To find a solution of DE (10) we can take the Al-Tememe transformation ($\mathcal{T}$) to both sides of (10), after substituting initial conditions and simplification we can put $\mathcal{T}(y)$ as follows:

$$\mathcal{T}(y) = \frac{h(p)}{k(p)} \;\dots(11)$$

Where h, k are polynomials, such that the degree of h is less than the degree of k and the polynomial k with known prime cofactors. By taking $\mathcal{T}^{-1}$ to both sides of equation (11) we will get:

$$y = \mathcal{T}^{-1}\left[\frac{h(p)}{k(p)}\right] \;\dots(12)$$

Equation (12) represents the general solution of the differential equation (10) which is form is given by:

$$y = A_0k_0(x) + A_1k_1(x) + \cdots + A_mk_m(x) \;\dots(13)$$

Such that $k_0, k_1, \dots, k_m$ are functions of x and that $A_0, A_1, \dots, A_m$ are constants, whose number equals to the degree of $k(p)$. To find the values of $A_0, A_1, \dots, A_m$ we will substituting the initial conditions, one of them is $y(1)$ so we will get:

$$A_0k_0(1) + A_1k_1(1) + \cdots + A_mk_m(1) = y(1) \;\dots(14)$$

Take derivatives of (13) m times to get

$$A_0k_0'(1) + A_1k_1'(1) + \cdots + A_mk_m'(1) = y'(1) \;\dots(15)$$

$$A_0k_0''(1) + A_1k_1''(1) + \cdots + A_mk_m''(1) = y''(1) \;\dots(16)$$

⋮ ⋮ ⋮ ⋮ ⋮

$$A_0k_0^{(m)}(1) + A_1k_1^{(m)}(1) + \cdots + A_mk_m^{(m)}(1) = y^m(1) \;\dots(k)$$

This is linear system can be solved to obtain $A_0, A_1, \dots, A_m$ and so we obtain the solution of the required differential equation (10).

In equation (12), The polynomial $h(p)$ is un necessary defined, it is indicated only by this symbol. While the polynomial $k(p)$ contains the multiplied $(a_0p^n + a_1p^{n-1} + \cdots + a_n)$ by the denominator of Al-Tememe transformation for the function $f(x)$.

Note: we can put

$\mathcal{T}(xy') = (p-1)\mathcal{T}(y) + a$; a constant

$\mathcal{T}(x^2y'') = (p-2)(p-1)\mathcal{T}(y) + h_1(p)$; degree of $h_1(p)$ less than 2.

$\mathcal{T}(x^3y''') = (p-3)(p-2)(p-1)\mathcal{T}(y) + h_2(p)$; degree of $h_2(p)$ less than 3.

⋮ ⋮ ⋮ ⋮ ⋮

$\mathcal{T}(x^my^{(m)}) = (p-m)(p-m-1)\dots(p-1)\mathcal{T}(y) + h_{m-1}(p)$; degree of $h_{m-1}(p)$ less than m.

Example 1: To solve the differential equation $xy' - 2y = x^5$; $y(1) = 0$ we take Al-Tememe transformation to both sides of above ODE and after substitution the initial condition $y(1) = 0$ we can write:

$$\mathcal{T}(y) = \frac{h(p)}{(p-3)(p-6)}$$

So we write the general solution, after taking inverse of Al-Tememe transformation, as follows:

$$y = Ax^2 + Bx^5 \;\dots(17)$$

Here the equation (17) contains two constant A and B so we need two linear algebra equations. We get one of them by the initial condition $y(1) = 0$ so we get the equation:

$$A + B = 0 \;\dots(18)$$

For finding the second equation we should find additional condition which be get it from the above differential equation by substituting the initial condition $y(1) = 0$ so we get:

$$y'(1) - 2y(1) = 1 \;\Rightarrow\; y'(1) = 1$$

And after taking derivative to equation (17) and substituted $y'(1) = 1$, We get:

$$2A + 5B = 1 \;\dots(19)$$

So, from (18) and as we get:

$$A = {}^{-1}/_3,\; B = {}^{1}/_3$$

And hence the solution is given by:

$$y = {}^{-1}/_3 x^2 + {}^{1}/_3 x^5$$

Example 2: To solve the differential equation:

$$x^2y'' + 3xy' - 3y = x^{-2}\ln x \;;\; y'(1) = y(1) = 0$$

We take $\mathcal{T}$ to both sides of the differential equation and after substitution the initial conditions we put:

$$\mathcal{T}(y) = \frac{h(p)}{(p^2 - 3p + 2 + 3p - 3 - 3)(p+1)^2}$$

$$\mathcal{T}(y) = \frac{h(p)}{(p-2)(p+2)(p+1)^2} \;\dots(20)$$

After taking $\mathcal{T}^{-1}$ to both sides, of (20), we can write the general solution as follows:

$$y = Ax + Bx^{-3} + Cx^{-2} + Dx^{-2}\ln x \;\dots(21)$$

Where A, B, C are constants.

To find the values of A, B, C and D we need four linear equations. We get the first equation by substituting $y(1) = 0$ in equation (21). For the second equation we derive the general solution (21) and substitute the initial condition $y'(1) = 0$. But the third equation we need two additional conditions. Now,

$y''(1) + 3y'(1) - 3y(1) = 0 \;\Rightarrow\; y''(1) = 0$

$y'''(1) + 2y''(1) + 3y''(1) + 3y'(1) - 3y'(1) = 1 \;\Rightarrow\; y'''(1) = 1$

After substituting

$y(1) = 0, y'(1) = 0, y''(1) = 0, y'''(1) = 1$

in y, y', y'', y''' respectively we get:

$A + B + C = 0 \;\dots(22)$

$A - 3B - 2C + D = 0 \;\dots(23)$

$12B + 6C - 5D = 0 \;\dots(24)$

$-60B - 24C + 26D = 1 \;\dots(25)$

So, from (22),(23),(24) and (25) we get:

$A = 0.027\bar{7}, B = -{}^{1}/_4, C = 0.2\bar{2}, D = -{}^{1}/_3$ so the solution of the differential equation takes the form:

$$y = 0.027\bar{7}x - {}^{1}/_4 x^{-3} + 0.2\bar{2}x^{-2} - {}^{1}/_3 x^{-2}\ln x$$

Example 3: To solve the differential equation

$$x^3y''' - 3xy' + 3y = \sin(\ln x) \;;\; y(1) = -2, y'(1) = 1, y''(1) = 0$$

We take $\mathcal{T}$ to both sides of differential equation and after substitution the initial conditions we put:

$$\mathcal{T}(y) = \frac{h(p)}{(p^3 - 6p^2 + 11p - 6 - 3p + 3 + 3)[(p-1)^2 + 1]} \;\dots(26)$$

$$\mathcal{T}(y) = \frac{h(p)}{(p^3 - 6p^2 + 8p)[(p-1)^2 + 1]}$$
$$\mathcal{T}(y) = \frac{h(p)}{p(p-2)(p-4)[(p-1)^2 + 1]}$$

After taking $\mathcal{T}^{-1}$ to both sides we get:

$$y = Ax^{-1} + Bx + Cx^3 + Dsin\,(ln\,x) + Ecos\,(ln\,x) \quad ...(27)$$

To find the values of A, B, C, D and E we need five linear equations. We get the first equation by substituting $y(1) = -2$ in equation (27). For the second equation we derive the general solution (27) and substitute the initial condition $y'(1) = 1$. For the third equation we derive the general solution (27) and substitute the initial condition $y''(1) = 0$. But the four equation we need two additional conditions. note that

$$y'''(1) - 3y'(1) + 3y(1) = 1 \implies y'''(1) = 9$$
$$y^{(4)}(1) + 3y'''(1) - 3y''(1) - 3y'(1) + 3y'(1) = 1 \implies y^{(4)}(1) = -26$$

So, we get the equations:

$A + B + C + E = -2 \quad ...(28)$

$-A + B + 3C + D = 1 \quad ...(29)$

$2A + 6C - D - E = 0 \quad ...(30)$

$-6A + 6C + D + 3E = 9 \quad ...(31)$

$12A + 5E = 13 \quad ...(32)$

We solve the system of equations (28), (29), (30), (31) and (32) we get:

$$A = {}^{17}/_{32}, B = {}^{-73}/_{16}, C = {}^{113}/_{160}, D = {}^{159}/_{40}, E = {}^{53}/_{40}$$

And hence the solution is given by:

$$y = {}^{17}/_{32}\,x^{-1} - {}^{73}/_{16}\,x + {}^{113}/_{160}\,x^3 + {}^{159}/_{40}\,sin(ln\,x) + {}^{53}/_{40}\,cos(ln\,x)$$

Example 4: To solve the differential equation:

$$x^4y^{(4)} + 5x^3y''' = 3x^{-3}, y(1) = y'(1) = y''(1) = y'''(1) = 0$$

After taking ($\mathcal{T}$) to both sides we can write:

$$\mathcal{T}(y) = \frac{h(p)}{(p-3)(p-2)(p-1)(p+1)(p+2)} \quad ...(33)$$

Taking $\mathcal{T}^{-1}$ to both sides of last equation, So we can write:

$$y = \mathcal{T}^{-1}\left[\frac{A}{(p-3)} + \frac{B}{(p-2)} + \frac{C}{(p-1)} + \frac{D}{(p+1)} + \frac{E}{(p+2)}\right]$$
$$y = Ax^2 + Bx + C + Dx^{-2} + Ex^{-3} \quad ...(34)$$

To find the values of A, B, C, D and E we need five linear equations. We get the first equation by substituting $y(1) = 0$ in equation (34). For the second equation we derive the general solution (34) and substitute the initial condition $y'(1) = 0$. For the third equation we derive again the general solution (34) and substitute the initial condition $y''(1) = 0$. For the four equation we derive again the general solution (34) and substitute the initial condition $y'''(1) = 0$. But for the five equation we need additional condition, see

$$y^{(4)}(1) + 5y'''(1) = 3 \implies y^{(4)}(1) = 3$$

So we get the equations:

$A + B + C + D + E = 0 \quad ...(35)$

$2A + B - 2D - 3E = 0 \quad ...(36)$

$2A + 6D + 12E = 0 \quad ...(37)$

$-24D - 60E = 0 \quad ...(38)$

$40D + 120E = 1 \quad ...(39)$

From above equations we get:

$$A = {}^{3}/_{40}, B = {}^{-1}/_{4}, C = {}^{1}/_{4}, D = {}^{-1}/_{8}, E = {}^{1}/_{20}$$

And hence the solution is given by:

$$y = {}^{3}/_{40}\,x^2 - {}^{1}/_{4}\,x + {}^{1}/_{4} - {}^{1}/_{8}\,x^{-2} + {}^{1}/_{20}\,x^{-3}$$

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