





$$= a^{-\alpha} b^{-\beta} \sum_{d=0}^{\infty} \frac{\prod_{j=1}^r \Gamma(m_j + M_j d) z^d}{\prod_{j=1}^s \Gamma(n_j + N_j d) d!} \times$$

$$H_{p_1+Q, Q+1; (p, q); (u, v)}^{0, N+2; (m, n); (g, h)} \left[ \begin{matrix} z_1 \\ z_2 \end{matrix} \left| \begin{matrix} (1-\alpha-d\rho, \lambda_1, \lambda_2), (1-\beta-d\sigma; \mu_1, \mu_2), ((a_p, \alpha_p, A_p)); ((c_p, C_p)); ((e_u, E_u)) \\ (b_2, \beta_2, B_2), (1-\alpha-\beta-d\rho-d\sigma; \lambda_1 + \mu_1, \lambda_2 + \mu_2); ((d_q, D_q)); ((f_v, F_v)) \end{matrix} \right. \right]$$

(2.6)

provided

$$\rho, \sigma, \lambda_1, \lambda_2, \mu_1, \mu_2 > 0 \quad \text{Re}\left(\alpha + \frac{\lambda_1 d_j}{D_j} + \frac{\lambda_2 f_k}{F_k}\right) > 0 \quad \text{Re}\left(\beta + \frac{\mu_1 d_j}{D_j} + \frac{\mu_2 f_k}{F_k}\right) > 0$$

(j = 1, …, m) and (k = 1, …, g)

$$1 + \sum_1^s N_j - \sum_1^r M_j > 0 \quad \left| \arg z_1 \right| < \frac{1}{2} \pi \Delta_1, \quad \left| \arg z_2 \right| < \frac{1}{2} \pi \Delta_2, \quad \Delta_1, \Delta_2 > 0$$

$\Delta_1, \Delta_2$  is defined by (1.6) and (1.7) respectively.

$H\{x, y\}$  defined by (1.1) and  $ax + b(1-x)$  is not zero for  $0 \leq x \leq 1, a, b \neq 0$

**Buschman, Kurl and Gupta [1]**

$$L\left\{ (x-1)^{\sigma-1} H_{p_1, q_1; p_2, q_2+1; p_3, q_3+1}^{0, 0; 1, n_2, 1, n_3} \left[ \begin{matrix} x-t \\ x-t \end{matrix} \left| \begin{matrix} (a_j, \alpha_j, A_j); (c_j, C_j); (e_j, E_j) \\ (b_j, \beta_j, B_j); (0, 1), (d_j, D_j); (0, 1), (f_j, F_j) \end{matrix} \right. \right] \right\}$$

$$= p^{-\sigma} H_2[p^{-1}, p^{-1}] \quad (2.7)$$

Provided  $\text{Re}(p) > 0, \quad \text{Re}\left(\sigma + \frac{d_j}{D_j} + \frac{f_k}{F_k}\right) > 0,$

Where  $= H_2[p^{-1}, p^{-1}] =$

$$H_{p_1+1, q_1; p_2, q_2+1; p_3, q_3+1}^{0, 1; 1, n_2, 1, n_3} \left[ \begin{matrix} p^{-1} \\ p^{-1} \end{matrix} \left| \begin{matrix} (1-\sigma, 1, 1), (a_j, \alpha_j, A_j); (c_j, C_j); (e_j, E_j) \\ (b_j, \beta_j, B_j); (0, 1), (d_j, D_j); (0, 1), (f_j, F_j) \end{matrix} \right. \right]$$

$$= \sum_{M, N=0}^{\infty} C_{M, N} (-p)^{-M-N} \Gamma(\sigma + M + N) / M! N! \quad (2.8)$$

Where  $C_{M, N} = \phi(N, M) \theta_2'(N) \theta_3'(M) \quad (2.9)$

$$H_2[p^{-1}, p^{-1}] = \sum_{v=0}^{\infty} h_v p^{-v} \quad (2.10)$$

Where  $h_v = (-1)^v \Gamma(\sigma + v) / v! \sum_{\mu=0}^v \binom{v}{\mu} C_{v-\mu, \mu} \quad (2.11)$

If k denotes the least value of v for which  $h_v \neq 0$ , then

$$H_2[p^{-1}, p^{-1}] = p^{-k} \sum_{n=0}^{\infty} h_{k+n} p^{-n} \quad (2.12)$$

So that if we let the coefficients  $H_\lambda$  be determined by the relation

$$\left[ \sum_{n=0}^{\infty} h_{k+n} p^{-n} \right]^{-1} = \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda} \quad (2.13)$$

$$t^{-p-k-\sigma-1} \sum_{\lambda=0}^{\infty} H_{\lambda} t^{\lambda} / \Gamma(\rho-k+\lambda-\sigma) \doteq p^{-(\rho-k-\sigma)} \left[ \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda} \right] \quad (2.14)$$

The coefficient  $H_{\lambda}$  being defined by the recurrences

$$h_k H_0 = 1 \text{ and for } \mu > 0 \text{ by } \sum_{\lambda=0}^{\mu} H_{\lambda} h_{\mu+k-\lambda} = 0 \text{ and the power series coefficients } h_{\nu} \text{ is given by} \quad (2.11).$$

### 3. The Integral Equation.

Theorem.

If  $2 > b > 0$ ,  $\text{Re}(1 + \alpha + a) > 0$ ,

$$AB\Gamma(d)\Gamma(-d) = 1, \quad a + c + 2 = m + l, \quad f^q(0) = 0, \quad 0 \leq q < m;$$

$g^s(0) = 0, \quad 0 \leq s < h$ ;  $m, l, h$  are integers and  $\text{Re}(h - l - k - \alpha + c + 1) > 0$ , then the integral equation:

$$g(x) = A \sum_{r=0}^{\infty} \frac{\Gamma(d+r) z^r}{r!} \int_0^x (x-t)^{a+\alpha+br} H_{p_1+1, q_1+1; p_2, q_2+1; p_3, q_3+1}^{0,1; 1, n_2; 1, n_3} \left[ \begin{array}{l} x-t \\ x-t \end{array} \left| \begin{array}{l} (1-\alpha, 1, 1), (a_j, \alpha_j, A_j); (c_j, C_j); (e_j, E_j) \\ (b_j, \beta_j, B_j), (-a-\alpha-br, 1, 1); (0, 1), (d_j, D_j); (0, 1), (f_j, F_j) \end{array} \right. \right] [D^m f(t)] dt. \quad (3.1)$$

Where the H- function of two variables involved in (3.1) satisfies all the conditions corresponding appropriately to the set of convergence conditions (1.5) to (1.7), the solution of the integral

equation is given by

$$f(t) = B \sum_{r=0}^{\infty} \Gamma(-d+r) \frac{z^r}{r!} \int_0^t (t-x)^{h-1+c+br-k-\alpha} V(t-x) [D^l g(x)] dx \quad (3.2)$$

where

$$V(x) = \sum_{\lambda=0}^{\infty} \frac{H_{\lambda} x^{\lambda}}{\Gamma(br+c+\lambda+h-k-\alpha)} \quad (3.3)$$

The coefficients

$$H_{\lambda} \text{ being defined by the recurrences } H_k h_0 = 1 \text{ and for } \mu > 0 \text{ by} \quad \sum_{\lambda=0}^{\infty} H_{\lambda} h_{\mu+k-\lambda} = 0 \text{ and the power series coefficients } h_{\nu} \text{ given by (2.10).}$$

**PROOF:**

Let  $f(t) \stackrel{\cdot}{=} F(p)$  and  $g(t) \stackrel{\cdot}{=} G(p)$  then

From (2.5), (2.7) and (2.11)

$$t^a {}_1\psi_1[(d,1), (a+1,b); zt^b] \stackrel{\cdot}{=} \Gamma(d) p^{-1-a} (1-zp^{-b})^d \quad (3.4)$$

$$t^c {}_1\psi_1[(-d,1), (c+1,b); zt^b] \stackrel{\cdot}{=} \Gamma(-d) p^{-1-c} (1-zp^{-b})^{-d} \quad (3.5)$$

$$t^{\alpha-1} H_{p_1, q_1; p_2, q_2+1; p_3, q_3+1}^{0,0; 1, n_2; 1, n_3} \left[ t \left| \begin{array}{l} ((a_j, \alpha_j, A_j)); ((c_j, C_j)); ((e_j, E_j)) \\ ((b_j, \beta_j, B_j)); (0,1), ((d_j, D_j)); (0,1), ((f_j, F_j)) \end{array} \right. \right] \stackrel{\cdot}{=} p^{-\alpha-k} \sum_{n=0}^{\infty} h_{k+n} p^{-n} \quad (3.6)$$

Using (2.3) in (3.4) and (3.6)

$$\Gamma(d) p^{-1-a} (1-zp^{-b})^d p^{-\alpha-k} \left[ \sum_{n=0}^{\infty} h_{k+n} p^{-n} \right] \stackrel{\cdot}{=} \int_0^x t^a {}_1\psi_1[(d,1), (a+1,b); zt^b] (x-t)^{\alpha-1} \times H_{p_1, q_1; p_2, q_2+1; p_3, q_3+1}^{0,0; 1, n_2; 1, n_3} \left[ x-t \left| \begin{array}{l} ((a_j, \alpha_j, A_j)); ((c_j, C_j)); ((e_j, E_j)) \\ ((b_j, \beta_j, B_j)); (0,1), ((d_j, D_j)); (0,1), ((f_j, F_j)) \end{array} \right. \right] dt \quad (3.7)$$

In (3.7) put  $t=ux$  and evaluate using (2.6) to get:

$$\Gamma(d) p^{-1-a-\alpha-k} (1-zp^{-b})^d \left[ \sum_{n=0}^{\infty} h_{k+n} p^{-n} \right] \stackrel{\cdot}{=} \sum_{r=0}^{\infty} \frac{(d+r)z^r}{r!} x^{a+\alpha+br} \times H_{p_1+1, q_1+1; p_2, q_2+1; p_3, q_3+1}^{0,1; 1, n_2; 1, n_3} \left[ x \left| \begin{array}{l} (1-\alpha, 1, 1), ((a_j, \alpha_j, A_j)); ((c_j, C_j)); ((e_j, E_j)) \\ ((b_j, \beta_j, B_j), (-a-\alpha-br, 1, 1)); (0,1), ((d_j, D_j)); (0,1), ((f_j, F_j)) \end{array} \right. \right] \quad (3.8)$$

From (2.14)

$$p^{-(k-\alpha-1+h)} \left[ \sum_{n=0}^{\infty} H_{\lambda} p^{-\lambda} \right] \stackrel{\cdot}{=} t^{h-1-k-\alpha} \sum_{\lambda=0}^{\infty} \frac{H_{\lambda} t^{\lambda}}{\Gamma(h-1-k+\lambda+\alpha)} \quad (3.9)$$

Now using (2.3) in (3.5) and (3.9), to get:

$$\Gamma(-d)p^{-1-c}(1-zp^{-b})^d p^{k+\alpha+1-h} \left[ \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda} \right] = \int_0^x t^c {}_1\Psi_1[(-d,1), (c+1,b); zt^b](x-t)^{h-1-k-\alpha-1} \sum_{\lambda=0}^{\infty} \frac{H_{\lambda}(x-t)^{\lambda}}{\Gamma(h-1-k+\lambda-\alpha)} dt \quad (3.10)$$

In (3.10) put  $t = ux$  and evaluate using (2.14), to get:

$$\Gamma(-d)p^{-1-c+k-\alpha-h}(1-zp^{-b})^{-d} \left[ \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda} \right] = \sum_{r=0}^{\infty} \frac{\Gamma(-d+r)z^r}{r!} x^{h-1+c+br-k-\alpha} \left[ \sum_{\lambda=0}^{\infty} \frac{H_{\lambda} x^{\lambda}}{\Gamma(br+c+\lambda+h-k-\alpha)} \right] \quad (3.11)$$

Using (2.3) and (3.8), the integral equation (3.1) becomes

$$G(p) = A\Gamma(d)p^{m-1-a-\alpha-k}(1-zp^{-b})^d \left[ \sum_{n=0}^{\infty} H_{k+n} p^{-n} \right] F(p) \quad (3.12)$$

Similarly using (2.3) and (3.10), the integral equation (3.2) becomes

$$F(p) = B\Gamma(-d)p^{l-1-c+\alpha+k}(1-zp^{-b})^{-d} \left[ \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda} \right] G(p) \quad (3.13)$$

The equations (3.12) and (3.13) can be obtained from each other when

$$AB\Gamma(-d)\Gamma(d) = 1 \quad \text{and} \quad m+1 = a+c+2.$$

Hence by LERCH'S theorem [4,p.5], it follows that each of the integral equation (3.1) and (3.2) is the solution of the other.

#### 4. Special Cases

In the theorem use [8,pp.88,89, eqn (6.4.3)] to get :

The integral equation

$$g(x) = A \sum_{r=0}^{\infty} \left[ \frac{\Gamma(d+r)\Gamma(\alpha)}{\Gamma(a+\alpha+br+1)} \frac{z^r}{r!} \int_0^t (x-t)^{a+\alpha+br} \times F_{q_1+1; q_2; q_3}^{p_1+1; p_2; p_3} \left[ \begin{matrix} x \\ x \end{matrix} \middle| \begin{matrix} \alpha, (a_{p_1}) : (c_{p_2}); (e_{p_3}); \\ (a+\alpha+br+1) : (0,1), (b_{q_1}); (d_{q_2}); (f_{q_3}) \end{matrix} \right. \right. \\ \left. \left. (-1)^{p_1+p_2-n_2+2} (x-t), (-1)^{p_1+p_3-n_3+2} (x-t) \right] \times [D^m f(t)] dt \right. \\ \left. = \prod_{j=1}^{p_1} \Gamma(a_j) \prod_{j=n_2+1}^{p_2} \Gamma(c_j) \prod_{j=n+1}^{p_3} \Gamma(e_j) \prod_{j=1}^{q_1} \Gamma(1-b_j) \prod_{j=1}^{q_2} \Gamma(1-d_j) \prod_{j=1}^{q_3} \Gamma(1-f_j) \right]$$

Where

$$2 > b > 0, \quad \text{Re}(1+\alpha+a) > 0, \quad \text{Re}(a_j) > 0, (j=1, \dots, p_1)$$

$$, \quad \text{Re}(c_j) > 0, (j=1, \dots, p_2)$$

$$, \quad \text{Re}(e_j) > 0, (j=1, \dots, p_3), \quad \text{Re}(1-b_j) > 0, (j=1, \dots, q_1), \quad \text{Re}(1-d_j) > 0, (j=1, \dots, q_2)$$

$$, \operatorname{Re}(1-f_j) > 0, (j=1, \dots, q_3) \quad AB\Gamma(d)\Gamma(-d) = 1, a+c+2 = m+1 \quad f^q(0) = 0, 0 \leq q \leq m$$

$gs(0) = 0, 0 \leq s \leq h$   $m, l, h$  are integers and  $\operatorname{Re}(h-l-k-\alpha+c+1) > 0$  has for its solution.

$$f(t) = B \sum_{r=0}^{\infty} \left[ \frac{\Gamma(-d+r)\Gamma(\alpha)z^r}{r!} \int_0^t (t-x)^{h-1+cr+lr-k-\alpha} W(t-x) [D^h g(x)] dx \right]$$

Where

$$W(x) = \sum_{\lambda=0}^{\infty} \left[ \frac{H_{\lambda} x^{\lambda}}{\Gamma(c+\lambda+br+h+l-1-k-\alpha)} \right]$$

Where

$H_{\lambda}$  being determined by (3.4) and  $h_{\nu}$  being given by (2.11) and the coefficients in (2.9) reducing to the form  $c_{v-\mu, \mu} = \theta_2'(v-\mu)\theta_3'(\mu)$ .

In (3.1) and (3.2) put  $z = m = l = 0, A = B = 1, a = c = -1, \alpha = \sigma, h = \rho$  to get the result given by Buschman, Kurl and Gupta[8, pp.226-228]

In (3.14) put  $z = m = l = 0, A = B = 1, a = c = -1, \alpha = \sigma, h = \rho$   
 $p_1 = q_1 = 0, (C_j) = (D_j) = (E_j) = (F_j) = 1, n_2 = n_3 = p_2 = p_3 = q_2 = q_3 = 1,$   
 $c_1 = 1 - a_1, d_1 = 1 - b_1, e_1 = 1 - a_2, f_1 = 1 - b_2$  and use the result [8, p.161] to get another result given by Buschman, Kurl and Gupta[8, p.230].

### References

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