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Al-Tememe Transformation for Solving Some LODE with Initial Conditions

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Abstract: ***Our aim is to apply Al-Tememe Transformations to solve linear ordinary differential equations (LODE)with variable coefficients by using initial conditions.***

Keywords: (T.T Al-Tememe transform),(T-1 Inverse of Al-Tememe transform),(ODE ordinary differential equation),(LODE linear ordinary differential equation), (ODE ordinary differential)

1. Introduction

The usual method of Al-Tememe transformation ($\mathcal{T}$.T) is to find solutions of linear ordinary differential equations with variable coefficients that satisfy some initial conditions is summarized as taking Al-Tememe transformation to both sides of differential equation and putting Al-Tememe transformation of general solution which is considered as a fraction whose numerator and denominator are polynomials. Then we can put it into partial fractions whose number equals to the number of prime factors of denominator that contain constants whose number equals to the degree of the polynomial which is in the denominator, and the values of these constants depend on coefficients of numerator, that can be evaluated by comparison of both sides, an during constructing system of linear equations which can be evaluated by known algebraic methods, and after taking the inverse of Al-Tememe transformation ($\mathcal{T}^{-1}$) we obtain the solution of the differential equation.

2. Preliminaries

Definition 1: [1]

Let f is defined function at period (a,b)then the integral transformation forf whose it's symbol $F(p)$ is defined as:

$$F(p)=\int_a^b k(p,x)f(x)dx$$

Where k is a fixed function of two variables, called the kernel of the transformation, and a, b are real numbers or $\mp\infty$, such that the integral above converges.

Definition 2:[3]

The Al-Tememe transformation for the function $f(x)$; $x>1$ is defined by the following integral:
$\mathcal{T}[f(x)]=\int_1^\infty x^{-p} f(x)dx=F(p)$,
such that this integral is convergent, p is positive constant.

Property of this transformation 1:[3]

This transformation is characterized by the linear property, that is

$\mathcal{T}[Af(x)+Bg(x)]=A\mathcal{T}[f(x)]+B\mathcal{T}[g(x)]$,

Where A, B are constants, the functions $f(x), g(x)$ are defined when ; $x>1$.

The Al-Tememe transform of some fundamental functions are given in table (1)[3]:

ID	Function,$\mathbf{f(x)}$	$\mathbf{F(p)=\int_1^\infty x^{-p}f(x)dx=\mathcal{T}[f(x)]}$	Regional of convergence
1	$k; k=\text{constant}$	$\frac{k}{p-1}$	$\mathbf{p}>1$
2	$x^n, n\in R$	$\frac{1}{p-(n+1)}$	$\mathbf{p>n+1}$
3	$\ln x$	$\frac{1}{(p-1)^2}$	$\mathbf{p}>1$
4	$x^n \ln x, n\in R$	$\frac{1}{[p-(n+1)]^2}$	$\mathbf{p>n+1}$
5	$\sin(a\ln x)$	$\frac{a}{(p-1)^2+a^2}$	$\mathbf{p}>1$
6	$\cos(a\ln x)$	$\frac{p-1}{(p-1)^2+a^2}$	$\mathbf{p}>1$
7	$\sinh(a\ln x)$	$\frac{a}{(p-1)^2-a^2}$	$\mathbf{\lvert p-1\rvert>a}$
8	$\cosh(a\ln x)$	$\frac{p-1}{(p-1)^2-a^2}$	$\mathbf{\lvert p-1\rvert>a}$

Table 1

From the Al-Tememe definition and the above table, we get:

Theorem1:

If $\mathcal{T}[f(x)]=F(p)$ and $a$is constant, then $\mathcal{T}[x^{-a}f(x)]=F(p+a)$.see [3]

Definition 3: [3]

Let $f(x)$ be a function where $(x > 1)$ and $\mathcal{T}[f(x)] = F(p)$, $f(x)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1}[F(p)] = f(x)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function.

Property 2:[3]

If $\mathcal{T}^{-1}[F_1(p)] = f_1(x)$, $\mathcal{T}^{-1}[F_2(p)] = f_2(x)$,…, $\mathcal{T}^{-1}[F_n(p)] = f_n(x)$ and $a_1, a_2, \dots, a_n$ are constants then,
$\mathcal{T}^{-1}[a_1F_1(p) + a_2F_2(p) + \cdots + a_nF_n(p)] = a_1f_1(x) + a_2f_2(x) + \cdots + a_nf_n(x)$,

Definition 4: [4]

The equation,

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = f(x),$$

Where $a_1, a_2, \dots, a_n$ are constants and $f(x)$ is a function of x, is called **Euler's equation**.

Theorem 2:[3]

If the function $f(x)$ is defined for $x > 1$ and its derivatives $f^{(1)}(x), f^{(2)}(x), \dots, f^{(n)}(x)$ are exist then:

$$\mathcal{T}[x^nf^{(n)}(x)] = -f^{(n-1)}(1) - (p-n)f^{(n-2)}(1) - \cdots - (p-n)\big(p-(n-1)\big)\dots((p-2)f(1) + (p-n)!\,F(p)$$

We will use Theorem (2) to prove that

$$\mathcal{T}(\ln x)^n = \frac{n!}{(p-1)^{n+1}};\ n \in \mathbb{N}$$

$$if\ n = 1 \Longrightarrow \mathcal{T}(\ln x) = \frac{1}{(p-1)^2}\ (Table1)\ \dots(1)$$

If $n = 2 \Longrightarrow y = (\ln x)^2 \Longrightarrow y(1) = 0$

$$y' = 2\ln x.\frac{1}{x} = \frac{2}{x}\ln x \Longrightarrow xy' = 2\ln x$$

$$\mathcal{T}(xy') = 2\mathcal{T}(\ln x) = 2\cdot\frac{1}{(p-1)^2} = \frac{2}{(p-1)^2}$$

$$\because \mathcal{T}(xy') = -y(1) + (p-1)\mathcal{T}(y) \Longrightarrow \mathcal{T}(xy') = (p-1)\mathcal{T}(y)$$

$$\therefore (p-1)\mathcal{T}(y) = \frac{2}{(p-1)^2} \Longrightarrow \mathcal{T}(y) = \frac{2}{(p-1)^3} = \frac{2!}{(p-1)^3}\dots(2)$$

If $n = 3 \Longrightarrow y = (\ln x)^3 \Longrightarrow y(1) = 0$

$$y' = 3(\ln x)^2\cdot\frac{1}{x} = \frac{3}{x}(\ln x)^2 \Longrightarrow xy' = 3(\ln x)^2$$

$$\mathcal{T}(xy') = 3\,\mathcal{T}(\ln x)^2 = 3.\frac{2}{(p-1)^3} = \frac{6}{(p-1)^3}$$

$$\because \mathcal{T}(xy') = (p-1)\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = \frac{6}{(p-1)^3} \Longrightarrow \mathcal{T}(y) = \frac{6}{(p-1)^4} = \frac{3!}{(p-1)^4}\dots(3)$$

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Also, $y = (\ln x)^n \Longrightarrow y(1) = 0$

$$y' = n(\ln x)^{n-1}\cdot\frac{1}{x} \Longrightarrow xy' = n\ (\ln x)^{n-1}$$

$$\mathcal{T}(xy') = n\,\mathcal{T}(\ln x)^{n-1} = n\cdot\frac{(n-1)!}{(p-1)^n} = \frac{n!}{(p-1)^n}$$

$$\because \mathcal{T}(xy') = (p-1)\mathcal{T}(y)$$

$$\therefore (p-1)\mathcal{T}(y) = \frac{n!}{(p-1)^n} \Longrightarrow \mathcal{T}(y) = \frac{n!}{(p-1)^{n+1}}\dots(n)$$

$$\therefore \mathcal{T}(\ln x)^n = \frac{n!}{(p-1)^{n+1}};\ n \in \mathbb{N}$$

Also we will use Theorem (2) to find $\mathcal{T}[x^m(\ln x)^n]$; $n, m \in \mathbb{N}$

The First Case: If $n = 1$

$$\mathcal{T}[x^m \ln x] = \frac{1!}{[p-(m+1)]^2}; m \in \mathbb{N}\ Table1$$

The Second Case: If $n = 2$ To find $\mathcal{T}[x^m (\ln x)^2]$

If $m = 1 \implies \mathcal{T}[x(\ln x)^2]$

Consider, $y = x(\ln x)^2 \implies y(1) = 0$

$$y' = x \cdot 2(\ln x) \cdot \frac{1}{x} + (\ln x)^2 \implies xy' = 2x(\ln x) + x(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x(\ln x)] + \mathcal{T}(y) = 2 \cdot \frac{1}{(p-2)^2} + \mathcal{T}(y)$$

$$\because \mathcal{T}(xy') = (p-1)\mathcal{T}(y) \Rightarrow (p-1)\mathcal{T}(y) = \frac{2}{(p-2)^2} + \mathcal{T}(y)$$

$$\implies (p-2)\mathcal{T}(y) = \frac{2}{(p-2)^2} \implies \mathcal{T}[x\,(\ln x)^2] = \frac{2}{(p-2)^3} \ \ldots (4)$$

If $m = 2 \implies \mathcal{T}[x^2(\ln x)^2]$

Consider, $y = x^2(\ln x)^2 \implies y(1) = 0$

$$y' = x^2 \cdot 2(\ln x) \cdot \frac{1}{x} + 2x(\ln x)^2 \implies xy' = 2x^2(\ln x) + 2x^2(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x^2(\ln x)] + 2\mathcal{T}(y) = 2 \cdot \frac{1}{(p-3)^2} + 2\mathcal{T}(y)$$

$$\because \mathcal{T}(xy') = (p-1)\mathcal{T}(y) \implies (p-1)\mathcal{T}(y) = \frac{2}{(p-3)^2} + 2\mathcal{T}(y)$$

$$\implies (p-3)\mathcal{T}(y) = \frac{2}{(p-3)^2} \implies \mathcal{T}[x^2(\ln x)^2] = \frac{2}{(p-3)^3} \ \ldots (5)$$

If $m = 3 \implies \mathcal{T}[x^3(\ln x)^2]$

Consider, $y = x^3(\ln x)^2 \implies y(1) = 0$

$$y' = x^3 \cdot 2(\ln x) \cdot \frac{1}{x} + 3x^2(\ln x)^2 \implies xy' = 2x^3(\ln x) + 3x^3(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x^3(\ln x)] + 3\mathcal{T}(y) = 2 \cdot \frac{1}{(p-4)^2} + 3\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 2 \cdot \frac{1}{(p-4)^2} + 3\mathcal{T}(y) \implies \mathcal{T}[x^3(\ln x)^2] = \frac{2}{(p-4)^3} \ \ldots (6)$$

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$$\mathcal{T}[x^m(\ln x)^2]; m \in \mathbb{N} \Rightarrow y = x^m(\ln x)^2 \implies y(1) = 0$$

$$y' = x^m \cdot 2(\ln x) \cdot \frac{1}{x} + mx^{m-1}(\ln x)^2 \implies xy' = 2x^m(\ln x) + mx^m(\ln x)^2$$

$$\mathcal{T}(xy') = 2\mathcal{T}[x^m(\ln x)] + m\mathcal{T}(y) = 2 \cdot \frac{1}{[p-(m+1)]^2} + m\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 2 \cdot \frac{1}{[p-(m+1)]^2} + m\mathcal{T}(y)$$

$$\Rightarrow \mathcal{T}[x^m(\ln x)^2] = \frac{2!}{[p-(m+1)]^3}; m \in \mathbb{N} \ \ldots (m)$$

Note:These cases are also true for $m \in \mathbb{Q}$

The Third Case: To find $\mathcal{T}[x^m(\ln x)^3]$

If $m = 1 \implies \mathcal{T}[x(\ln x)^3]$

Consider, $y = x(\ln x)^3 \implies y(1) = 0$

$$y' = x \cdot 3(\ln x)^2 \cdot \frac{1}{x} + (\ln x)^3 \implies xy' = 3x(\ln x)^2 + x(\ln x)^3$$

$$\mathcal{T}(xy') = 3\mathcal{T}[x(\ln x)^2] + \mathcal{T}(y) = 3 \cdot \frac{2}{(p-2)^3} + \mathcal{T}(y)$$

$$\implies (p-1)\mathcal{T}(y) = \frac{3!}{(p-2)^3} + \mathcal{T}(y)$$

$$\Longrightarrow (p-2)\mathcal{T}(y) = \frac{3!}{(p-2)^3} \Longrightarrow \mathcal{T}[x\,(ln\,x)^2] = \frac{3!}{(p-2)^4} \ldots (7)$$

If $m = 2 \Longrightarrow \mathcal{T}[x^2(ln\,x)^3]$

Consider, $y = x^2(ln\,x)^3 \Longrightarrow\ y(1) = 0$

$$y' = x^2 \cdot 3(ln\,x)^2 \cdot \frac{1}{x} + 2x(ln\,x)^3 \Longrightarrow xy' = 3x^2(ln\,x)^2 + 2x^2(ln\,x)^3$$

$$\mathcal{T}(xy') = 3\mathcal{T}[x^2(ln\,x)^2] + 2\mathcal{T}(y) = 3 \cdot \frac{2}{(p-3)^3} + 2\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 3 \cdot \frac{2}{(p-3)^3} + 2\mathcal{T}(y)\ \Longrightarrow \mathcal{T}(x^2(ln\,x)^3) = \frac{3!}{(p-3)^4} \ldots (8)$$

If $m = 3 \Longrightarrow\ \mathcal{T}[x^3(ln\,x)^3]$

Consider, $y = x^3(ln\,x)^3 \Longrightarrow\ y(1) = 0$

$$y' = x^3 \cdot 3(ln\,x)^2 \cdot \frac{1}{x} + 3x^2(ln\,x)^3$$

$$xy' = 3x^3(ln\,x)^2 + 3x^3(ln\,x)^3 \Longrightarrow\ \mathcal{T}(xy') = 3\mathcal{T}[x^3(ln\,x)^2] + 3\mathcal{T}(y)$$

$$= 3 \cdot \frac{2}{(p-4)^3} + 3\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 3 \cdot \frac{2}{(p-4)^3} + 3\mathcal{T}(y)\ \Longrightarrow \mathcal{T}[x^3(ln\,x)^3] = \frac{3!}{(p-4)^4} \ldots (9)$$

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$\mathcal{T}[x^m(ln\,x)^3]$

consider $\Longrightarrow\ y = x^m(ln\,x)^3 \Longrightarrow\ y(1) = 0$

$$y' = x^m \cdot 3(ln\,x)^2 \cdot \frac{1}{x} + mx^{m-1}(ln\,x)^3 \Longrightarrow xy' = 3x^m(ln\,x)^2 + mx^m(ln\,x)^3$$

$$\mathcal{T}(xy') = 3\mathcal{T}[x^m(ln\,x)^2] + m\mathcal{T}(y) = 3 \cdot \frac{2}{[p-(m+1)]^3} + m\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = 3 \cdot \frac{2}{[p-(m+1)]^3} + m\mathcal{T}(y)$$

$$\Rightarrow \mathcal{T}[x^m(ln\,x)^3] = \frac{3!}{[p-(m+1)]^4}\ ; m \in \mathbb{N}\ \ldots (h)$$

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Gradually we find now

$\mathcal{T}[x^m(ln\,x)^n]\ ; m \in \mathbb{N}, n \in \mathbb{N}$

Consider, $y = x^m(ln\,x)^n \Longrightarrow y(1) = 0$

$$y' = x^m \cdot n(ln\,x)^{n-1} \cdot \frac{1}{x} + mx^{m-1}(ln\,x)^n \Longrightarrow xy' = nx^m(ln\,x)^{n-1} + mx^m(ln\,x)^n$$

$$\mathcal{T}(xy') = n\mathcal{T}[x^m(ln\,x)^{n-1}] + m\mathcal{T}(y) = n \cdot \frac{(n-1)!}{[p-(m+1)]^n} + m\mathcal{T}(y)$$

$$(p-1)\mathcal{T}(y) = n \cdot \frac{(n-1)!}{[p-(m+1)]^n} + m\mathcal{T}(y)$$

$$\mathcal{T}[x^m(ln\,x)^n] = \frac{n!}{[p-(m+1)]^{n+1}}\ ;\ m \in \mathbb{N}, n \in \mathbb{N}$$

The New Method of Al-Tememe Transformation:

We shell study a new method of Al-Tememe transformation to obtain a solution for a linear ordinary differential equation LODE with variables coefficients (**Euler's equation**) that satisfies some initial conditions and we can write this method as follows:

We can generalize the method of the researcher Mohammed [2] that he used it to solve the ODE with constant coefficients subject to initial conditions by using Laplace transform.

Suppose we have a linear ordinary differential equation (LODE) of order (n) with variables coefficients and due to certain initial conditions, which general form can be written as follow:

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = f(x)\ \ldots (10)$$

Where $a_0,\ a_1, \ldots, a_n$ are constants, $y^{(n)}$ the nth derivative of the function $y(x), f(x)$ is a continuous function whose Al-Tememe transformation can be determined, and $y(1), \ldots, y^{(n-1)}(1)$are defined.

To find a solution of DE (10) we can take the Al-Tememe transformation ($\mathcal{T}$) to both sides of (10), after substituting initial conditions and simplification we can put $\mathcal{T}(y)$ as follows:

$$\mathcal{T}(y) = \frac{h(p)}{k(p)} \quad \dots (11)$$

Where h, k are polynomials, such that the degree of h is less than the degree of k and the polynomial k with known prime cofactors. By taking $\mathcal{T}^{-1}$ to both sides of equation (11) we will get:

$$y = \mathcal{T}^{-1}\left[\frac{h(p)}{k(p)}\right] \quad \dots (12)$$

Equation (12) represents the general solution of the differential equation (10) which is form is given by:

$$y = A_0k_0(x) + A_1k_1(x) + \cdots + A_mk_m(x) \quad \dots (13)$$

Such that $k_0, k_1, \dots, k_m$ are functions of x and that $A_0, A_1, \dots, A_m$ are constants, whose number equals to the degree of $k(p)$. To find the values of $A_0, A_1, \dots, A_m$ we will substituting the initial conditions, one of them is $y(1)$ so we will get:

$$A_0k_0(1) + A_1k_1(1) + \cdots + A_mk_m(1) = y(1) \quad \dots (14)$$

Take derivatives of (13) m times to get

$$A_0k_0'(1) + A_1k_1'(1) + \cdots + A_mk_m'(1) = y'(1) \quad \dots (15)$$

$$A_0k_0''(1) + A_1k_1''(1) + \cdots + A_mk_m''(1) = y''(1) \quad \dots (16)$$

$$\vdots$$

$$A_0k_0^{(m)}(1) + A_1k_1^{(m)}(1) + \cdots + A_mk_m^{(m)}(1) = y^m(1) \quad \dots (k)$$

This is linear system can be solved to obtain $A_0, A_1, \dots, A_m$ and so we obtain the solution of the required differential equation (10).

In equation (12), The polynomial $h(p)$ is un necessary defined, it is indicated only by this symbol. While the polynomial $k(p)$ contains the multiplied $(a_0p^n + a_1p^{n-1} + \cdots + a_n)$ by the denominator of Al-Tememe transformation for the function $f(x)$.

Note: we can put

$\mathcal{T}(xy') = (p-1)\mathcal{T}(y) + a\,;\, a\ constant$

$\mathcal{T}(x^2y'') = (p-2)(p-1)\mathcal{T}(y) + h_1(p)$; degree of $h_1(p)$ less than 2.

$\mathcal{T}(x^3y''') = (p-3)(p-2)(p-1)\mathcal{T}(y) + h_2(p)$; degree of $h_2(p)$ less than 3.

$$\vdots$$

$\mathcal{T}(x^my^{(m)}) = (p-m)(p-m-1)\dots(p-1)\mathcal{T}(y) + h_{m-1}(p)$; degree of $h_{m-1}(p)$ less than m.

Example 1: To solve the differential equation $xy' - 2y = x^5\,; y(1) = 0$ we take Al-Tememe transformation to both sides of above ODE and after substitution the initial condition $y(1) = 0$ we can write:

$$\mathcal{T}(y) = \frac{h(p)}{(p-3)(p-6)}$$

So we write the general solution, after taking inverse of Al-Tememe transformation, as follows:

$$y = Ax^2 + Bx^5 \quad \dots (17)$$

Here the equation (17) contains two constant $A\ and\ B$ so we need two linear algebra equations. We get one of them by the initial condition $y(1) = 0$ so we get the equation:

$$A + B = 0 \quad \dots (18)$$

For finding the second equation we should find additional condition which be get it from the above differential equation by substituting the initial condition $y(1) = 0$ so we get:

$$y'(1) - 2y(1) = 1 \implies y'(1) = 1$$

And after taking derivative to equation (17) and substituted $y'(1) = 1$, We get:

$$2A + 5B = 1 \quad \dots (19)$$

So, from (18) and as we get:

$$A = {-1}/{3},\ B = {1}/{3}$$

And hence the solution is given by:

$$y = {-1}/{3}\,x^2 + {1}/{3}\,x^5$$

Example 2: To solve the differential equation:

$$x^2y'' + 3xy' - 3y = x^{-2}\ln x\ ;\ y'(1) = y(1) = 0$$

We take $\mathcal{T}$ to both sides of the differential equation and after substitution the initial conditions we put:

$$\mathcal{T}(y) = \frac{h(p)}{(p^2 - 3p + 2 + 3p - 3 - 3)(p+1)^2}$$

$$\mathcal{T}(y) = \frac{h(p)}{(p-2)(p+2)(p+1)^2} \quad \dots (20)$$

After taking $\mathcal{T}^{-1}$ to both sides, of (20), we can write the general solution as follows:

$$y = Ax + Bx^{-3} + Cx^{-2} + Dx^{-2}\ln x \quad \dots (21)$$

Where A, B, C are constants.

To find the values of A, B, C and D we need four linear equations. We get the first equation by substituting $y(1) = 0$ in equation (21). For the second equation we derive the general solution (21) and substitute the initial condition $y'(1) = 0$. But the third equation we need two additional conditions. Now,

$y''(1) + 3y'(1) - 3y(1) = 0 \implies y''(1) = 0$

$y'''(1) + 2y''(1) + 3y''(1) + 3y'(1) - 3y'(1) = 1 \implies y'''(1) = 1$

After substituting

$y(1) = 0, y'(1) = 0, y''(1) = 0, y'''(1) = 1$

in y, y', y'', y''' respectively we get:

$A + B + C = 0 \quad \dots (22)$

$A - 3B - 2C + D = 0 \quad \dots (23)$

$12B + 6C - 5D = 0 \quad \dots (24)$

$-60B - 24C + 26D = 1 \quad \dots (25)$

So, from (22),(23),(24) and (25) we get:

$A = 0.027\bar{7}, B = -{1}/{4}, C = 0.2\bar{2}, D = -{1}/{3}$ so the solution of the differential equation takes the form:

$$y = 0.027\bar{7}x - {1}/{4}\,x^{-3} + 0.2\bar{2}x^{-2} - {1}/{3}\,x^{-2}\ln x$$

Example 3: To solve the differential equation

$$x^3y''' - 3xy' + 3y = \sin(\ln x)\ ; y(1) = -2, y'(1) = 1, y''(1) = 0$$

We take $\mathcal{T}$ to both sides of differential equation and after substitution the initial conditions we put:

$$\mathcal{T}(y) = \frac{h(p)}{(p^3 - 6p^2 + 11p - 6 - 3p + 3 + 3)[(p-1)^2 + 1]} \quad \dots (26)$$

$$\mathcal{T}(y) = \frac{h(p)}{(p^3 - 6p^2 + 8p)[(p-1)^2 + 1]}$$

$$\mathcal{T}(y) = \frac{h(p)}{p(p-2)(p-4)[(p-1)^2 + 1]}$$

After taking $\mathcal{T}^{-1}$ to both sides we get:

$$y = Ax^{-1} + Bx + Cx^3 + Dsin\,(ln\,x) + Ecos\,(ln\,x) \quad \dots (27)$$

To find the values of A, B, C, D and E we need five linear equations. We get the first equation by substituting $y(1) = -2$ in equation (27). For the second equation we derive the general solution (27) and substitute the initial condition $y'(1) = 1$. For the third equation we derive the general solution (27) and substitute the initial condition $y''(1) = 0$. But the four equation we need two additional conditions. note that

$$y'''(1) - 3y'(1) + 3y(1) = 1 \implies y'''(1) = 9$$

$$y^{(4)}(1) + 3y'''(1) - 3y''(1) - 3y'(1) + 3y'(1) = 1 \implies y^{(4)}(1) = -26$$

So, we get the equations:

$A + B + C + E = -2 \quad \dots (28)$

$-A + B + 3C + D = 1 \quad \dots (29)$

$2A + 6C - D - E = 0 \quad \dots (30)$

$-6A + 6C + D + 3E = 9 \quad \dots (31)$

$12A + 5E = 13 \quad \dots (32)$

We solve the system of equations (28), (29), (30), (31) and (32) we get:

$$A = {}^{17}/_{32}, B = {}^{-73}/_{16}, C = {}^{113}/_{160}, D = {}^{159}/_{40}, E = {}^{53}/_{40}$$

And hence the solution is given by:

$$y = {}^{17}/_{32}\,x^{-1} - {}^{73}/_{16}\,x + {}^{113}/_{160}\,x^3 + {}^{159}/_{40}\,sin(ln\,x) + {}^{53}/_{40}\,cos(ln\,x)$$

Example 4: To solve the differential equation:

$$x^4y^{(4)} + 5x^3y''' = 3x^{-3}, y(1) = y'(1) = y''(1) = y'''(1) = 0$$

After taking ($\mathcal{T}$) to both sides we can write:

$$\mathcal{T}(y) = \frac{h(p)}{(p-3)(p-2)(p-1)(p+1)(p+2)} \quad \dots (33)$$

Taking $\mathcal{T}^{-1}$ to both sides of last equation, So we can write:

$$y = \mathcal{T}^{-1}\left[\frac{A}{(p-3)} + \frac{B}{(p-2)} + \frac{C}{(p-1)} + \frac{D}{(p+1)} + \frac{E}{(p+2)}\right]$$

$$y = Ax^2 + Bx + C + Dx^{-2} + Ex^{-3} \quad \dots (34)$$

To find the values of A, B, C, D and E we need five linear equations. We get the first equation by substituting $y(1) = 0$ in equation (34). For the second equation we derive the general solution (34) and substitute the initial condition $y'(1) = 0$. For the third equation we derive again the general solution (34) and substitute the initial condition $y''(1) = 0$. For the four equation we derive again the general solution (34) and substitute the initial condition $y'''(1) = 0$. But for the five equation we need additional condition, see

$$y^{(4)}(1) + 5y'''(1) = 3 \implies y^{(4)}(1) = 3$$

So we get the equations:

$A + B + C + D + E = 0 \quad \dots (35)$

$2A + B - 2D - 3E = 0 \quad \dots (36)$

$2A + 6D + 12E = 0 \quad \dots (37)$

$-24D - 60E = 0 \quad \dots (38)$

$40D + 120E = 1 \quad \dots (39)$

From above equations we get:

$$A = {}^{3}/_{40}, B = {}^{-1}/_{4}, C = {}^{1}/_{4}, D = {}^{-1}/_{8}, E = {}^{1}/_{20}$$

And hence the solution is given by:

$$y = {}^{3}/_{40}\,x^2 - {}^{1}/_{4}\,x + {}^{1}/_{4} - {}^{1}/_{8}\,x^{-2} + {}^{1}/_{20}\,x^{-3}$$

References

[1] Gabriel Nagy, “Ordinary Differential Equations” Mathematics Department, Michigan State University, East Lansing, MI, 48824.October 14, 2014

[2] Mohammed, A.H., “Linear Differential Equations” Al-Qadisiya university magazine for completely sciences, volume (7), number (2). (2002)

[3] Mohammed, A.H., Athera Nema Kathem, “Solving Euler’s Equation by Using New Transformation”, Karbala university magazine for completely sciences, volume (6), number (4. (2008)

[4] PhD, Andr’as Domokos, “Differential Equations Theory and Applications”, California State University, Sacramento, Spring, 2015