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Conditions of Boundedness for Compact operators on Banach spaces

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Abstract: ***In this paper we study Compact operator between Banach spaces which are bounded . We establish two conditions that must be satisfies for boundedness of Compact operator.***

Keywords: Closure of image ; Continuous at every points ; Compact operator ; Bounded

1. Introduction

A Banach space is a normed linear space that a complete metric space with respect to the metric derived from its norm . See [7] .This space has studied by several authors in different context such a in [3] , [1] and [6] . We give a brief discussion of conditions for boundedness of Compact operator. First Condition is the existence of the constant M ≥ 0 as we done in Remark 3 that the operator $T: X \to Y$ between two normed operator bounded by letting T is compact which associate with the closures of the images of $B(0,2)$ and $B(0,n)$ producing $\|T\| = \sup_{\|x\|=1} \|Tx\| \leq \sup_{x \in B(0,2)}$ .Since $\sup_{x \in B(0,2)} \in \mathbb{R}_{\geq 0}$ we use a constant M instead of $\sup_{x \in B(0,2)}$ for some M $\in \mathbb{R}_{\geq 0}$, hence $\|Tx\| \leq M$ is bounded as long as $TB(0,2) \subset \overline{TB(0,2)}$ and $TB(0,2)$ is compact and hence bounded .In Lemma 4 we put $S = \{X \in x: \|x\| = 1\}$. Since T(S) is compact and $\|x\| = 1$ this result $\|T\| \leq M$ which is bounded by constant M .Second condition is the operator T must be continuous at every point . See Theorem 6 we show that the operator $T: X \to Y$ is bounded by letting T is continuous for all $x, y \in X$ we have $\|Tx - Ty\| = \|T(x-y)\| \leq M\|x-y\|$ where M is a constant, then by letting T is continuous at 0. Since T is linear we have $T(0) = 0$ there exists $\delta > 0$ with $\|x\| \leq \delta$ and $x \neq 0$ we define $\tilde{x}$ by $\tilde{x} = \frac{\delta x}{\|x\|}$. Then $\|\tilde{x}\| \leq \delta$. From linearity of T that $\|Tx\| = \frac{\|x\|}{\delta}\|T\tilde{x}\| \leq M\|x\|$ where M = $\frac{1}{\delta}$. Thus T is bounded and continuous at zero , then at every point . (See Remark 3, Lemma 4, Definition 5and Theorem 6) .

2. Compact operators on Banach space

A continuous linear operator $T : X \to Y$ on Banach spaces is compact when T maps bounded sets in X pre-compact sets in Y , that is , sets with compact closure . Since bounded sets lies in some ball in X , and since T is linear , its suffices to verify that T maps the unit ball in X to a pre-compact set in Y . Finite –rank operators are clearly compact . Both right and left compositions ToS and SoT of compact T with continuous S produce compact operators . Recall a criterion for pre-compactness : a set E in a complete metric space is pre-compact if and only if its totally bounded , sense that , given $\varepsilon > 0$, E is covered by finitely-many open balls of radius ε . We claim that operator-norm limits T = $\lim_i T_i$ of compact operators T_i are compact : given $\varepsilon > 0$,choose T_i so that$|T_i - T|_{OP} < \varepsilon$, cover the image of the Unit ball B_1 under T_i by finitely-many open ball U_k of radius ε .

Since $|T_i x - Tx| < \varepsilon$ for all $\in X$, enlarging the balls U_k to radius 2ε covers TB_1 .
(If desired , rewrite the proof replacing ε by $\frac{\varepsilon}{2}$) . So there exist Banach spaces with compact operators which are not norm-limits of finite –rank operators . Recall that every compact operators $T: X \to Y$ on Hilbert space is an operator norm limit of finite rank operators . Given $> o$, let $y_1, \ldots, y_n$ be the centers of open $\varepsilon -$ balls in Y covering TB_1 , where B_1 is the unit ball in X . Let $T_\varepsilon = PoT$ where P is the projection of Y to the span of the y_i , then similarly , the sum of two compact operator is compact .

Definition 1

A compact operator is a linear operator T from Banach space X to anther Banach space Y , such that the image under T of any bounded subset of X is necessarily a bounded operator , and so continuous . Any bounded operator Y that has finite rank is a compact operator , the class of compact operators is a natural generalization of the class of finite –rank operators in an infinite-dimensional setting . When Y is a Hilbert space , it is true that any compact is a limit of finite-rank operators can be defined alternatively as the closure in the operator norm of the finite –rank operators .

Lemma 2 : Let K be a compact operator on $\mathcal{H}$ and suppose (T_n) is a bounded sequence in $\mathcal{B}(\mathcal{H})$ such that , for each $x \in \mathcal{H}$ the sequence $(T_n x)$ converges to Tx , where $T \in \mathcal{B}(\mathcal{H})$. Then $(T_n K)$ converges to TK in norm. Briefly, the above can be rephrased as:
If $K \in \mathcal{K}(\mathcal{H})$ and $\|T_n x - Tx\| \to 0$ for all $x \in \mathcal{H}$then $\|T_n K - TK\| \to 0$.

In words: Multiplying by a compact operator on the right converts a pointwise convergent sequence of operators into a norm convergent one .

Proof
Since (T_n) is bounded sequence , $\|T_n\| \leq M$ for some constant M . Then for all $x \in \mathcal{H}$, $\|Tx\| = \lim_n \|T_n x\| \leq M\|x\|$ and so $\|T\| \leq M$.
Let K be compact and suppose that $\|TK - T_n K\| \nrightarrow 0$. Then there exists some $\delta > 0$.

And a subsequence $(T_{ni}K)$ such that $\|TK - T_{ni}K\| > \delta$ .Choose unit vectors (x_{ni}) of $\mathcal{H}$ such that $\|(TK - T_{ni}K)x_{ni}\| > \delta$. [That this can be done follows directly from the definition of the norm of an operator .]. Using the fact that K is compact , we can find a subsequence (x_{nj}) of (x_{ni}) such that (Kx_{nj}) is convergent . Let the limit of this sequence by . Then for all j

$\delta < \|(TK - T_{nj}K)x_{nj}\| \le \|(T - T_{nj})(Kx_{nj} - y)\| + \|(T - T_{nj})y\|$.

Now , using the convergence of (Kx_{nj}) to y, there exists n so that , for $n_j > n$,

$\|Kx_{nj} - y\| < \frac{\delta}{8M}$. Also , using the convergence of (T_{nj}) to T , there exists m so that , for $n_j > m$, $\|(T - T_{nj})y\| < \frac{\delta}{4}$. Then , for $j > \max[n, m]$ that right hand side of the displayed inequality is less than $\frac{\delta}{2}$, and this contradiction shows that the supposition that $\|TK - T_nK\| \nrightarrow 0$ is false .

3. First Condition of Boundedness of Compact operator

Remark 3: A compact operator is bounded .

Proof

First we show that a compact linear operator $T : X \rightarrow Y$ between normed spaces is bounded , if T is compact , then the closure of the image of B(0,1) and B(0, n) is compact , then

$$\|T\| = sup_{\|x\|=1}\|Tx\| \le sup_{x\in B(0,2)} \|Tx\| \le M$$

For some M $\in \mathbb{R}_{\ge 0}$ since

$TB(0,2) \subset \overline{TB(0,2)}$ and $\overline{TB(0,2)}$ is compact hence bounded .

Lemma 4 : Every compact operator is bounded .

Proof

Put S = $\{x \in X : \|x\| = 1\}$, then T(S) is relatively compact , hence bounded (by M) therefore , $\|T\| \le M$.

Definition 5

Let X and Y be two normed linear spaces . We denote that both X and Y norms by $\|.\|$. A linear operator T : X$\rightarrow Y$ is bounded if there is constant $M \ge 0$ such that

$\|Tx\| \le M\|x\|$ for all $x \in X$ (1)

If no such constant exists , the we say that t is unbounded .

If $T: X \rightarrow Y$ is a bounded linear compact operator we define operator norm $\|T\|$ of T by

$\|T\| = in\{M | \|Tx\| \le M\|x\| \; for \; all \; x \in X\}$ (2) .

We denote the set of all linear maps $T: X \rightarrow Y$ by $\mathcal{L}(X, Y)$, and the set of all bounded linear maps by $\mathcal{B}(X, Y)$. Then the domain and range spaces are the same , we write

$\mathcal{L}(X, X) = \mathcal{L}(X)$ and $\mathcal{B}(X, X) = \mathcal{B}(X)$ equivalent expressions for $\|Tx\| \; are$:

$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$; $\|T\| = \sup_{\|x\|<1} \|Tx\|$;

$\|T\| = \sup_{\|x\|=1} \|Tx\|$ (3)

4. Second Condition of Boundedness of Compact operator

Theorem 6

Linear compact operator is bounded if it is continuous .

Proof

First , suppose that $T : X \rightarrow Y$ is bounded .

Then , for all $x, y \in X$, we have

$\|Tx - Ty\| = \|T(x - y)\| \le M\|x - y\|$.

Where M is a constant for which (1) holds, therefore we can take $= \in/M$.

In the definition of continuity, T is continuous.

Second , suppose that T is continuous at 0 . Since T is linear, we have T(0) = 0 .

Choosing $\in = 1$ in the definition of continuity , we conclude that there is a $\delta > 0$.

Such that $\|Tx\| \le 1$ whenever $\|x\| \le \delta$ for any $\in X$, with $x \neq 0$, we define

$\tilde{x}$ by $\tilde{x} = \delta \frac{x}{\|x\|}$.

Then $\|\tilde{x}\| \le \delta$, so $\|T\tilde{x}\| \le 1$.

It follow from linearity of T that $\|Tx\| = \frac{\|x\|}{\delta}\|T\tilde{x}\| \le M\|x\|$.

Where M = $\frac{1}{\delta}$. Thus T is bounded . The proof shows that if any compact linear map is continuous at zero , then its continuous at every point .

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