Conditions of Boundedness for Compact operators on Banach spaces

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Abstract: In this paper we study Compact operator between Banach spaces which are bounded. We establish two conditions that must be satisfied for boundedness of Compact operator.

Keywords: Closure of image ; Continuous at every points ; Compact operator ; Bounded

1. Introduction

A Banach space is a normed linear space that a complete metric space with respect to the metric derived from its norm. See [7]. This space has been studied by several authors in different context such as in [3], [1] and [6]. We give a brief discussion of conditions for boundedness of Compact operator. First Condition is the existence of the constant M ≥ 0 as we done in Remark 3 that the operator T: X → Y between two normed operator bounded by letting T is compact which associate with the closures of the images of B(0,2) and B(0, n) producing ||T|| = sup x∈B(0,2)||Tx|| ≤ sup x∈B(0,2)||x|| . Since x∈B(0,2) we use a constant M instead of x∈B(0,2) , hence ||Tx|| ≤ M is bounded as long as TB(0,2) ⊂ (TB(0,2)) and T(B(0,2)) is compact and hence bounded. In Lemma 4 we put S = {x ∈ x ||x|| = 1}. Since T(S) is compact and ||x|| = 1 this result ||T|| ≤ M which is bounded by constant M. Second condition is the operator T must be continuous at every point. See Theorem 6 we show that the operator T: X → Y is bounded by letting T is continuous for all x, y ∈ X we have ||Tx − Ty|| = ||T(x − y)|| ≤ M||x − y|| where M is a constant, then by letting T is continuous at every point. Since T is linear we have T(0) = 0 there exists δ > 0 with ||x|| ≤ δ and x ≠ 0 we define x = δx. Then ||−x|| ≤ δ. From linearity of T that ||Tx|| = ||(x/δ)||T(δx)|| ≤ M||x|| where M = 1/δ. Thus T is bounded and continuous at zero , then at every point. See Remark 3, Lemma 4, Definition 5 and Theorem 6).

2. Compact operators on Banach space

A continuous linear operator T : X → Y on Banach spaces is compact when T maps bounded sets in X pre-compact sets in Y , that is, sets with compact closure. Since bounded sets lies in some ball in X , and since T is linear , its sufficient to verify that T maps the unit ball in X to a pre-compact set in Y . Finite –rank operators are clearly compact. Both right and left compositions ToS and S oTo of compact T with continuous S produce compact operators. Recall a criterion for pre-compactness: a set E in a complete metric space is pre-compact if and only if its totally bounded, sense that, given ε > 0 , E is covered by finitely-many open balls of radius ε. We claim that operator-norm limits T = limi Ti of compact operators Ti are compact : given ε > 0 , choose Ti so that||Ti − T|| ≤ ε , cover the image of the Unit ball B1 under Ti by finitely-many open ball Uk of radius ε .

Since ||Ti x − Ty|| ≤ M||x − y|| hence bounded. In Lemma 4 we put S = {x ∈ x ||x|| = 1}. Since T(S) is compact and ||x|| = 1 this result ||T|| ≤ M which is bounded by constant M. Second condition is the operator T must be continuous at every point. See Theorem 6 we show that the operator T: X → Y is bounded by letting T is continuous for all x, y ∈ X we have ||Tx − Ty|| = ||T(x − y)|| ≤ M||x − y|| where M is a constant, then by letting T is continuous at every point. Since T is linear we have T(0) = 0 there exists δ > 0 with ||x|| ≤ δ and x ≠ 0 we define x = δx. Then ||−x|| ≤ δ. From linearity of T that ||Tx|| = ||(x/δ)||T(δx)|| ≤ M||x|| where M = 1/δ. Thus T is bounded and continuous at zero , then at every point. See Remark 3, Lemma 4, Definition 5 and Theorem 6).

A compact operator is a linear operator T from Banach space X to another Banach space Y, such that the image under T of any bounded subset of X is necessarily a bounded operator, and so continuous. Any bounded operator Y that has finite rank is a compact operator, the class of compact operators is a natural generalization of the class of finite –rank operators in an infinite-dimensional setting. When Y is a Hilbert space, it is true that any compact is a limit of finite-rank operators can be defined alternatively as the closure in the operator norm of the finite –rank operators.

Lemma 2 : Let K be a compact operator on H and suppose (Kx) is a bounded sequence in B(H) such that, for each x ∈ H the sequence (Kx) converges to Tx, where T ∈ B(H). Then (Kx) converges to TK in norm.Briefly, the above can be rephrased as: If K ∈ K(H) and ||Tn x − Tx|| → 0 for all x ∈ H then ||TK − Kx|| → 0 .

In words: Multiplying by a compact operator on the right converts a pointwise convergent sequence of operators into a norm convergent one .

Proof since (Kx) is bounded sequence, ||Kx|| ≤ M for some constant M. Then for all x ∈ H, ||Tx|| = limn ||Tnx|| ≤ ||M||||x|| and so ||T|| ≤ M . Let K be compact and suppose that ||TK − Kx|| → 0 . Then there exists some δ > 0 .
And a subsequence \((T_{n_j}K)\) such that \(\|TK - T_{n_j}K\| > \delta\). Choose unit vectors \((x_{n_j})\) of \(\mathcal{H}\) such that \(\|(TK - T_{n_j}K))x_{n_j}\| < \|T(T - T_{n_j})y\| + \|T(T - T_{n_j})y\|\). 

Now, using the convergence of \((Kx_{n_j})\) to \(y\), there exists \(n\) so that \(n > k\), \(\|Kx_{n_j} - y\| < \frac{\delta}{\|M\|}\). Also, using the convergence of \((T_{n_j})\) to \(T\), there exists \(m\) so that, for \(n_j > m\), \(\|(T_{n_j} - T)y\| < \frac{\delta}{4}\). Then, for \(j > \max[n,m]\) that right hand side of the displayed inequality is less than \(\frac{\delta}{2}\), and this contradiction shows that the supposition that \(\|TK - T_{n_j}K\| \to 0\) is false.

3. First Condition of Boundedness of Compact operator

Remark 3: A compact operator is bounded.

Proof

First we show that a compact linear operator \(T : X \to Y\) between normed spaces is bounded, if \(T\) is compact, then the closure of the image of \(B(0,1)\) and \(B(0,n)\) is compact, \(\Rightarrow\) compact hence bounded. 

For some \(M \in \mathbb{R}_{>0}\) since \(\overline{T}B(0,2) \subset \overline{T}B(0,2)\) and \(\overline{T}B(0,2)\) is compact hence bounded.

Lemma 4: Every compact operator is bounded.

Proof

Put \(S = \{x \in X : \|x\| = 1\}\), then \(T(S)\) is relatively compact, hence bounded (by \(M\)) therefore, \(\|T\| \leq M\).

Definition 5

Let \(X\) and \(Y\) be two normed linear spaces. We denote that both \(X\) and \(Y\) norms by \(\|\cdot\|\). A linear operator \(T : X \to Y\) is bounded if there is constant \(M \geq 0\) such that \(\|T x\| \leq M \|x\|\) for all \(x \in X\) \((1)\).

If no such constant exists, we say that \(T\) is unbounded. If \(T : X \to Y\) is a bounded linear compact operator we define operator norm \(\|T\|\) of \(T\) by \(\|T\| = \text{inf}\{M\|T\| \leq M\|\cdot\|\} \) for all \(x \in X\) \((2)\).

We denote the set of all linear maps \(T : X \to Y\) by \(\mathcal{L}(X,Y)\), and the set of all bounded linear maps by \(\mathcal{B}(X,Y)\). Then the domain and range spaces are the same, we write \(\mathcal{L}(X,Y) = \mathcal{L}(X)\) and \(\mathcal{B}(X,Y) = \mathcal{B}(X)\).

Then the closure of \(\mathcal{B}(X,Y)\) is compact, hence bounded.

4. Second Condition of Boundedness of Compact operator

Theorem 6

Linear compact operator is bounded if it is continuous.

Proof

First, suppose that \(T : X \to Y\) is bounded. Then, for all \(x, y \in X\), we have \(\|Tx - Ty\| \leq M\|x - y\|\). Where \(M\) is a constant for which \((1)\) holds, therefore we can take \(\varepsilon = M\).

In the definition of continuity, \(T\) is continuous.

Second, suppose that \(T\) is continuous at \(0\). Since \(T\) is linear, we have \(T(0) = 0\).

Choosing \(\varepsilon = 1\) in the definition of continuity, we conclude that there is a \(\delta > 0\).

Such that \(\|Tx\| \leq 1\) whenever \(\|x\| \leq \delta\) for any \(x \in X\), with \(x \neq 0\), we define \(\tilde{x}\) by \(\tilde{x} = \frac{x}{\delta}\).

Then \(\|\tilde{x}\| \leq \delta\), so \(\|\tilde{T}\| \leq 1\).

It follow from linearity of \(T\) that \(\|T\| = \frac{\|\tilde{T}\|}{\delta}\) \(\leq M\|x\|\).

Where \(M = \frac{1}{\delta}\). Thus \(T\) is bounded. The proof shows that if any compact linear map is continuous at zero, then it is continuous at every point.

References


