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Convolution of Al-Tememe Transformation

Ali Hassan Mohammed[1], Alaa Sallh Hadi[2], Hassan Nadem Rasoul[3]

[1]University of Kufa, Faculty of Education for Girls, Department of Mathematics, Iraq

[2]University of Kufa, Faculty of Education for Girls, Department of Mathematics, Iraq

[3]University of Kufa. Faculty of Computer Science and Math. Department of Mathematics, Iraq

Abstract: ***Our aim in this paper is to find the convolution of Al-Tememe transformation for a function and use it to solve PDEs.***

Keywords: PDE, Transformation, ODE

1. Introduction

The Al-Tememe transformation plays an important role to solve ODE and PDE with variable coefficients and this transformation appears for the first time at 2008.We will use the new idea in [4] to find the convolution of Al-Tememe transformation which we will defined it in a new method and so it will give us the ability to find ($\mathcal{T}^{-1}$)by new method for some functions more easily without using the partition method.

2. Preliminaries

Definition 1: [1]

Let f is defind function at period (a, b)then the integral transformation for$f$ whose it′s symbol $F(p)$ is defined as :

$$F(p) = \int_a^b k(p, x) f(x) dx$$

Where k is a fixed function of two variables, called the kernel of the transformation, and a, b are real numbers or $\mp\infty$, such that the integral above converges.

Definition 2:[2]

The Al-Tememe transformation for the function $f(x)$where $(x > 1)$ is defined by the following integral :

$$\mathcal{T}[f(x)] = \int_1^\infty x^{-s} f(x) dx = F(s)$$

Such that this integral is convergent ,s is positive constant. From the above definition we can write

$$\mathcal{T}[u(x,t)] = \int_1^\infty t^{-s} u(x,t) dt = v(x,s)$$

Such that $u(x,t)$ is a function of $x$and t.

Property 1:[2]

This transformation is characterized by the linear property ,that is

$\mathcal{T}[Au_1(x,t) + Bu_2(x,t)] = A\mathcal{T}[u_1(x,t)] + B\mathcal{T}[u_2(x,t)]$,

Where A ,B are constants ,the functions$u_1(x,t)$, $u_2(x,t)$ are defined when $t > 1$.

The Al-Tememe transform of some fundamental functions are given in table(1)[2]

Table 1

ID	*Function* ,$f(x)$	$F(p) = \int_1^\infty x^{-s} f(x) dx = \mathcal{T}[f(x)]$	*Region of convergence*
1	$k; k = constant$	$\frac{k}{s-1}$	$s > 1$
2	$x^n , n \in R$	$\frac{1}{s-(n+1)}$	$s > n+1$
3	$\ln x$	$\frac{1}{(s-1)^2}$	$s > 1$
4	$x^n \ln x , n \in R$	$\frac{1}{[s-(n+1)]^2}$	$s > n+1$
5	$\sin\ (a \ln x)$	$\frac{a}{(s-1)^2 + a^2}$	$s > 1$
6	$\cos\ (a \ln x)$	$\frac{p-1}{(s-1)^2 + a^2}$	$s > 1$
7	$\sinh\ (a \ln x)$	$\frac{a}{(s-1)^2 - a^2}$	$\lvert s-1 \rvert > a$
8	$\cosh\ (a \ln x)$	$\frac{p-1}{(s-1)^2 - a^2}$	$\lvert s-1 \rvert > a$
9	$(\ln x)^n , n \in N,$	$\frac{n!}{(s-1)^{n+1}}$	$s > 1$

10	$x^m(\ln x)^n$, $n \in N, , m \in Q$	$\frac{n!}{[s-(m+1)]^{n+1}}$	$s > m+1$

Definition 3: [2]
Let $u(x,t)$ be a function where $(t > 1)$ and $\mathcal{T}[u(x,t)] = v(x,s), u(x,t)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1}[v(x,s)] = u(x,t)$, where $\mathcal{T}^{-1}$returns the transformation to the original function.

Definition 4:[3]
A function $f(x)$ is piecewise continuous on an interval $[a,b]$ if the interval can be partitioned by a finite number of points$a = x_0 < x_1 < \cdots < x_n = b$ such that:
1. $f(x)$ is continuous on each subinterval $(x_i,\ x_{i+1})$, for $i = 0, 1, 2, \ldots, n-1$
2. The function f has jump discontinuity at x_i , thus
$$\left|\lim_{x\to x_i^+} f(x)\right| < \infty , i = 0,1,2,\ldots,n-1 ; \left|\lim_{x\to x_i^-} f(x)\right| < \infty , i = 0,1,2,\ldots,n$$
Note: A function is piecewise continuous on $[0,\infty)$ if it is piecewise continuous in $[0,A]$ for all $A > 0$

Definition 5 :Al-TememeConvolution:
Al-Tememe convolution of two functions,$f(x,t)$and$g(x,t)$, is defined for$t > 1$by:
$$(f*g)(x,t) = \int_1^x f(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot\frac{du}{u} ;\ u \neq 0$$
; f and$g$ are piecewise continuous on $[1,\infty)$.

Theorem 1:Let $f(x,t)$ and $g(x,t)$ be two functions of $x$and t. The Al-Tememe convolution of $f(x,t)$ and $g(x,t)$is also a function of x, denoted by $\mathcal{T}(f * g)(x,t)$ and is given by the relation:
$$\mathcal{T}[(f*g)(x,t)] = \mathcal{T}[f(x,t)]\cdot\mathcal{T}[g(x,t)]$$
Proof:
$$\mathcal{T}[f(x,t)]\cdot\mathcal{T}[g(x,t)]=\left(\int_1^\infty u^{-s}f(x,u)du\right)\cdot\left(\int_1^\infty v^{-s}g(x,v)dv\right)$$
$$= \int_1^\infty\left(\int_1^\infty (uv)^{-s}f(x,u)\cdot g(x,v)dv\right)du$$
Let $uv = t$ and noting that $u$is fixed in the interior integral.$\Rightarrow dt = udv$
$$\therefore \mathcal{T}[f(x,t)]\cdot\mathcal{T}[g(x,t)] = \int_1^\infty\left[\int_u^\infty t^{-s}f(x,u)\cdot g\left(x,\frac{t}{u}\right)\frac{dt}{u}\right]du$$

If$g(x,t) = 0$ for$t < 1 \Rightarrow g\left(x,\frac{t}{u}\right) = 0forx < u$
$$\Rightarrow \mathcal{T}[f(x,t)]\cdot\mathcal{T}[g(x,t)] =$$
$$\int_1^\infty\int_1^\infty t^{-s}f(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot dt\cdot\frac{du}{u}$$
$$= \int_1^\infty\int_1^\infty \left|t^{-s}f(x,u)\cdot g\left(x,\frac{t}{u}\right)\right|\cdot dt\cdot\frac{du}{u}$$
$$= \int_1^\infty\left[\int_1^t t^{-s}f(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot\frac{du}{u}\right]dt$$
$$= \int_1^\infty t^{-s}\left(\int_1^x f(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot\frac{du}{u}\right)dt$$
$(f*g)(x,t)$
$= \mathcal{T}[(f*g)(x,t)]$ ∎.
Note: If $\mathcal{T}[f(x,t)] = F(x,s)$and $\mathcal{T}[g(x,t)] = G(x,s)$then:
$$\mathcal{T}^{-1}[F(x,s)]\cdot[G(x,s)] = (f*g)(x,t)$$
Other basic properties of the convolution are as follows:
1) $* g = g * f$, the convolution is commutative
2) $c(f*g)(x) = cf*g = f*cg$, c constant;
3) $f*(g*h) = (f*g)*h$(associative property);
4) $f*(g+h) = (f*g)+(f*h)$ (distributive property)

Prove (1):
$$(f*g)(x,t) = \int_1^t f(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot\frac{du}{u} ;\ u\neq 0$$
; f and$g$ are piecewise continuous on $[1,\infty)$.
Now, let
$$v = \frac{t}{u} \Rightarrow u = \frac{t}{v}$$
$$\therefore uv = t \Rightarrow udv + vdu = 0 \Rightarrow \frac{du}{u} = -\frac{dv}{v}$$
If $u = 1 \Rightarrow v = t$ and if $u = t \Rightarrow v = 1$
$$\therefore (f*g)(x,t) = \int_t^1 f\left(x,\frac{t}{v}\right)\cdot g(x,v)\cdot\frac{-dv}{v}$$
$$= \int_1^t g(x,v)\cdot f\left(x,\frac{t}{v}\right)\cdot\frac{dv}{v} = (g*f)(x,t)$$
$$\Rightarrow (f*g)(x,t) = (g*f)(x,t)$$
So, this convolution is commutative.

Prove (2):
$$c(f*g)(x,t) = c\left(\int_1^t f(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot\frac{du}{u}\right)$$
; $u \neq 0$, c constant
$$= \int_1^x cf(x,u)\cdot g\left(x,\frac{t}{u}\right)\cdot\frac{du}{u} = cf*g$$
By the same method we can prove $c(f*g)(x,t) = f*cg$

Prove (3):
$$[f*(g*h)](x,t) =$$
$$\int_1^t f(x,u)\cdot(g*h)\left(x,\frac{t}{u}\right)\cdot\frac{du}{u} ;u\neq 0 \text{ by dif. (5)}$$
$$= \int_1^t f(x,u)\left(\int_1^{\frac{t}{u}} g(x,v)h\left(\frac{t/u}{v}\right)\frac{dv}{v}\right)\frac{du}{u} ;\ u\neq 0 , v\neq 0$$
$$let\ v = \frac{k}{u} \Rightarrow dv = \frac{dk}{u} ;\ u\neq 0$$
$$= \int_1^t\left(\int_u^t f(x,u)g\left(x,\frac{k}{u}\right)h\left(x,\frac{t}{k}\right)\frac{dk}{uv}\right)\frac{du}{u}$$

$$= \int_1^t \left(\left(\int_1^k f(x,u)\, g\left(x, \frac{k}{u}\right) \frac{du}{u} \right) h\left(x, \frac{t}{k}\right) \frac{dk}{k} \right.$$

$$= [(f * g) * h](x,t)$$

So,thisconvolution is associative.

Prove (4):

$$f * (g + h) = \int_1^t f(x,u) \cdot (g + h)\left(x, \frac{t}{u}\right) \cdot \frac{du}{u}\ ;\ u \neq 0$$

$$= \int_1^t f(x,u) \cdot \left[g\left(x, \frac{t}{u}\right) + h\left(x, \frac{t}{u}\right) \right] \cdot \frac{du}{u}$$

$$= \int_1^t f(x,u) \cdot g\left(x, \frac{t}{u}\right) \cdot \frac{du}{u} + \int_1^t f(x,u) \cdot h\left(x, \frac{t}{u}\right) \cdot \frac{du}{u}$$

$$= (f * g) + (f * h)$$

<u>Example 1:</u>To find

$$\mathcal{T}^{-1}\left[\frac{x^2}{(s-5)(s+2)} \right]$$

Firstly, we use the usual method:

$$\frac{x^2}{(p-5)(p+2)} = \frac{A}{p-5} + \frac{B}{p+2}$$

$$\Rightarrow A = x^2/7\ , B = -x^2/7$$

$$\therefore \mathcal{T}^{-1}\left[\frac{x^2}{(s-5)(s+2)} \right] = \mathcal{T}^{-1}\left(\frac{x^2/7}{s-5} \right) - \mathcal{T}^{-1}\left(\frac{x^2/7}{s+2} \right)$$

$$= x^2/7\, t^4 - x^2/7\, t^{-3}$$

Now, we will use the convolution method and we get :

$$\mathcal{T}^{-1}\left[\frac{x^2}{(s-5)(s+2)} \right] = \mathcal{T}^{-1}\left[\frac{x^2}{(s-5)} \cdot \frac{x^2}{(s+2)} \right]$$

$$= x^2 t^4 * t^{-3}$$

$f(x,t)$ $g(x,t)$

$$= \int_1^t f(x,u) \cdot g\left(x, \frac{t}{u}\right) \cdot \frac{du}{u} = \int_1^t x^2 t^4 \cdot \left(\frac{t}{u}\right)^{-3} \cdot \frac{du}{u}$$

$$= \int_1^t x^2 t^4 \cdot \frac{t^{-3}}{u^{-3}} \cdot \frac{1}{u} \cdot du = x^2 t^{-3} \int_1^t u^6 du$$

$$= x^2 t^{-3} \left(\frac{u^7}{7} \right) \Big|_1^t = x^2 t^{-3} \left(\frac{t^7}{7} - \frac{1}{7} \right)$$

$$= (x^2/7) t^4 - (x^2/7) t^{-3}$$

<u>Example 2 :</u>To find

$$\mathcal{T}^{-1}\left[\frac{2 sinx}{(s+2)(s-4)^2} \right]$$

We note that

$$\frac{2 sinx}{(s+2)(s-4)^2} = \frac{A}{(s+2)} + \frac{B}{(s-4)} + \frac{C}{(s-4)^2}$$

$$\Rightarrow A = \frac{sinx}{18}, B = \frac{-sinx}{18}, C = \frac{sinx}{3}$$

$$= \frac{sinx}{18} t^{-3} - \frac{sinx}{18} t^3 + \frac{sinx}{3} t^3 \ln t$$

Now, we will use the convolution method and we get :

$$\mathcal{T}^{-1}\left[\frac{2 sinx}{(s+2)(s-4)^2} \right] = \mathcal{T}^{-1}\left[\frac{2 sinx}{(s-4)^2} \cdot \frac{1}{s+2} \right]$$

$$= 2(sinx) t^3 \ln t * t^{-3}$$

$f(x,t)$ $g(x,t)$

$$= \int_1^t 2(sinx) u^3 \ln u \cdot \left(\frac{t}{u}\right)^{-3} \cdot \frac{du}{u} =$$

$$\int_1^t 2(sinx) u^3 \ln u \cdot \frac{t^{-3}}{u^{-3}} \cdot \frac{1}{u} \cdot du$$

$$= 2(sinx) t^{-3} \int_1^t u^5 \cdot \ln u \cdot du$$

$$= 2(sinx) t^{-3} \left[\frac{u^6}{6} \ln u \Big|_1^t - \int_1^t \frac{u^5}{6} du \right]$$

$$= 2(sinx) t^{-3} \left[\frac{t^6}{6} \ln t - 0 - \left(\frac{u^6}{36} \right) \Big|_1^t \right]$$

$$= 2(sinx) t^{-3} \left[\frac{t^6}{6} \ln t - \frac{t^6}{36} + \frac{1}{36} \right]$$

$$= \frac{sinx}{18} t^{-3} - \frac{sinx}{18} t^3 + \frac{sinx}{3} t^3 \ln t$$

<u>Example 3:</u>To solve the following PDE

$$t u_t + u_{xx} = xt\ ;\ u(x,1) = 0$$

By using T-transformation.

We take $\mathcal{T}$ to both sides and we will get :

$$-u(x,1) + (s-1)\mathcal{T}[u] + \frac{d^2 v}{dx^2} = \frac{x}{(s-2)}$$

$$\frac{d^2 v}{dx^2} + (s-1) v = \frac{x}{(s-2)} \Rightarrow v = \frac{1}{D^2 + (s-1)} \cdot \frac{x}{(s-2)}$$

$$v = \frac{1}{(s-1)(1 + \frac{D^2}{(s-1)})} \cdot \frac{x}{(s-2)}$$

$$v = \frac{1}{(s-1)(s-2)} \cdot \left(1 - \frac{D^2}{(s-1)}\right) x$$

$$v = \frac{x}{(s-1)(s-2)}$$

By taking $\mathcal{T}^{-1}$to both sides we get

$$u = \mathcal{T}^{-1}\left[\frac{x}{(s-1)} \cdot \frac{1}{(s-2)} \right]$$

Now, we will use the convolution method and we get:

$$u = \mathcal{T}^{-1}\left[\frac{x}{(s-1)} \cdot \frac{1}{(s-2)} \right]$$

$f(x,t) = x\ g(x,t) = t$

$$u = \int_1^t f(x,t) \cdot g\left(x, \frac{t}{u}\right) \cdot \frac{du}{u}$$

$$u = \int_1^t x \cdot \left(\frac{t}{u}\right) \cdot \frac{du}{u}$$

$$= xt \int_1^t (x)^{-2} du = xt \left[(u)^{-1} \Big|_1^t \right]$$

$$= xt[-t^{-1} + 1]$$

$$\therefore u = -x + xt$$

Now, we will use the usual method:

$$\frac{x}{(s-1)(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-2)}$$

$A = -x$and $B = x$.

So,

$$\mathcal{T}^{-1}\left[\frac{x}{(s-1)(s-2)}\right] = \mathcal{T}^{-1}\left[\frac{-x}{(s-1)} + \frac{x}{(s-2)}\right]$$

$$u = -x + xt$$

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