

$$\underline{z}(t_{n+1} : r) - \underline{z}(t_n : r) = \sum_{i=1}^4 w_i \underline{l}_i$$

$$\bar{z}(t_{n+1} : r) - \bar{z}(t_n : r) = \sum_{i=1}^4 w_i \bar{l}_i \text{ where } l_i\text{'s are constant}$$

$$[l_i(t, z(t, r))]_r = [l_i(t, z(t, r)), \bar{l}_i(t, z(t, r))] , i = 1, 2, 3, 4 \text{ and } 5$$

$$\underline{k}_1(t, y(t : r)) = hf(t_n, \underline{y}(t_n : r))$$

$$\underline{l}_1(t, z(t : r)) = hf(t_n, \underline{z}(t_n : r))$$

$$\bar{k}_1(t, y(t : r)) = hf(t_n, \bar{y}(t_n : r))$$

$$\bar{l}_1(t, z(t : r)) = hf(t_n, \bar{z}(t_n : r))$$

$$\underline{k}_2(t, y(t : r)) = hf\left(t_n + \frac{h}{3}, \underline{y}(t_n : r) + \frac{1}{3}\underline{k}_1\right)$$

$$\underline{l}_2(t, z(t : r)) = hf\left(t_n + \frac{h}{3}, \underline{z}(t_n : r) + \frac{1}{3}\underline{l}_1\right)$$

$$\bar{k}_2(t, y(t : r)) = hf\left(t_n + \frac{h}{3}, \bar{y}(t_n : r) + \frac{1}{3}\bar{k}_1\right)$$

$$\bar{l}_2(t, z(t : r)) = hf\left(t_n + \frac{h}{3}, \bar{z}(t_n : r) + \frac{1}{3}\bar{l}_1\right)$$

$$\underline{k}_3(t, y(t : r)) = hf\left(t_n + \frac{h}{3}, \underline{y}(t_n : r) + \frac{1}{6}(\underline{k}_1 + \underline{k}_2)\right)$$

$$\underline{l}_3(t, z(t : r)) = hf\left(t_n + \frac{h}{3}, \underline{z}(t_n : r) + \frac{1}{6}(\underline{l}_1 + \underline{l}_2)\right)$$

$$\bar{k}_3(t, y(t : r)) = hf\left(t_n + \frac{h}{3}, \bar{y}(t_n : r) + \frac{1}{6}(\bar{k}_1 + \bar{k}_2)\right)$$

$$\bar{l}_3(t, z(t : r)) = hf\left(t_n + \frac{h}{3}, \bar{z}(t_n : r) + \frac{1}{6}(\bar{l}_1 + \bar{l}_2)\right)$$

$$\underline{k}_4(t, y(t : r)) = hf\left(t_n + \frac{h}{2}, \underline{y}(t_n : r) + \frac{1}{8}(\underline{k}_1 + 3\underline{k}_3)\right)$$

$$\underline{l}_4(t, z(t : r)) = hf\left(t_n + \frac{h}{2}, \underline{z}(t_n : r) + \frac{1}{8}(\underline{l}_1 + 3\underline{l}_3)\right)$$

$$\bar{k}_4(t, y(t : r)) = hf\left(t_n + \frac{h}{2}, \bar{y}(t_n : r) + \frac{1}{8}(\bar{k}_1 + 3\bar{k}_3)\right)$$

$$\bar{l}_4(t, z(t : r)) = hf\left(t_n + \frac{h}{2}, \bar{z}(t_n : r) + \frac{1}{8}(\bar{l}_1 + 3\bar{l}_3)\right)$$

$$\underline{k}_5(t, y(t : r)) = hf\left(t_n + h, \underline{y}(t_n : r) + \frac{1}{2}(\underline{k}_1 - 3\underline{k}_3 + 4\underline{k}_4)\right)$$

$$\underline{l}_5(t, z(t : r)) = hf\left(t_n + h, \underline{z}(t_n : r) + \frac{1}{2}(\underline{l}_1 - 3\underline{l}_3 + 4\underline{l}_4)\right)$$

$$\bar{k}_5(t, y(t : r)) = hf\left(t_n + h, \bar{y}(t_n : r) + \frac{1}{2}(\bar{k}_1 - 3\bar{k}_3 + 4\bar{k}_4)\right)$$

$$\bar{l}_5(t, z(t : r)) = hf\left(t_n + h, \bar{z}(t_n : r) + \frac{1}{2}(\bar{l}_1 - 3\bar{l}_3 + 4\bar{l}_4)\right)$$

$$F(t, y(t : r)) = \underline{k}_1(t, y(t : r)) + 4\underline{k}_4(t, y(t : r)) + \underline{k}_5(t, y(t : r))$$

$$G(t, y(t : r)) = \bar{k}_1(t, y(t : r)) + 4\bar{k}_4(t, y(t : r)) + \bar{k}_5(t, y(t : r))$$

and

$$P(t, z(t : r)) = \underline{l}_1(t, z(t : r)) + 4\underline{l}_4(t, z(t : r)) + \underline{l}_5(t, z(t : r))$$

$$Q(t, z(t : r)) = \bar{l}_1(t, z(t : r)) + 4\bar{l}_4(t, z(t : r)) + \bar{l}_5(t, z(t : r))$$

The exact and approximate solution at $t_n, 0 \leq n \leq N$ are denoted by

$$[Y(t_n)]_r = [Y(t_n : r), \bar{Y}(t_n : r)], \quad [y(t_n)]_r = [y(t_n : r), \bar{y}(t_n : r)] \text{ and}$$

$$[Z(t_n)]_r = [Z(t_n : r), \bar{Z}(t_n : r)], \quad [z(t_n)]_r = [z(t_n : r), \bar{z}(t_n : r)] \text{ respectively.}$$

The solution calculated by the grid points at $a = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_N = b, h = \frac{b-a}{N} = t_{n+1} - t_n$. Therefore we have

$$\underline{Y}(t_{n+1} : r) = \underline{Y}(t_n : r) + \frac{1}{6}F[t_n, \underline{Y}(t_n : r)]$$

$$\bar{Y}(t_{n+1} : r) = \bar{Y}(t_n : r) + \frac{1}{6}G[t_n, \bar{Y}(t_n : r)]$$

$$\underline{Z}(t_{n+1} : r) = \underline{Z}(t_n : r) + \frac{1}{6}P[t_n, \underline{Z}(t_n : r)]$$

$$\bar{Z}(t_{n+1} : r) = \bar{Z}(t_n : r) + \frac{1}{6}Q[t_n, \bar{Z}(t_n : r)]$$

$$\underline{y}(t_{n+1} : r) = \underline{y}(t_n : r) + \frac{1}{6}F[t_n, \underline{y}(t_n : r)]$$

$$\bar{y}(t_{n+1} : r) = \bar{y}(t_n : r) + \frac{1}{6}G[t_n, \bar{y}(t_n : r)]$$

$$\underline{z}(t_{n+1} : r) = \underline{z}(t_n : r) + \frac{1}{6}P[t_n, \underline{z}(t_n : r)]$$

$$\bar{z}(t_{n+1} : r) = \bar{z}(t_n : r) + \frac{1}{6}Q[t_n, \bar{z}(t_n : r)]$$

To show the convergence of these approximation

$$\lim_{h \rightarrow 0} \underline{y}(t : r) = \underline{Y}(t : r), \quad \lim_{h \rightarrow 0} \bar{y}(t : r) = \bar{Y}(t : r),$$

$$\lim_{h \rightarrow 0} \underline{z}(t : r) = \underline{Z}(t : r), \quad \lim_{h \rightarrow 0} \bar{z}(t : r) = \bar{Z}(t : r)$$

Lemma 4.1:

Let a sequence of numbers $\{W_n\}_{n=0}^N$ satisfies $|W_{n+1}| \leq A|W_n| + B$, $0 \leq n \leq N-1$ for some given positive constants A and B, then $|W_n| \leq A^n |W_0| + B \frac{A^n - 1}{A - 1}$, $0 \leq n \leq N-1$. [11]

Lemma 4.2:

Let a sequence of numbers $\{W_n\}_{n=0}^N$ and $\{V_n\}_{n=0}^N$ satisfies the conditions

$$|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B$$

$$\text{and } |V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B$$

for some given positive constants A and B and denote $U_n = |W_n| + |V_n|$, $0 \leq n \leq N$ then

$$U_n \leq \bar{A}^n U_0 + \bar{B} \frac{\bar{A}^n - 1}{\bar{A} - 1}, 1 \leq n \leq N$$

where $\bar{A} = 1 + 2A$ and $\bar{B} = 2B$

Proof: Consider,

$$|W_{n+1}| + |V_{n+1}| \leq |W_n| + |V_n| + 2A \{|W_n| + |V_n|\} + 2B$$

$$= (1 + 2A) \{|W_n| + |V_n|\} + 2B$$

By lemma 4.1, for W_n , $0 \leq n \leq N$ hence it is valid.

Theorem - 4.1

Let $F(t, u, v)$ & $G(t, u, v)$ belong to $C^1(K)$ and let the partial derivatives of F & G be bounded over K , then for arbitrary fixed value r , $0 \leq r \leq 1$ are approximate solutions converge to the exact solutions of $\underline{Y}(t_n : r)$ and $\bar{Y}(t_n : r)$ uniformly in t . Similarly, $P(t, u, v)$ & $Q(t, u, v)$ belong to $C^1(K)$ and let the partial derivatives of P and Q be bounded over K , then for arbitrary fixed value r , $0 \leq r \leq 1$ are approximate solutions converge to the exact solutions of $\underline{Z}(t_n : r)$ and $\bar{Z}(t_n : r)$ uniformly in t .

6. Numerical Example

Consider $\frac{dy}{dt} = -4y + 5z$ and $\frac{dz}{dt} = 8y - 6z$ with an initial conditions $y(0) = (7.6 + 0.4r, 8.2 - 0.2r)$ and $z(0) = (3.8 + 0.2r, 4.3 - 0.3r)$; $0 \leq r \leq 1$.

Solution:

The exact solution is $\underline{Y}(t : r) = 3\underline{y}_1(t : r)e^t + 5\underline{y}_2(t : r)e^{-3t}$,

$$\underline{Z}(t : r) = 3\underline{z}_1(t : r)e^t + \underline{z}_2(t : r)e^{-3t}$$

when $t = 1$ then the exact solution is given by,

$$\underline{Y}(1 : r) = (2.75 + 0.25r)e + 5(0.8 + 0.2r)e^{-3}$$

$$\text{and } \bar{Y}(1 : r) = (3.25 - 0.25r)e + (1.2 - 0.2r)e^{-3}$$

$$\underline{Z}(1 : r) = (2.75 + 0.25r)e + (0.8 + 0.2r)e^{-3}$$

$$\text{and } \bar{Z}(1 : r) = (3.25 - 0.25r)e + (1.2 - 0.2r)e^{-3}$$

The exact and approximate solutions obtained by the fifth order Runge-Kutta Merson method with initial condition for taking $k = 0.1$

Table 5.1

Exact Solution with h = 0.1				
r	[Y Y]		[Z Z]	
0.0	7.6744233017	9.1331383527	7.5151046830	8.8941604245
0.1	7.7473590543	9.0602026002	7.5840574700	8.8252076375
0.2	7.8202948068	8.9872668476	7.6530102571	8.7562548504
0.3	7.8932305594	8.9143310951	7.7219630442	8.6873020633
0.4	7.9661663119	8.8413953425	7.7909158313	8.6183492762
0.5	8.0391020645	8.7684595900	7.8598686184	8.5493964891
0.6	8.1120378170	8.6955238374	7.9288214054	8.4804437021
0.7	8.1849735696	8.6225880849	7.9977741925	8.4114909150
0.8	8.2579093221	8.5496523323	8.0667269796	8.3425381279
0.9	8.3308450747	8.4767165798	8.1356797667	8.2735853408
1.0	8.4037808272	8.4037808272	8.2046325537	8.2046325537

Table 5.2

Approximation Solution by fifth order Runge-Kutta Merson with h = 0.1				
r	[y y]		[z z]	
0.0	7.9835987091	9.2810068130	7.7944021225	9.0868310928
0.1	8.0256175995	9.1932840347	7.8354244232	8.9986104965
0.2	8.0676355362	9.1055622101	7.8764467239	8.9103908539
0.3	8.1096553802	9.0178413391	7.9174709320	8.8221721649
0.4	8.1516742706	8.9301195145	7.9584946632	8.7339515686
0.5	8.1936931610	8.8423986435	7.9995169640	8.6457328796
0.6	8.2357130051	8.7546758652	8.0405406952	8.5575122833
0.7	8.2777309418	8.6669540405	8.0815629959	8.4692935944
0.8	8.3197507858	8.5792322159	8.1225881577	8.3810720444
0.9	8.3617696762	8.4915103912	8.1636104584	8.2928533554
1.0	8.4037885666	8.4037885666	8.2046337128	8.2046337128

Table 5.3

Complete Error Analysis with h = 0.1				
r	[y y]		[z z]	
0.0	0.3091754074	0.1478684603	0.2792974395	0.1926706683
0.1	0.2782585452	0.1330814345	0.2513669532	0.1734028590
0.2	0.2473407294	0.1182953625	0.2234364668	0.1541360035
0.3	0.2164248208	0.1035102440	0.1955078878	0.1348701016
0.4	0.1855079587	0.0887241720	0.1675788319	0.1156022924
0.5	0.1545910965	0.0739390535	0.1396483456	0.0963363905
0.6	0.1236751881	0.0591520278	0.1117192898	0.0770685812
0.7	0.0927573722	0.0443659556	0.0837888034	0.0578026794
0.8	0.0618414637	0.0295798836	0.0558611781	0.0385339165
0.9	0.0309246015	0.0147938114	0.0279306917	0.0192680146
1.0	0.0000077394	0.0000077394	0.0000011591	0.0000011591

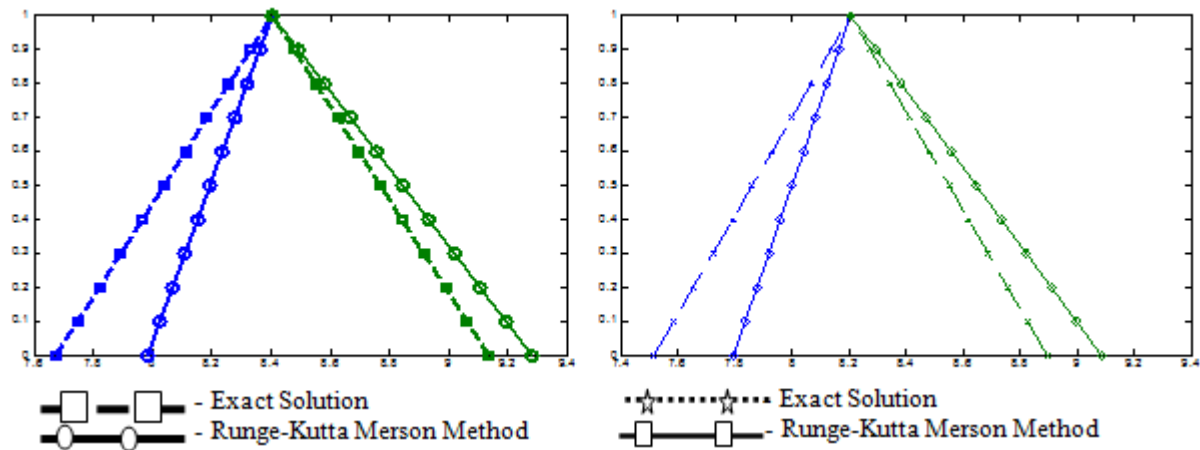


Figure 5.1

7. Conclusion

In this paper we have found the iterative solution of first order simultaneous fuzzy differential equation using fifth order Runge-Kutta Merson method.

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