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Survey of Mathematical Programming and Related Concepts

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Abstract: ***The Theory of Mathematical Programming plays a important role in mathematics for solving the problems , as well as being a full fill area of research in Abstract Mathematics. In this "tutorial" on Abstract Mathematical Programming , we will try to point out the different types of Mathematical Programming .***

Keywords: Mathematical programming , Multi objective programming ,Fractional programming , Optimality, Duality.

1. Introduction

Some Notations, definitions are given which are subsequently used in this paper.

1.1 Notations

Let x , y $\in R^n$ where

x= $(x_1, x_2, \ldots\ldots, x_n)^T$ and

y= $(y_1, y_2, \ldots\ldots, y_n)^T$

Where T denote the transpose operation.

(i) x > y iff $x_i > y_i$, i=1,2,...,n

(ii) x$\geq$ y iff $x_i \geq y_i$, i=1,2,...,n and x $\neq$ y

(iii) x >=y iff $x_i >= y_i$, i=1,2,...,n

(iv) x =y iff $x_i = y_i$, i=1,2,...,n

For x , y$\in$ R , x $\geq$ y has the ususal meaning.

Definition 1.1.1: A set X$\subseteq R^n$ is said to be convex if for all x , $\bar{x} \in$X

tx + (1-t) $\bar{x} \in$ X, for t$\in$[0,1]

Definition 1.1.2: A numerial function f defined on a convex set X$\subseteq R^n$ is said to be convex at $\bar{x} \in$X with respect to X if f((1-t) $\bar{x}$ + tx)$\leq$(1-t)f($\bar{x}$) + tf(x) ,

For every x$\in$X and t$\in$[0,1].

f is said to be convex on X if it is convex at each $\bar{x} \in$X .

1.1.3 Definition : A set X$\subseteq R^n$ is said to be invex (η -vex) if there exists a vector function η : X×X$\rightarrow R^n$ such that x , $\bar{x} \in$X , $\lambda \in [0,1] \Rightarrow \bar{x}+\lambda\eta(x,\bar{x}) \in X$.

1.1.4 Definition: A differentiable real valued function f defined on a non- empty open set X$\subseteq R^n$ is said to be invex if there exists an n - dimensional vector function η: X×X$\rightarrow R^n$ such that for all x , $\bar{x} \in$X , f(x)-f($\bar{x}$)$\geq \eta(x,\bar{x})^t \nabla f(\bar{x})$

1.1.5 Definition: Let X$\subseteq R^n$ be an invex set with respect to η : X×X$\rightarrow R^n$. We say that f is Preinvex with respect to η If

f($\bar{x}+\lambda\eta(x,\bar{x})$)$\leq \lambda f(x)+(1-\lambda) f(\bar{x})$, for all x , $\bar{x} \in$X and $\lambda \in$[0,1].

1.1.6 Definition : Let X$\subseteq R^n$ be an invex set with respect to η : X×X$\rightarrow R^n$. We say that f is Preincave with respect to η If

f($\bar{x}+\lambda\eta(x,\bar{x})$)$\geq \lambda f(x)+(1-\lambda) f(\bar{x})$,

for all x , $\bar{x} \in$X and $\lambda \in$[0,1].

1.1.7 Definition : A subset X$\subseteq R^n$ is said to be a locally starshaped at $\bar{x} \in$X , if for every $x \in X$, there exists a positive number $a(\bar{x},x) \leq 1$ such that

$(1-\lambda)\bar{x}+\lambda x \in X,\ for\ 0<\lambda<a(\bar{x},x).$

1.1.8 Definition : A numerial function f defined on a set X$\subseteq R^n$ is said to be semilocally convex (slc) at $\bar{x}$ if X is locally starshaped at $\bar{x}$ and corresponding to $\bar{x}$ and each $x \in X$, there exists a positive number $d(\bar{x},x) \leq a(\bar{x},x)$ Such that

$f((1-\lambda)\bar{x}+\lambda x) \leq (1-\lambda) f(\bar{x})+\lambda f(x),$

$0<\lambda<d(\bar{x},x)$

1.2 Mathematical Programming

A general mathematical programming problem is of the form

Maximize (Minimize) f(x)

Subject to ($\leq,\geq$,=) b_j **; j=1,2,...,m..................(1.2.1)**

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The function f(x) is called the objective function and $g_j(x)$ (j=1,2,...,m) are called Constraint functions of the given problem. A vector which satisfies **(1.2.1)** is called a feasible solution. Any feasible solution which maximize (minimize) the objective function is called an optimal solution of the problem.

If all the relations among the variables are linear, both in constraints and the function to be optimized, we call it a linear programming problem. If one or more of the functions appearing in mathematical programming problem are non - linear, then it is known as a non - linear programming problem and is given as follows:

(NP) Minimize f(x)
Subject to
g(x)<=0
x$\in X_0$

where $X_0 \subseteq R^n$ is a non empty set and

f : $X_0 \rightarrow$ R and g: $X_0 \rightarrow R^m$

Let X= {x$\in X_0$**: g(x) <=0**} be the set of all feasible solutions of (NP)

Definition 1.2.1 : $\bar{x}$ is said to be a local minimum solution tom problem (NP) if $\bar{x}$ $\in$X and there exists $\in$>0 such that X $\in$ $N_\in(\bar{x}) \cap X \Rightarrow f(x) \geq f(\bar{x})$

Where $N \in (\bar{x})$ = {x$\in R^n$:llx- $\bar{x}$ ll<$\in$ }

Definition 1.2.2 : $\bar{x}$ is said to be minimum sol*ution to the problem (NP) if* $\bar{x}$ $\in$*X and f(*$\bar{x}$*) =* $\min_{x \in X} f(x)$

1.3 Multi Objective Programming

In actual practice these are problems when more than one conflicting objectives come into existance . Such problems are grouped into the class of multi objective programming problems . Unlike in the single objective case , a solution does not exist that maximizes or minimizes all the objective simultaneously . A good decision is based on the principle that there is no other alternate that can be better in some aspects and not worse in every aspect of consideration. At present, many solutions , weak efficient solutions , Pareto optimal solutions exist for such types of problems.

The general multi objective programming problem can be written as

(VOP)
Minimize f(x) = ($f_1(x), f_2(x),\ldots\ldots, f_p(x))$
Subject to g(x)<=0
x$\in X_0$

If all the objective and constraint function in a multi objective programming problem are linear , such a programming problem is called linear multi objective programming problem , Otherwise it is called non-linear multi objective programming problem.

efinition 1.3.1: An element x^0 $\in$X is said to be an efficient (Pareto Optimal, non dominated, non - inferior) solution of (VOP) if there exists no x $\in$X such that

$f(x) \leq f(x^0)$.

1.4 Fractional Programming Problem

Here, we consider a special class of programming problems where the objective function function is the ratio of two functions. These ratios represent some kind of efficiency measure for a system and arise whenever an optimization of ratios such as Performance / cost , income / invesment , cost / time etc. is required . Such programming problems and they are of the form :

Maximize $\dfrac{f(x)}{g(x)}$

Subject to

$h_j(x) \leq 0$
$x \in X_0$ **; j=1,2,.......,m.............................. (1.4.1)**

Where g(x)>0 on X_0

(1.4.1) is called a linear fractional program if all functions f(x) , g(x) and $h_j(x)$, j=1,2,......,m are linear.

A multiobjective fractional programming is of the form :

f

(VFP) Minimize $\left(\dfrac{f_1(x)}{g_1(x)}, \dfrac{f_2(x)}{g_2(x)}, \ldots\ldots\ldots, \dfrac{f_p(x)}{g_p(x)}\right)$

Subject to $h_j(x) \leq 0$, $x \in X_0$ **; j=1,2,.......,m............(1.4.2)**

(1.4.2) is called linear multi objective fractional programming problem if all the objective and constraint functions are linear. Otherwise, it is called non - linear multi objective fractional programming problem .

1.5 Min-max Fractional programming problem:

A Min-max fractional programming problem (p) as follows:

p) $\min_{x \in X}(\max_{y \in Y} \dfrac{f(x,y)}{g(x,y)} = F(x))$

Subject to X = {$x \in R^n : h(x) \in R_+^p$**} and Y is compact subset of** R^m **.**

Where functions f(. , .) and

g(. , .) : $R^n \times R^m \rightarrow$ R are continous on X ×Y , and for each y$\in$Y , f(. , y) , g (. , y) and

h(.) : $R^n \rightarrow R^m$ are locally lipschitz functions without loss of generality , we may assume that

f(x ,y) $\geq$ 0

g(x ,y) $\geq$ 0 for all (x , y) $\in$ X ×Y .

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2. Optimality

Optimality conditions are very important because they lead to the identification of optimal solution. Fritz John [1] gave necessary optimality criterion for a non - linear programming problem without imposing any constraint Qualification . Kuhn and Tucker [2] obtained and necessary optimality condition by imposing restrictions on constraints called constraint Qualification . They also obtained sufficient optimality conditions by assuming the functions to be convex. Optimality conditions for a non - linear programming problem involving invex functions have been obtained by Hanson [3],[4] and Kaul and Kaul [5]. For a multi objective programming problem , invexity has been applied by Weir [6] and preinvexity by Weir and Mond [7] etc. Optimality conditions for a fractional programming with semilocally preinvex functions have been optained by Stancu - Minasian [8].

3. Duality

The duality theory is an important part of the optimization theory . A main question which is investigation in duality is that under what assumptions it is possible to associate an equivalent maximization problem to a given minimization problem . Duality in linear programming was first introduced by Neumann [9] and was later studied by Dantzig and Orden [10] , Isermann [11] developed multi objective duality in linear case have been given by Jahn [12] others.

Duality for non - linear programming and multi objective programming problems involving semilocally preinvex functions have been studied by Preda et al . [13] and Weir and Mond .

4. Convexity

Convexity plays a key role in mathematical programming . Though many significant results in mathematical programming have been derived under convexity assumptions , still most of the real world problems are non - convex in nature. The concept of semilocally convex functions was introduced by Ewing and was further extended to semilocally quasi-convex , semilocally pseudo-convex functions by Kaul and Kaur. Suneja and Gupta defined the (strict) semilocally pseudo-convexity at a point with respect to a set . By using these concepts Kaul and Kaur and Suneja and Gupta obtained optimality conditions and duality results for non-linear programming problems. These results have been extended by Preda , for multiple objective programming problems involving semilocally convex and related functions has been studied by Lyall et al.

Certain important results for the non - linear programming problem (NP) are proved in the form of the following theorems.

1 Theorem: The set X of all feasible solutions of(NP) is invex with respect to η at each of the points at which g is r-preinvex with respect to η .

2 Theorem: Let f and g be r - preinvex at $\bar{x}$ on X with respect to the same function η ,and let $\bar{x}$ be a local minimum in (NP). Then $\bar{x}$ is a global minimum in (NP).

3 Theorem: Let $\bar{x}$ be a global minimum in (NP) and $\eta : X \times X \rightarrow R^n$ be a vector valued function satisfying the condition $\eta(x,u) \neq 0$, for all x , u ∈ X x ≠ u . If f is strictly r - preinvex at $\bar{x}$ on X with respect to and g is r - preinvex at on X with respect to η , then $\bar{x}$ is a unique optimal solution in (NP).

4 Theorem: Assume that a point $\bar{x}$ is feasible for problem (NP) and let the following Kuhn - Tucker optimality condition be satisfied at this point.

$$\nabla f(\bar{x}) + \bar{\xi}^t \nabla g(\bar{x}) = 0$$

$$\bar{\xi}^t g(\bar{x}) = 0$$

$$\bar{\xi} \in R^m , \bar{\xi} >= 0$$

If the Lagrangian is an r-invex function with respect to η at $\bar{x}$ on X, then $\bar{x}$ is a global minimum point in (NP).

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