^{*}B^{*} = E^{*}E$$
$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = E^{*}E$$

$$D^{*}D - W - MM^{-}(W - D^{*}D(B^{-})^{*}B^{*}) = 0$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left[\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right]$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C^{*}C - T + (T - AA^{-}C^{*}C)N^{-}N = 0$$
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \left[\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right]\begin{bmatrix} 0.5 & 1 \\ 0.25 & 0.5 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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$$F^*F - MM^-D^*D(B^-)^*A^*(I - N^-N) - MM^-F^*FN^-N - (I - MM^-)(Y(I - N^-N)$$
$$+ (I - MM^-)BA^-C^*CN^-N = 0$$

$$\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & -1\\-0.25 & 0.5\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 1\\0.25 & 0.5\end{bmatrix}$$
$$- \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1.5 & 3\\0 & 0\end{bmatrix}\begin{bmatrix}0.5 & -1\\-0.25 & 0.5\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}0.5 & 1\\0.25 & 0.5\end{bmatrix}$$
$$= \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

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