International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2013): 4.438

An Integral Equation Involving H- Function of Two Variables as Its Kernel

Dr. P. Prasanth

Governemnt HSS, Ramanthali, Kannur, Kerala, India

Abstract: ***The object of this paper is to solve an integral equation of convolution form having the H- function of two variables as its kernel. It generalizes the result given by Buschman, Kurl and Gupta [8, pp. 226-228] .A few other special cases are also given.***

1. Introduction

The H- function of two variables defined by Mittal and Gupta [5, p.117] is represented here in contracted form as

$$H[X,Y] = H^{0N:m,n;g,h}_{P,Q:p,q;u,v}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_P,\alpha_P,A_P):(c_p,C_p);(e_u,E_u) \\ (b_Q,\beta_Q,B_Q):(d_q,D_q);(f_v,F_v) \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\phi_1(s)\phi_2(t)\psi(s,t)x^s y^t dsdt \quad (1.1)$$

Where $\psi(s,t) = \dfrac{\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=1}^{Q}\Gamma(1-b_j+\beta_j s+B_j t)\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j s-A_j t)}$ (1.2.)

$$\phi_1(s) = \frac{\prod_{j=1}^{m}\Gamma(d_j-D_j s)\prod_{j=1}^{n}\Gamma(1-c_j+C_j s)}{\prod_{j=m+1}^{q}\Gamma(1-d_j+D_j s)\prod_{j=n+1}^{p}\Gamma(c_j-C_j s)} \quad (1.3)$$

$$\phi_2(t) = \frac{\prod_{j=1}^{g}\Gamma(f_j-F_j t)\prod_{j=1}^{h}\Gamma(1-e_j+E_j t)}{\prod_{j=g+1}^{v}\Gamma(1-f_j+F_j t)\prod_{j=h+1}^{u}\Gamma(e_j-E_j t)} \quad (1.4)$$

Where $x, y \neq 0$, an empty product is intyerpreted as unity.

N,P,Q,m,n,p,q,g,h,u,v are all non-negetive integers such that

$0 \le N \le P, Q \ge 0, 0 \le m \le q, o \le n \le p, 0 \le g \le v, 0 \le h \le u$ and $\alpha_j,\beta_j,A_j,B_j,C_j,D_j,E_j,F_j$ are all

positive. The sequence of the parameters $(a_p),(b_q),(c_p),(d_q),(e_u),(f_v)$ are so resricted that none

of the poles of the integrand coincide.

The double integrand in (1.2.1) converges absolutely if

$$|\arg x| < \frac{1}{2}\Delta_1\pi, \quad |\arg y| < \frac{1}{2}\Delta_2\pi \quad (1.5)$$

Where $\Delta_1 = \sum_{j=1}^{N}(\alpha_j) - \sum_{j=N+1}^{P}(\alpha_j) - \sum_{j=1}^{Q}(\beta_j) + \sum_{j=1}^{m}(D_j) - \sum_{j=m+1}^{q}(D_j) + \sum_{j=1}^{n}(C_j) - \sum_{j=n+1}^{p}(C_j) > 0$ (1.6)

$$\Delta_1 = \sum_{j=1}^{N}(A_j) - \sum_{j=N+1}^{P}(A_j) - \sum_{j=1}^{Q}(B_j) + \sum_{j=1}^{g}(F_j) - \sum_{j=g+1}^{v}(F_j) + \sum_{j=1}^{h}(E_j) - \sum_{j=h+1}^{u}(E_j) > 0 \quad (1.7)$$

2. Notations and Results Used

The Lplace transform

$$F(p) = \int_0^\infty e^{-pt} f(t)dt, \quad \mathrm{Re}(p) > 0$$

is represented by $F(p) \doteq f(t)$ (2.1)

ERDELYI [2,p.129-131]

If $f(t) \doteq F(p), \quad f(0) = f^{/}(0) = \ldots\ldots.. f^{n-1}(0) = 0$ and $f^n(t)$ is continuous , then

$$f^n(t) \doteq p^n F(p) \quad (2.2)$$

If $f_1(t) \doteq F_1(p)$ and $f_2(t) \doteq F_2(p)$, then $\int_0^t f_1(u) f_2(t-u)du \doteq F_1(p)F_2(p)$ (2.3)

SRIVASTAVA[8, P.19.]

$$H_{p,q+1}^{1,p}\left[-x\left|\begin{matrix}(1-a_j, A_j) \\ (0,1),(1-b_j, B_j)\end{matrix}\right.\right] = {}_p\psi_q\left[\begin{matrix}(a_j, A_j) \\ (b_j, B_j)\end{matrix} x\right] = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j + A_j r)x^r}{\prod_{j=1}^{q}\Gamma(b_j + B_j r)r!} \quad (2.4)$$

NAIR[7,p.10]

$$t^\alpha {}_1\psi_1\left[(d,1);(1+\alpha,b);-ct^b\right] \doteq \Gamma(d)p^{-1-\alpha}(1+cp^{-b})^{-d} \quad (2.5)$$

Provided $\mathrm{Re}(p) > 0, 2 > b > 0, \mathrm{Re}(1+\alpha) > 0, \quad \left|\arg cp^{-b}\right| < \frac{\pi(2-b)}{2}$.

MUHAMMED [6,P.109]

$$\int_0^x x^{\alpha-1}(1-x)^{\beta-1}[(ax+b)(1-x)^{-\alpha-\beta} {}_r\psi_s[(m_r, M_r);(n_s, N_s); z(ax)^\rho \{b(1-x)^\sigma\}\{(ax+b)(1-x)^{-\rho-\varsigma}\}]$$

$$\times H[z_1(ax)^{\lambda_1}\{b(1-x)\}^{\mu_1}\{(ax+b)(1-x)^{\lambda_1-\mu_1}\}, z_2(ax)^{\lambda_2}\{b(1-x)^{\mu_2}\}\{(ax+b)(1-x)^{-\lambda_2-\mu_2}\}]dx$$

$$= a^{-\alpha} b^{-\beta} \sum_{d=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(m_j + M_j d) z^d}{\prod_{j=1}^{s} \Gamma(n_j + N_j d) d!} \times$$

$$H^{0,N+2:(m,n);(g,h)}_{p_1+Q,Q+1_1:(p,q);(u,v)} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (1-\alpha-d\rho,\lambda_1,\lambda_2),(1-\beta-d\sigma;\mu_1,\mu_2),((a_p,\alpha_p,A_p));((c_p,C_p));((e_u,E_u)) \\ (b_2,\beta_2,B_2),(1-\alpha-\beta-d\rho-d\sigma;\lambda_1+\mu_1,\lambda_2+\mu_2):((d_q D_q));((f_v,F_v)) \end{matrix} \right]$$

(2.6)

provided

$$\rho,\sigma,\lambda_1,\lambda_2,\mu_1,\mu_2 > 0 \quad \mathrm{Re}(\alpha + \frac{\lambda_1 d_j}{D_j} + \frac{\lambda_2 f_k}{F_k} > 0 \quad \mathrm{Re}(\beta + \frac{\mu_1 d_j}{D_j} + \frac{\mu_2 f_k}{F_k} > 0,$$

(j = 1,…….,m) and (k = 1,……,g)

$$1 + \sum_1^s N_j - \sum_1^r M_j > 0, \quad |\arg z_1| < \frac{1}{2}\pi\Delta_1, \quad |\arg z_2| < \frac{1}{2}\pi\Delta_2, \quad \Delta_1, \Delta_2 > 0$$

Δ_1, Δ_2 is defined by (1.6) and (1.7) respectively.

$H\{x,y\}$ defined by (1.1)and $ax + b(1-x)$ is not zero for $0 \le x \le 1, \quad a,b \ne 0$

Buschman, Kurl and Gupta [1]

$$L\left\{(x-1)^{\sigma-1} H^{0,0:\ 1,n_2\ 1,n_3}_{p_1,q_1:p_2,q_2+1;p_3,q_3+1} \left[\begin{matrix} x-t \\ x-t \end{matrix} \middle| \begin{matrix} (a_j,\alpha_j,A_j);(c_j,C_j);(e_j,E_j) \\ (b_j,\beta_j,B_j):(0,1),(d_j,D_j);(0,1),(f_j,F_j) \end{matrix} \right] \right\}$$

$$= p^{-\sigma} H_2[p^{-1}, p^{-1}] \quad (2.7)$$

Provided $\mathrm{Re}(p) > 0, \quad \mathrm{Re}(\sigma + \frac{d_j}{D_j} + \frac{f_k}{F_k}) > 0,$

Where $= H_2[p^{-1}, p^{-1}] =$

$$H^{0,1:\ 1,n_2\ 1,n_3}_{p_1+1,q_1:p_2,q_2+1;p_3,q_3+1} \left[\begin{matrix} p^{-1} \\ p^{-1} \end{matrix} \middle| \begin{matrix} (1-\sigma,1,1),(a_j,\alpha_j,A_j);(c_j,C_j);(e_j,E_j) \\ (b_j,\beta_j,B_j):(0,1),(d_j,D_j);(0,1),(f_j,F_j) \end{matrix} \right]$$

$$= \sum_{M,N=0}^{\infty} C_{M,N} (-p)^{-M-N} \Gamma(\sigma + M + N)/M!N! \quad (2.8)$$

Where $C_{M,N} = \phi(N,M)\theta_2^{/}(N)\theta_3^{/}(M)$ (2.9)

$$H_2[p^{-1}, p^{-1}] = \sum_{v=0}^{\infty} h_v p^{-v} \quad (2.10)$$

Where $h_v = (-1)^v \Gamma(\sigma + v)/v! \sum_{\mu=0}^{v} \binom{v}{\mu} C_{v-\mu,\mu}$ (2.11)

If k denotes the least value of v for which $h_v \ne 0$, then

$$H_2[p^{-1}, p^{-1}] = p^{-k} \sum_{n=0}^{\infty} h_{k+n} p^{-n} \quad (2.12)$$

So that if we let the coeffiecients H_λ be determined by the relation

$$[\sum_{n=0}^{\infty} h_{k+n} p^{-n}]^{-1} = \sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda} \quad (2.13)$$

$$t^{-p-k-\sigma-1} \sum_{\lambda=0}^{\infty} H_\lambda t^\lambda / \Gamma(\rho - k + \lambda - \sigma) \doteq p^{-(\rho-k-\sigma)} \left[\sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda} \right] \quad (2.14)$$

The coeffiecient H_λ being defined by the recurrences

$h_k H_0 = 1$ and for $\mu > 0$ by $\sum_{\lambda=0}^{\mu} H_\lambda h_{\mu+k-\lambda} = 0$ and the power series coeffiecients h_ν is given by (2.11).

3. The Integral Equation.

Theorem.

If $2 > b > 0, \quad \mathrm{Re}(1+\alpha+a) > 0$,

$$AB\Gamma(d)\Gamma(-d) = 1, \quad a+c+2 = m+l, \quad f^q(0) = 0, \quad 0 \le q < m;$$

$g^s(0) = 0, \quad 0 \le s < h; \quad m, l, h$ are integers and $\mathrm{Re}(h-l-k-\alpha+c+1) > 0$, then the integral equation:

$$g(x) = A \sum_{r=0}^{\infty} \frac{\Gamma(d+r) z^r}{r!} \int_0^x (x-t)^{a+\alpha+br} H^{0,1:1,n_2;1,n_3}_{p_1+1,q_1+1:p_2,q_2+1;p_3,q_3+1}$$

$$\left[\begin{matrix} x-t \\ x-t \end{matrix} \middle| \begin{matrix} (1-\alpha,1,1),(a_j,\alpha_j,A_j):(c_j,C_j);(e_j,E_j) \\ (b_j,\beta_j,B_j),(-a-\alpha-br,1,1):(0,1),(d_j,D_j);(0,1),(f_j,F_j) \end{matrix} \right] [D^m f(t) dt. \quad (3.1)$$

Where the H- function of two variables involved in (3.1) satisfies all the conditions corresponding appropriately to the set oif convergence conditions (1.5) to (1.7) , the solution of the integral

equation is given by

$$f(t) = B \sum_{r=0}^{\infty} \Gamma(-d+r) \frac{z^r}{r!} \int_0^t (t-x)^{h-1+c+br-k-\alpha} V(t-x)[D^l g(x)] dx \quad (3.2)$$

where

$$V(x) = \sum_{\lambda=0}^{\infty} \frac{H_\lambda x^\lambda}{\Gamma(br+c+\lambda+h-k-\alpha)} \quad (3.3)$$

The coefficients

H_λ being defined by the recurrences $H_k h_0 = 1$ and for $\mu > 0$ by

$\sum_{\lambda=0}^{\infty} H_\lambda h_{\mu+k-\lambda} = 0$ and the power series coefficients h_ν given by (2.10).

PROOF:

Let $f(t) \doteq F(p)$ and $g(t) \doteq G(p)$ then

From (2.5) , (2.7) and (2.11)

$$t^{a}\,{}_1\psi_1[(d,1),(a+1,b); zt^{b}] \doteq \Gamma(d)\,p^{-1-a}(1-zp^{-b})^{d} \quad (3.4)$$

$$t^{c}\,{}_1\psi_1[(-d,1),(c+1,b); zt^{b}] \doteq \Gamma(-d)\,p^{-1-c}(1-zp^{-b})^{-d} \quad (3.5)$$

$$t^{\alpha-1} H^{0,0\,:1,n_2\,;1,n_3}_{p_1,q_1:p_2,q_2+1;p_3,q_3+1}\left[\begin{matrix} t \\ t \end{matrix}\middle| \begin{matrix} ((a_j,\alpha_j,A_j)):((c_j,C_j));((e_j,E_j)) \\ ((b_j,\beta_j,B_j)),:(0,1),((d_j,D_j));(0,1),((f_j,F_j)) \end{matrix}\right]$$

$$\doteq p^{-\alpha-k}\sum_{n=0}^{\infty} h_{k+n}\,p^{-n} \quad (3.6)$$

Using (2.3) in (3.4)and (3.6)

$$\Gamma(d)\,p^{-1-a}(1-zp^{-b})^{d}\,p^{-\alpha-k}[\sum_{n=0}^{\infty} h_{k+n}\,p^{-n}] \doteq \int_0^x t^{a}\,{}_1\psi_1[(d,1),(a+1,b); zt^{b}](x-t)^{\alpha-1}$$

$$\times H^{0,0\,:1,n_2\,;1,n_3}_{p_1,q_1:p_2,q_2+1;p_3,q_3+1}\left[\begin{matrix} x-t \\ x-t \end{matrix}\middle| \begin{matrix} ((a_j,\alpha_j,A_j)):((c_j,C_j));((e_j,E_j)) \\ ((b_j,\beta_j,B_j)),:(0,1),((d_j,D_j));(0,1),((f_j,F_j)) \end{matrix}\right]dt \quad (3.7)$$

In (3.7) put t=ux and evaluate using (2.6) to get:

$$\Gamma(d)\,p^{-1-a-\alpha-k}(1-zp^{-b})^{d}[\sum_{n=0}^{\infty} h_{k+n}\,p^{-n}] \doteq \sum_{r=0}^{\infty}\frac{(d+r)z^{r}}{r!}x^{a+\alpha+br} \times H^{0,1\,:1,n_2\,;1,n_3}_{p_1+1,q_1+1:p_2,q_2+1;p_3,q_3+1}$$

$$\left[\begin{matrix} x \\ x \end{matrix}\middle| \begin{matrix} (1-\alpha,1,1),((a_j,\alpha_j,A_j)):((c_j,C_j));((e_j,E_j)) \\ (b_j,\beta_j,B_j),(-a-\alpha-br,1,1):(0,1),((d_j,D_j));(0,1),((f_j,F_j)) \end{matrix}\right] \quad (3.8)$$

From (2.14)

$$p^{-(-k-\alpha--1+h)}[\sum_{n=0}^{\infty} H_{\lambda}\,p^{-\lambda}] \doteq t^{h-1-k-\alpha}\sum_{\lambda=0}^{\infty}\frac{H_{\lambda}t^{\lambda}}{\Gamma(h-1-k+\lambda+\alpha)} \quad (3.9)$$

Now using (2.3) in (3.5) and (3.9), to get:

$$\Gamma(-d)p^{-1-c}(1-zp^{-b})^{d}\,p^{k+\alpha+1-h}[\sum_{\lambda=0}^{\infty}H_{\lambda}p^{-\lambda}] \doteqdot$$

$$\int_0^x t^{c}\,{}_1\psi_1[(-d,1),(c+1,b);zt^{b}](x-t)^{h-1-k-\alpha-1}\sum_{\lambda=0}^{\infty}\frac{H_{\lambda}(x-t)^{\lambda}}{\Gamma(h-1-k+\lambda-\alpha)}dt \quad (3.10)$$

In (3.10) put t = ux and evaluate using (2.14), to get:

$$\Gamma(-d)p^{-1-c+k-\alpha-h}(1-zp^{-b})^{-d}[\sum_{\lambda=0}^{\infty}H_{\lambda}p^{-\lambda}] \doteqdot$$

$$\sum_{r=0}^{\infty}\frac{\Gamma(-d+r)z^{r}}{r!}x^{h-1+c+br-k-\alpha}[\sum_{\lambda=0}^{\infty}\frac{H_{\lambda}x^{\lambda}}{\Gamma(br+c+\lambda+h-k-\alpha)}] \quad (3.11)$$

Using (2.3) and (3.8), the integral equation (3.1) becomes

$$G(p)=A\Gamma(d)p^{m-1-a-\alpha-k}(1-zp^{-b})^{d}[\sum_{n=0}^{\infty}H_{k+n}p^{-n}]F(p) \quad (3.12)$$

Similarly using (2.3) and (3.10), the integral equation (3.2) becomes

$$F(p)=B\Gamma(-d)p^{l-1-c+\alpha+k}(1-zp^{-b})^{-d}[\sum_{\lambda=0}^{\infty}H_{\lambda}p^{-\lambda}]G(p) \quad (3.13)$$

The equations (3.12) and (3.13) can be obtained from each other when

$AB\Gamma(-d)\Gamma(d)=1$ and m+1= a+c+2.

Hence by LERCH'S theorem [4,p.5], it follows that each of the integral equation (3.1) and (3.2) is the solution of the other.

4. Special Cases

In the theorem use [8,pp.88,89, eqn (6.4.3)] to get :

The integral equation

$$g(x)=A\sum_{r=0}^{\infty}[\frac{\Gamma(d+r)\Gamma(\alpha)}{\Gamma(a+\alpha+br+1)}\frac{z^{r}}{r!}\int_0^t(x-t)^{a+\alpha+br}\times F_{q_1+1:q_2;q_3}^{p_1+1:p_2;p_3}$$

$$\left[\begin{array}{c|c} x & \alpha,(a_{p_1}):(c_{p_2});(e_{p_3}); \\ x & (a+\alpha+br+1):(0,1),(b_{q_1});(d_{q_2});(f_{q_3}) \end{array}(-1)^{p_1+p_2-n_2+2}(x-t),(-1)^{p_1+p_3-n_3+2}(x-t)\right]$$

$$\times\left[D^{m}f(t)\right]dt$$

$$=\prod_{j=1}^{p_1}\Gamma(a_j)\prod_{j=n_2+1}^{p_2}\Gamma(c_j)\prod_{j=n\ +1}^{p_3}\Gamma(e_j)\prod_{j=1}^{q_1}\Gamma(1-b_j)\prod_{j=1}^{q_2}\Gamma(1-d_j)\prod_{j=1}^{q_3}\Gamma(1-f_j)$$

Where

$2>b>0$, $\mathrm{Re}(1+\alpha+a)>0$, $\mathrm{Re}(a_j)>0,(j=1,\ldots\ldots,p_1)$

, $\mathrm{Re}(c_j)>0,(j=1,\ldots\ldots,p_2)$

, $\mathrm{Re}(e_j)>0,(j=1,\ldots\ldots,p_3)$, $\mathrm{Re}(1-b_j)>0,(j=1,\ldots\ldots,q_1)$, $\mathrm{Re}(1-d_j)>0,(j=1,\ldots\ldots,q_2)$

, $\mathrm{Re}(1-f_j)>0,(j=1,.....,q_3)$ $AB\Gamma(d)\Gamma(-d)=1, a+c+2=m+1$ $f^q(0)=0, 0\le q\le m$

$gs(0)=0, 0\le s\le h$ m, l,h are integers and $\mathrm{Re}(h-l-k-\alpha+c+1)>0$ has for its solution.

$$f(t)=B\sum_{r=0}^{\infty}[\frac{\Gamma(-d+r)\Gamma(\alpha)z^r}{r!}\int_0^t(t-x)^{h-1+cr+lr-k-\alpha}W(t-x)[D^h g(x)]dx$$

Where

$$W(x)=\sum_{\lambda=0}^{\infty}[\frac{H_\lambda x^\lambda}{\Gamma(c+\lambda+br+h+l-1-k-\alpha)}$$

Where

H_λ being determined by (3.4) and h_v being given by (2.11) and the coefficients in (2.9) reducing to the form $c_{v-\mu,\mu}=\theta_2^/(v-\mu)\theta_3^/(\mu)$.

In (3.1) and (3.2) put $z=m=l=0, A=B=1, a=c=-1, \alpha=\sigma, h=\rho$ to get the result given by Buschman, Kurl and Gupta[8,pp.226-228]

In (3.14) put $z=m=l=0, A=B=1, a=c=-1, \alpha=\sigma, h=\rho$
$p_1=q_1=0,(C_j)=(D_j)=(E_j)=(F_j)=1, n_2=n_3=p_2=p_3=q_2=q_3=1,$
$c_1=1-a_1, d_1=1-b_1, e_1=1-a_2, f_1=1-b_2$ and use the result [8,p.161] to get another result given by Buschman, Kurl and Gupta[8,p.230].

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